MODELS FOR TYPE THEORY

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PLAN

TYPE THEORIES

CATEGORICAL MODEL THEORY

MODELS OF TYPE THEORY

THE POLYNOMIAL MODEL

FUNCTIONAL EXTENSIONALITY

DIALECTICA INTERPRETATION

TYPE THEORIES

1910 RUSSELL & WHITEHEAD

Predicative Theory

1930 CHURCH

Simple Theory

1970 GIRARD, REYNOLDS

Type quantification

1970 MARTIN-LÖF

Dependent types

WHAT IS TYPE THEORY FOR?

Foundation of Mathematics

Programming Language

Constructive Mathematics

Proof assistant

Univalent foundations (again!)

This talk:-

WHAT IS TYPE THEORY?

DEPENDENT TYPES

QUANTIFICATION

Space of sections

$$\frac{B(a) \text{ type } [a \in A]}{\Pi a \in A \,.\, B(a) \text{ type}}$$

Total space

$$\frac{B(a) \text{ type } [a \in A]}{\Sigma a \in A \,.\, B(a) \text{ type}}$$

(Type formation)

DEPENDENT TYES

THEORY OF EQUALITY

$$\frac{A \text{ type}}{Id_A(a,a') \text{ type } [a,a' \in A]}$$ Formation

$a \in A \vdash r(a) \in Id_A(a,a)$ Introduction

$b \in \Pi a. B(a,a,r(a))$ Elimination

$\vdash J(b) \in \Pi a,a',p.\, B(a,a',p)$

(Here for all

$B(a,a',p)$ type $[a,a' \in A, p \in Id_A(a,a')]$)

DEPENDENT TYPES

INDUCTIVE DEFINITIONS

$\mathbb{N}$ type $\quad \vdash 0 \in \mathbb{N} \quad n \in \mathbb{N} \vdash Sn \in \mathbb{N}$

$a \in P0,\ f \in \Pi n. Pn \to PSn \vdash$

$\mathrm{rec}(a,f) \in \Pi n\, Pn$

$$\mathrm{rec}(a,f)(0) = a$$

$$\mathrm{rec}(a,f)(Sn) = f(\mathrm{rec}(a,f)\,n)$$

Computation rules

[= is a relation ; Id is a type]

UNIVALENCE PROJECT

Idea of $Id_A(a,a')$ as homotopies/paths from a to a'. (Folklore.)

h-hierarchy of types:

$\Sigma a \in A \ \Pi a \in A' \ Id_A(a,a')$ says

A is contractible etc.

univalent fibrations $B(a)$ $[a \in A]$

$$Id_A(a,a') \longrightarrow HIso(Ba, Ba')$$

is itself a homotopy equivalence.

(Echoes of higher categories.)

SEMANTICS

MODEL THEORY: First order logic as a misleading success story.

PROBLEMATIC THEORIES

Intuitionistic logic/theories

Lambda calculus

Programming languages

Linear logic

Theories of proof

[Process calculi; quantum protocols ....]

CATEGORICAL MODEL THEORY

Flexible framework in which
to answer

What is an interpretation?

Benefits

Proofs that one has a
semantics

Possible theory of models

(Neither a priori obvious.)

MAIN THEMES

Theories represented as categories

Models of theories in general categories

Formal syntax	Structured categories	Specific semantics

Induction on the structure of terms done once and for all.

EXAMPLES

Higher order intuitionistic type theory

$\sim$ Toposes

Programming with effects

$\sim$ Monadic semantics

Linear logic

$\sim$ $*$ autonomous categories (plus)

BROADER CONTEXT

(cf. Mathematical Structures in Computer Science)

NOT Categories per se

BUT Abstract mathematics

so parallel to

Combinatorial } structure in proof
Geometric } theory.

· Structured operational semantics.

DEPENDENT TYPES

Seely's early attempt

Locally cartesian closed categories

(Fine for Π, Σ but not faithful to intended meaning of Id)

MODELS OF TYPE THEORY

(ancient history)

Category $\mathbb{C}$ with 1

Class $\mathbb{E}$ of squarable maps :

given $e \in \mathbb{E}$ and f the pullback

$$\begin{tikzcd} \cdot \arrow[r, "f'"] \arrow[d, "e'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \cdot \arrow[d, "e"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$

exists with $e' \in \mathbb{E}$

$\mathbb{E}$ contains all isomorphisms and all $A \to 1$ are in $\mathbb{E}$.

MEANING

$A \in \mathbb{C}$ $\qquad$ A type

$B \xrightarrow{f} A \in \mathbb{C}$ $\qquad$ $b \in B \vdash f(b) \in A$

$B \longrightarrow A \in \mathbb{E}$ $\qquad$ $B(a)$ type $[a \in A]$

Main issue: Coherence in the sense of category theory ignored except by Maietti.

FIBRATION VIEW

Categories with display maps

Comprehension categories

(Unresolved folklore)

In effect

$$\begin{tikzcd} \mathbb{E} \arrow[r, hook] \arrow[dr] & \mathbb{C}^2 \arrow[d, "\mathrm{cod}"] \\ & \mathbb{C} \end{tikzcd}$$

Subfibration with $\mathbb{C} \to \mathbb{E}(1) \to \mathbb{C}$
an isomorphism.

QUANTIFICATION

(still ancient history)

Π Fibration $\mathbb{E}$ equipped with right adjoints to pull back satisfying Beck-Chevalley

Σ Class $\mathbb{E}$ closed under composition so fibration equipped with left adjoints etc ....

AXIOM OF CHOICE

$$\prod a \in A \sum b \in Ba . \; C(a,b)$$

$$\longrightarrow$$

$$\sum f \in \prod a \in A . Ba . \; \prod a \in A . \; C(a, f(a))$$

This should be a categorical isomorphism.

IDENTITY TYPES

(a modern view)

Category $\mathbb{C}$ equipped with a weak factorization system

$$
\begin{tikzcd}
B \arrow[rr, "f"] \arrow[dr] & & A \\
& P_f \arrow[ur] &
\end{tikzcd}
$$

with lifting property so acyclic cofibration

in $\mathbb{E}$ so "fibration"

POLYNOMIALS

Map
or better
type in context

$S \longleftarrow P$

$P(s)$ type $[\, s \in S\,]$

(Polynomial functor

$X \longmapsto \sum s \in S.\ X^{Ps}$)

W-TYPES

If $S \leftarrow P$ a polynomial then (in good cases) the free monad generated by the polynomial functor is given by another polynomial.

[Terms generated by a signature: first general form of induction]

BICATEGORY OF POLYNOMIALS

(Gambino – H)

$$\Sigma a \in A \left(\Sigma c \in C.\, X^{D_c} \right)^{B_a}$$

$$= \sum (a,f) \in \sum a \in A.\, C^{B_a}.\; X^{\sum b \in B_a.\, D_{f(b)}}$$

$$= \sum s \in S\;\; X^{P_s}$$

with

$$S = \sum a \in A.\, C^{B_a} \qquad P_s = \sum b \in B_a\, D_{f(b)}$$

MAPS OF POLYNOMIALS

$$\begin{tikzcd}
U & X \arrow[l] & f^*Y \arrow[l, "F"'] \arrow[d] \arrow[dll, phantom, very near start, "\lrcorner"] \\
V \arrow[from=u, "f"'] & & Y \arrow[ll]
\end{tikzcd}$$

(2-cells of the bicategory.)

EXPLANATION OF MAPS

Freeness! What maps

$$\Sigma_{u \in U} A^{X_u} \longrightarrow \Sigma_{v \in V} A^{Y_v}$$

must exist?

A choice $f : U \longrightarrow V$ and then

$A^{X_u} \longrightarrow A^{Y_{f(u)}}$ which must be

given by $F_u : Y_{f(u)} \longrightarrow X_u$.

CATEGORICAL VIEW

$\mathbb{E} \longrightarrow \mathbb{C}$ fibration then

$\mathrm{Poly}\,\mathbb{E} \longrightarrow \mathbb{C}$ is the result of freely adding sums to

$$\mathbb{E}^{op} \longrightarrow \mathbb{C}.$$

That is

$$\mathrm{Poly}(\mathbb{E}) = \sum \mathbb{E}^{op}.$$

EXTRAORDINARY FACT

(Altenkirch, Levy, Staton for the case of Sets)

Poly $\mathbb{E}$ is cartesian closed

(and with Π Σ along display maps).

(Related earlier model/calculations by Birkedahl – Rovelli.)

WEAK QUESTION

If $\mathbb{C}$ a cartesian closed category then is it the base category of types for some model $\mathbb{E} \to \mathbb{C}$ of type theory?

Unfortunately YES.

Displays
"
Fibrations

} take product projections

(That tells us something!)

BETTER QUESTION

Take $\mathbb{E} \longrightarrow \mathbb{C}$: get $\mathrm{Poly}(\mathbb{E}) \longrightarrow \mathbb{C}$ fibration with cartesian closed fibres and with Π and Σ. Is it part of a model $\mathrm{Poly}(\mathbb{E} \to \mathbb{C})$ of type theory?

Answer (T. von Glehn) YES

Can extend along $\mathbb{C} \longrightarrow \mathrm{Poly}\,\mathbb{C}$ and define $\begin{matrix}\Pi \\ \Sigma\end{matrix}$, weak factorization

FUNCTIONAL EXTENSIONALITY

$$B(a) \text{ type } [a \in A]$$

$$\Pi a \in A \; \mathrm{Id}_{B(a)}(fa, ga)$$

$$\to \mathrm{Id}_{\Pi a \in A . B(a)}(f, g)$$

(Many equivalent formulations)

GENERAL ISSUE

Type constructions

$\prod a.B_a \quad \sum a.B_a \quad Id_A(a,a') \quad \mathbb{N}$

"determines" the elements/terms of types T, but do they determine the (elements of) $Id_T(t,t')$ etc etc ?

(Types carry the structure of a weak ω-groupoid !)

COMPETING VIEWS

Homotopy Type Theory

Univalence ~ as much identification of types as possible: so identities as rigid as possible.

Pure Type Theory

Rules determine types so nothing more is true of identities than has to be

Pragmatic Type Theory

Lots of different models. So who cares?

IS THERE AN ISSUE?

Theorem (Voevodsky) Univalence implies functional extensionality. So it holds in the simplicial set model.

Theorem (von Glehn) Functional extensionality fails in polynomial models.

SO YES!

DIALECTICA INTERPRETATION

(GÖDEL 1942/1958)

FORMULAE OF HA

$\longmapsto$ FORMULAE $\exists u \forall x\, A(u,x)$

Functionals of GÖDEL'S T

QF

Free ccc with nno

CRUCIAL STEP

$$\exists u \forall x A(u,x) \longrightarrow \exists v \forall y B(v,y)$$

$$\Downarrow$$

$$\exists f : U \Rightarrow V \quad \exists F : U \times Y \longrightarrow X$$

$$\forall u\, y \quad A(u, F(u,y)) \longrightarrow B(f(u), y)$$

VARIANT DIALECTICA

$U \leftarrow X \leftarrow A \sim \Sigma_{u \in U} \Pi_{x \in X(u)} A(u,x)$

Maps

$U \leftarrow X \leftarrow A$ to $V \leftarrow Y \leftarrow B$

are

$f : U \Rightarrow V \quad F : \Pi u \in U . y \in Y(f(u)) . X(u)$

$$\phi : \Pi u \in U . y \in Y(f(u)) . a \in A(u, F(u,y)) . B(f(u), y)$$

CATEGORICAL INTERPRETATION

Fibred category $\quad \mathbb{E} \longrightarrow \mathbb{C}$

first add products $\quad \Pi\mathbb{E} \longrightarrow \mathbb{C}$

then add sums $\quad \Sigma\Pi\mathbb{E} \longrightarrow \mathbb{C}$

the Dialectica fibration

Compare

$$\mathrm{Poly}(\mathbb{E}) = \Sigma(\mathbb{E}^{op}) \longrightarrow \mathbb{C}$$

so $\mathrm{Poly}^2(\mathbb{E}) = \Sigma((\Sigma\,\mathbb{E}^{op})^{op})$

$$= \Sigma\Pi\mathbb{E}$$

DELICATE POINT

Start with model $\mathbb{E} \to \mathbb{C}$ of type theory. Then

$$\mathrm{Poly}^2(\mathbb{E} \to \mathbb{C})$$

is (a little) more complicated than that arising from

$$\mathrm{Poly}^2(\mathbb{E}) \longrightarrow \mathbb{C}.$$

Is there a significant difference?

THE FUTURE

There are many analogous functional interpretation models:

- Diller-Nahm
- "Copenhagen Interpretation"

To do

- Compare and contrast
- Develop tools for calculation
- Create a model theory of

MODELS FOR TYPE THEORY