Functional Interpretations of Type Theory

CMI Workshop: Quantum Mechanics and Computation

J. M. E. Hyland

University of Cambridge

3rd October 2013

Outline

Contents

General Aim

In the spirit of Categorical Logic, to sketch an interpretation of Type Theory based on the idea of the Dialectica Interpretation.

What is Category Theory for?

O why bother with Abstract Mathematics?

How Others See Us

- What has Category Theory got to do with Computer Science?
- What has Category Theory got to do with Quantum Mechanics?

What One Could Ask

- On the one hand what is the point of abstract algebra?
- On the other why fuss about the theory of stacks?

Could we ban the phrase abstract nonsense?

Categorical Logic

Main Themes include

- Models of theories in general categories
- Theories represented as categories

A siginificant contribution is to give accounts of the interpretion of a formal system. Categories with structure act as intermediaries between some syntax and some specific semantics.

- To show that one has a model of a theory one typically runs through an induction over the structure of terms, propositions, types.
- A categorical analysis does the induction once and for all when the characterization is determined: after that one has direct access to models.
- Is this honest? Where in the literature is there a proof that small categories and profunctors give a bicategory?

First Order Model Theory

An interpretation of first order logic is a first order structure i.e. a set equipped with functions and relations. Meaning is determined by 'Tarski's definition of truth' e.g.

$$A \wedge B \textit{ is true if and only if } A \textit{ is true and } B \textit{ is true}$$

$$A \vee B \textit{ is true if and only if } A \textit{ is true or } B \textit{ is true}$$

and so on. The apparent tiresome tautology makes this aspect of the subject straightforward. That is misleading: it is not like that most of the time.

Other systems

- Constructive logic: first order, higher order.
- Lambda calculus: functional programming.
- Proof theory, type theory.
- Linear logic, operads.
- Process calculi, quantum protocols.

In these case the distinction syntax/semantics or theory/model is not so clear cut. But

- when we have the theory, we need an analysis of what it is to be a model;
- when we have the models we seek a language in which to articulate their structure.

Example: Higher-order constructive logic

Exploiting categorical constructions

An interpretation of higher order impredicative type theory is given by an elementary topos. (Or vice versa?)

Definition

An elementary topos is a category with finite limits and power objects.

Theorem

Let G be a left exact comonad on a topos $\mathcal{E}$. Then the category $\mathcal{E}_G$ of coalgebras is a topos with a surjective geometric morphism $\mathcal{E} \to \mathcal{E}_G$.

We do not have as ready access to $\mathcal{E}_G$ from the point of view of type theory.

Example: Pure Lambda Calculus

Making a precise definition

What is an interpretation of the λ-calculus.
An algebraic theory or abstract clone $\mathcal{L}$ equipped with a natural retraction

$$\mathcal{L}(n+1) \lhd \mathcal{L}(n)$$

in the category $[\mathbb{F}, \mathbf{Set}]$.
This definition was implicit in Scott's talk. Usually when a proof that we have an interpretation is given then it either uses this definition or else the Scott Representation Theorem.

Theorem

A reflexive object U with $(U \Leftarrow U) \lhd U$ in a cartesian (closed) category gives an interpretation of the λ-calculus; and any interpretation arises in this way.

Example: Operads

- Let S be 2-monad on the 2-categories **Cat** $\hookrightarrow$ **CAT** equipped with a distributive law $SP \to PS$ over the presheaf construction.
- Then S extends to a pseudomonad $\hat{S}$ on the bicategory **Prof** of small categories and profunctors (distributeurs).
- A (coloured) S-operad is a normal monad in the Kleisli bicategory $\mathrm{Kleisli}(\hat{S})$.

This provides a concise defintion and a clear theory of change of base. Sadly it appears to be an abstraction too far for the operads community.

Example: Categorical Quantum Mechanics

- A dagger category is a category $\mathbb{C}$ equipped with an identity on objects functor $\dagger : \mathbb{C}^{op} \to \mathbb{C}$ which is involutive.
- A dagger symmetric monoidal category is a weak monoid in dagger categories.
- A dagger compact (closed) category is a dagger symmetric monoidal category in which all objects have duals compatible with the dagger structure.

Higher dimensional versions featured in Baez's talk. The basic ideas have the enormous benefit of demystifying quantum information protocols e.g. quantum teleportation, entanglement swapping (work of Abramsky, Coecke and the Oxford group).

Abstraction in Elementary Mathematics

So pervasive that we do not remark on it

Two questions

- Two old women set out at dawn, each walking at constant velocity one from A to B the other from B to A. They pass each other at noon and arrive respectively at A at 4pm and at B at 9pm. What time was dawn?
- I conceal a 10 kopeck or 20 kopeck coin and you guess its value. If you are right then you get the coin and if you are wrong then you pay me 15 kopecks. Is this a fair game?

What is the connection?

There is a mathematical and an accidental (cultural) question!

Categorical Proof Theory

Propositions vs Proofs

Realizability and Functional Interpretations are usually thought of as interpretations of predicate logic i.e. they concentrate on the entailment relation $\phi \vdash \psi$. That is proof theory as a study of provability not of proofs themselves.

- Traditional Proof Theory: Indexed Preorders
- Categorical Proof Theory: Indexed Categories

Many traditional interpretations are the preordered set reflection of a natural categorical proof theory.

Examples

- modified realizability
- extensional realizability
- van den Berg's Herbrand realizability

The Dialectica Interpretation

Origins

K. Gödel. *Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes*. Dialectica, 1958.

Interpretation of Heyting arithmetic in a system of primitive recursive functionals of finite type (Gödel's system T) via formulae

$$\exists u.\forall x.A(u,x)$$

with A decidable. Crucial ingredient: the interpretation of the implication

$$\exists u.\forall x.A(u,x) \to \exists v.\forall y.B(v,y)$$

is given by

$$\exists f \in U \to V, F \in U \times Y \to X.$$

$$\forall u,y.A(u,F(u,y)) \to B(f(u),y)$$

The Dialectica Interpretation

Developments in mathematical logic

- C. Spector. *Provably recursive functionals of analysis: a consistency proof of analysis by an extension of principles formulated in current intuitionistic mathematics*. AMS, 1962. (Extension of the interpretation to analysis)
- J.-Y. Girard. *Interprétation fonctionelle et élimination des coupures de l'arithmetique d'ordre superieure*, Paris VII, 1972.
  (2nd order impredicative system F: extension to it.)
- A. S. Troelstra. *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*. SLN 334, 1974. (Miscellaneous applications.)
- U. Kohlenbach. Monotone interpretation: proof mining. 1990-today. (Applications to principles in analysis. Connection with the idea of hard vs soft analysis?)

The Dialectica Interpretation

Simple example: intellectual hygiene for first year undergraduates

Take $\mathbf{x} = (x_n) \in \mathbb{N} \Rightarrow \mathbb{N}$ an infinite sequence of natural numbers. Then

$$\exists N, K.\forall n \geq N.x_n \leq K \rightarrow \exists M.\forall m.x_m \leq M$$

is constructively valid.

This interprets as

$$\exists \mu : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^2 \rightarrow \mathbb{N} \qquad \exists \nu : \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^2 \times \mathbb{N} \rightarrow \mathbb{N}$$

$$\forall \mathbf{x}, N, K, m.(\nu \geq N \wedge x_\nu > K) \vee x_m \leq \mu$$

Here

$$\mu = \mu(\mathbf{x}, N, K) \quad \text{and} \quad \nu = \nu(\mathbf{x}, N, K, m)$$

The Dialectica Interpretation

Intuitive Reading

For μ consider $x_0, \cdots, x_{N-1}$ and take the maximum of those and K.
Now if $x_m \leq \mu$, we will be done and so we can set $\nu = 0$; but otherwise $x_m > \mu$ and so necessarily $m \geq N$ and so we output $\nu = m$ and then certainly $\nu \geq N$ and $x_\nu > \mu \geq K$.

Note that we could always set $\nu = m$ outright.

Basic proof theory

The point is that from a proof of the proposition we extract functionals μ and ν definable in Gödel's T. A simple direct proof will produce intuitive functionals.

Dialectica Categories

de Paiva, 1986: The Dialectica implication as maps in a category:

- objects $U \leftarrow A \rightarrow X$
- maps $U \leftarrow A \rightarrow X$ to $V \leftarrow B \rightarrow Y$
  - $f : U \rightarrow V$
  - $F : U \times Y \rightarrow X$
  - $\phi : \Pi u \in U y \in Y . A(u, F(u, y)) \rightarrow B(f(u), y)$

Originally $U \leftarrow A \rightarrow X$ was a relation between U and X and so ϕ an inclusion.

Variants

- Girard Categories and Linear Logic.
- Diller-Nahm monad: cartesian closed categories

Folklore Understanding of the Dialectica

Read the object

$$U \leftarrow A \rightarrow X \qquad \text{as} \qquad \Sigma u \in U.\Pi x \in X.A\,.$$

The Dialectica maps say that Σ and Π have been added freely.

Freely adding sums

A map of formal sums $\Sigma_{i \in I} A_i$ to $\Sigma_{j \in J} B_j$ is given by

$$f : I \longrightarrow J \quad \text{and} \quad \phi_i : A_i \longrightarrow B_{f(i)} \quad \text{all } i \in I\,.$$

Freely adding products

A map of formal products $\Pi_{i \in I} A_i$ to $\Pi_{j \in J} B_j$ is given by

$$g : J \longrightarrow I \quad \text{and} \quad \psi_j : A_{g(j)} \longrightarrow B_j \quad \text{all } j \in J\,.$$

Dependent Dialectica

The natural extension freely adding sums of products

Regard $U \leftarrow A \rightarrow X$ as $U \leftarrow U \times X \leftarrow A$ so that we see a constant type indexed over U. Natural to relax that condition and consider a category

- objects indexed families $U \leftarrow X \leftarrow A$, or in type theory,

$$X(u) \text{ type } [u \in U] \quad A(u, x) \text{ type } [u \in U,\ x \in X(u)]$$

- maps $U \leftarrow X \leftarrow A$ to $V \leftarrow Y \leftarrow B$
  - $f : U \Rightarrow V$
  - $F : \Pi u \in U. \times Y(f(u) \Rightarrow X(u)$
  - $\phi : \Pi u \in U, y \in Y(f(u)).A(u, F(u, y)) \Rightarrow B(f(u), y)$

written in Type Theory. (There is a categorical diagram but I could not typeset it.)

Some Related Ideas

Game Semantics

- Simple games as free Πs of free Σs of free Πs ... .
  (Many papers by Cockett, Seely and others.)

Free bicompletions

- A. Joyal. *Free bicompletion of enriched categories* and *Free bicomplete categories.* C. R. M. Rep. Acad. Sci. Canada, 1995.
  (Information on the additives of Linear Linear Logic.)

Free sums and products

- P. Hofstra. *The dialectica monad and its cousins.* CRM Proceedings and Lecture Notes 53.
  (Analysis of free addition of simple products and sums.)

A Dialectica Interpretation of Type Theory?

Raised by Per Martin-Löf at the Troelstra meeting in 1999

Possible Motivation

A judgement

$$t(a_1, \cdots a_n) \in B \quad [a_1 \in A_1 . \cdots a_n \in A_n]$$

already has the shape

$$\exists t . \forall \mathbf{a} . B$$

of the Dialectica Interpretation. So prima facie Type Theory renders the interpetation redundant.

There is a sharper form of this worry arising from the validity of the so-called Axiom of Choice in Type Theory.

Dependent Type Theory

Unrealised intention: a general theory of inductive definitions

Main Ingredients

- Typed terms: $a \in A \vdash t(a) \in B$
- Types indexed over types: $a \in A \vdash B(a)\ \mathrm{type}$
- Implicit Substitution: From $a \in A \vdash t(a) \in B$ and $b \in \vdash C(b)\ \mathrm{type}$ get $a \in C(t(a))\ \mathrm{type}$.
- Sums: If $a \in A \vdash B(a)\ \mathrm{type}$ then $\Sigma a \in A.B(a)\ \mathrm{type}$ with familiar rules (left adjoint with Beck-Chevalley)
- Products: If $a \in A \vdash B(a)\ \mathrm{type}$ then $\Pi a \in A.B(a)\ \mathrm{type}$ with familiar rules (right adjoint with Beck-Chevalley)
- Identity types $a, a' \in A \vdash \mathrm{Id}_A(a, a')\ \mathrm{type}$ with identity rule

Categorical Models

Dependent types, sums and products

Category $\mathbb{C}$ with collection of maps $\mathbb{F}$ in $\mathbb{C}$: in accord with HTT call these fibrations.

- $\mathbb{F}$ closed under pullback so we form a fibration $\mathbb{F} \longrightarrow \mathbb{C}$ (Assume pullback chosen and so this is a cloven fibration - corresponding to a pseudofunctor.)
- All maps to 1 are in $\mathbb{F}$
  So the fibre over 1 is $\mathbb{C}$.
- $\mathbb{F}$ contains all isos and is closed under composition
  This gives
  - type theoretic strong sums
  - ensures Beck-Chevalley ($=$ good substitution) for sums and for products
- Pullbacks along maps in $\mathbb{F}$ have right adjoints
  This gives products.

Categorical Models

Identity Types; coproducts

Identity

Maps with the llp wrt fibrations are trivial cofibrations.

- Every map factorizes as a trivial cofibration followed by a fibration (Trivial cofibration = llp wrt fibrations.)

Note that since we have products this factorization is preserved by pulling back along fibrations. Else it should be an additional condition (cf Joyal's recent formulation).

Coproducts

- Each slice $\mathbb{F}/I$ has finite coproducts and these are stable

I would hope to find a version of the theory with a milder assumption here.

Basic Framework

Definition

For now call a (cloven) fibration ($\mathbb{F} \to \mathbb{C}$) satisfying the assumptions regarding fibred sums and products, identities and copruducts a *category with fibrations*.

Terminological problem

- $\mathbb{F} \to \mathbb{C}$ is a categorical fibration
- The arrows in $\mathbb{F}$ are called fibrations by reason of a topological intuition.

What is to be done? The old display terminology is not attractive.

The interaction of identity and existence

$$\begin{tikzcd}
B \arrow[rr, "f"] \arrow[dd, equal] & & A \arrow[dd, equal] \\
& \Sigma a \in A, b \in B . \mathrm{Id}_A(f(b), a) \arrow[ul] \arrow[ur] \arrow[u, phantom, "\simeq"] & \\
B \arrow[rr, "f"'] \arrow[ur] & & A
\end{tikzcd}$$

I would like to see this in a more general context.

The Sceptics Case

The Type Theoretic Axiom of Choice (AC)

$$\Pi a \in A.\Sigma b \in B(a).C(b) \to$$

$$\Sigma f \in (\Pi a \in A.B(a)).\Pi a \in A.C(f(a))$$

In the interpretations we consider the implied map is an isomorphism.

Consequence of AC

Inductively all types have the form

$$\Sigma u \in U.\Pi x \in X(u).A(x)$$

which is already the form of the Dialectica interpretation.

Constructing Interpretations of Dependent Types

This is an underdeveloped subject. Here is just one aspect.

Taking the Fibration seriously

Suppose that we have a construction on categories which we apply fibrewise to $\mathbb{F} \to \mathbb{C}$ giving a fibration $\Phi(\mathbb{F}) \to \mathbb{C}$ which is an extension $\mathbb{F} \to \Phi\mathbb{F}$ over $\mathbb{C}$. keeping the base fixed. We need to extend this fibration along the functor

$$\mathbb{C} = \mathbb{F}(1) \longrightarrow \Phi\mathbb{F}(1)$$

to give a model of Type Theory. This is not straightforward.

One possibility is a kind of *internalisation* (cf the Dialectica Interpretation of System F in Girard's Thesis, or Robinson-Rosolini on relational parametricity.

Another technique lies behind this talk. There is at least one more.

Polynomials

Also known as Containers

Polynomials and maps of polynomials in **Sets**

A polynomial is a map $U \leftarrow X$thought of as a general signature: a collection U of function symbols with X_u the arity of $u \in U$. A map of polynomials from $U \leftarrow X$ to $V \rightarrow Y$ is

$$\begin{tikzcd}
U \arrow[d, "f"'] & X \arrow[l] & f^*Y \arrow[l, "F"'] \arrow[d] \\
V & & Y \arrow[ll]
\end{tikzcd}$$

This gives a category **Pol** of polynomials or in a different culture containers.

The fibred category of polynomials

There is an evident fibred version of the polynomial construction.

We can identify that with

$$\Sigma(\mathbf{Sets}^2 \to \mathbf{Sets})^{op} ,$$

the result of freely adding sums to the *opposite* of **Sets** indexed over **Sets**. So **Pol** is the fibre over 1.

Note the use of the opposite of a fibred category!

For a fibration $\mathbb{E} \to \mathbb{B}$ we define

$$\mathrm{Pol}(\mathbb{E}) = \Sigma(\mathbb{E}^{op})$$

the polynomial construction.

The bicategory

A polynomial $U \leftarrow X$ induces a functor

$$\mathbf{Sets} \to \mathbf{Sets} : S \mapsto \sum u \in U.(X_u \Rightarrow S)$$

In a more general perspective indexed polynomials

$$I \leftarrow U \leftarrow X \to J$$

are the 1-cells of a bicategory with polynomial maps as 2-cells.

Polynomial operads

Monads in the polynomial bicategory correspond exactly to rigid operads, equivalently (Zawadowski) to the operads with non-standard amalgamation of Hermida-Makkai-Power.

Cartesian closure

A Little Miracle

T. Altenkirch, P. Levy and S. Staton. *Higher Order Containers*. In CiE'10, LNCS 6158.

Theorem

The category **Pol** *of containers/polynomials is cartesian closed.*

However it is not locally cartesian closed.

Further analysis

P. Hyvernat. *A linear category of polynomial diagrams*. To appear in Mathematical Structures in Computer Science.

- Linear logic background.
- Direct formulation of ALS in type theory. (Removed from final version!) So ALS in any *suitable interpretation*, (e.g. locally cartesian closed categories with coproducts.)

Computing with coproducts

Concrete explanation of Altenkirch-Levy-Staton

Take a coproduct $A + B$. Write it as

$$A + B = \Sigma x \in A + 1. \star(x) \Rightarrow B$$

where $\star(x)$ is represented by the obvious $1 \to A + 1$.
By (AC) we can write the type of maps into a coproduct

$$\begin{aligned} C \Rightarrow A + B &= C \Rightarrow \big(\Sigma x \in A + 1. \star(x) \Rightarrow B\big) \\ &= \Sigma f \in (C \Rightarrow A + 1). \Pi c \in C. (\star(f(c)) \Rightarrow B) \end{aligned}$$

This is the key idea also in the interpretation of Type Theory.

Natural Question on Polynomials

Is there a (simple) notion of fibration for polynomials giving a model of type theory?

Answer of Tamara von Glehn: YES!

Take fibrations to be the maps $U \leftarrow X$ to $V \leftarrow Y$

$$\begin{tikzcd}
U \arrow[d, "f"'] & X \arrow[l] & f^*Y \arrow[l, "F"'] \arrow[d] \\
V & & Y \arrow[ll]
\end{tikzcd}$$

where $F : f^*Y \to X$ is a coproduct inclusion.

The Polynomial Interpretation

Given $\mathbb{F} \to, \mathbb{C}$ a category with fibrations, the polynomial category with fibrations has

- objects the fibrations $U \leftarrow X$ from $\mathbb{F}$;
- maps the standard maps of polynomials;
- fibrations the von Glehn fibrations.

Theorem

Suppose $\mathbb{F} \to \mathbb{C}$ is a category with fibrations. Then the corresponding polynomial category is itself a category with fibrations.

We are developing technology to make the proof easier!

The von Glehn Factorization

For the locally cartesian closed case

$$
\begin{tikzcd}
U \arrow[d, equal] & X \arrow[l] & X + f^* Y \arrow[l] \arrow[d] & \\
U \arrow[d, "f"'] & & X + f^* Y \arrow[ll] & f^* Y \arrow[l] \arrow[d] \\
V & & & Y \arrow[lll]
\end{tikzcd}
$$

Functional extensionality

A calculation in the model

For types A and $B(a)$ $[a \in A]$ consider the identity type on $\Pi a \in A.B(a)$. For $f, g \in \big(\Pi a \in A.B(a)\big)$ function extensionality is

$$(\Pi a \in A.\mathrm{Id}_{B(a)}(f(a), g(a)) \Rightarrow \mathrm{Id}_{\Pi a \in A.B(a)}(f, g)$$

Tamara von Glehn has shown the following.

Theorem

The axiom of extensionality fails in polynomial models.

There are notorious issues re extensionality with the original Dialectica Interpretation. This is different and a trickier calculation.

The Dialectica Interpretation of Type Theory

Suppose we have $\mathbb{F} \to \mathbb{C}$ a category with fibrations. The Dialectica fibration is by definition $\Sigma\Pi\mathbb{F}$.

Theorem

The Dialectica fibration extends to a category with fibrations

- Either by adapting the von Glehn analysis.
- Or since

$$\mathrm{Pol}(\mathrm{Pol}\mathbb{F}) = \Sigma(\Sigma\mathbb{F}^{op})^{op} = \Sigma\Pi\mathbb{F}$$

one can find the Dialectica model inside the iterated polynomial model.

Thoughts for the future

- Computing in the Polynomial Interpretation is daunting and even more so in the Dialectica. We need a deeper categorical analysis.
- There are variations on the construction to investigate. Is there anything precise to say about the connection with Gödel's original interpretation.
- In a different direction there should be a variation restoring extensionality.
- More generally the model theory of interpretations needs investigation.

Last slide!

Many thanks for this inaugural workshop of the Centre for Quantum Mathematics and Computation

Very best wishes for the future