

SEQUENTIALITY

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1. Causal Interaction.
2. Sequential Interaction.
3. Towards Games.

ABSOLUTE CAUSALITY

SPACES

DEFINITION

An (absolute) causality space is

$$\langle \cdot | \cdot \rangle : U \times X \rightarrow R$$

such that

$$\langle u|x \rangle = \langle u'|x' \rangle$$

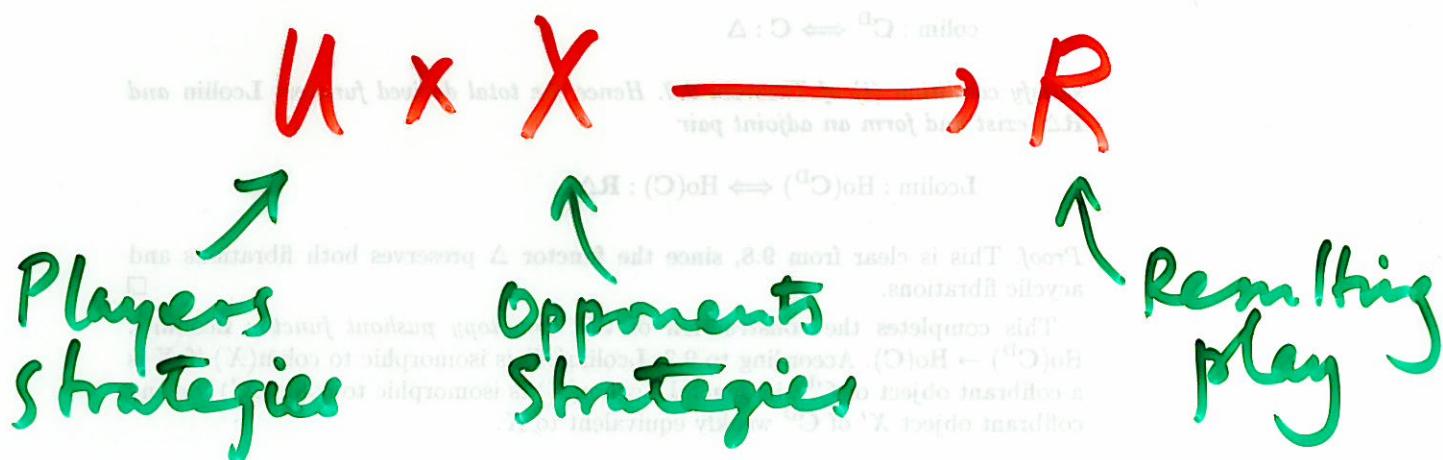
implies

$$\langle u|x' \rangle = \langle u'|x' \rangle$$

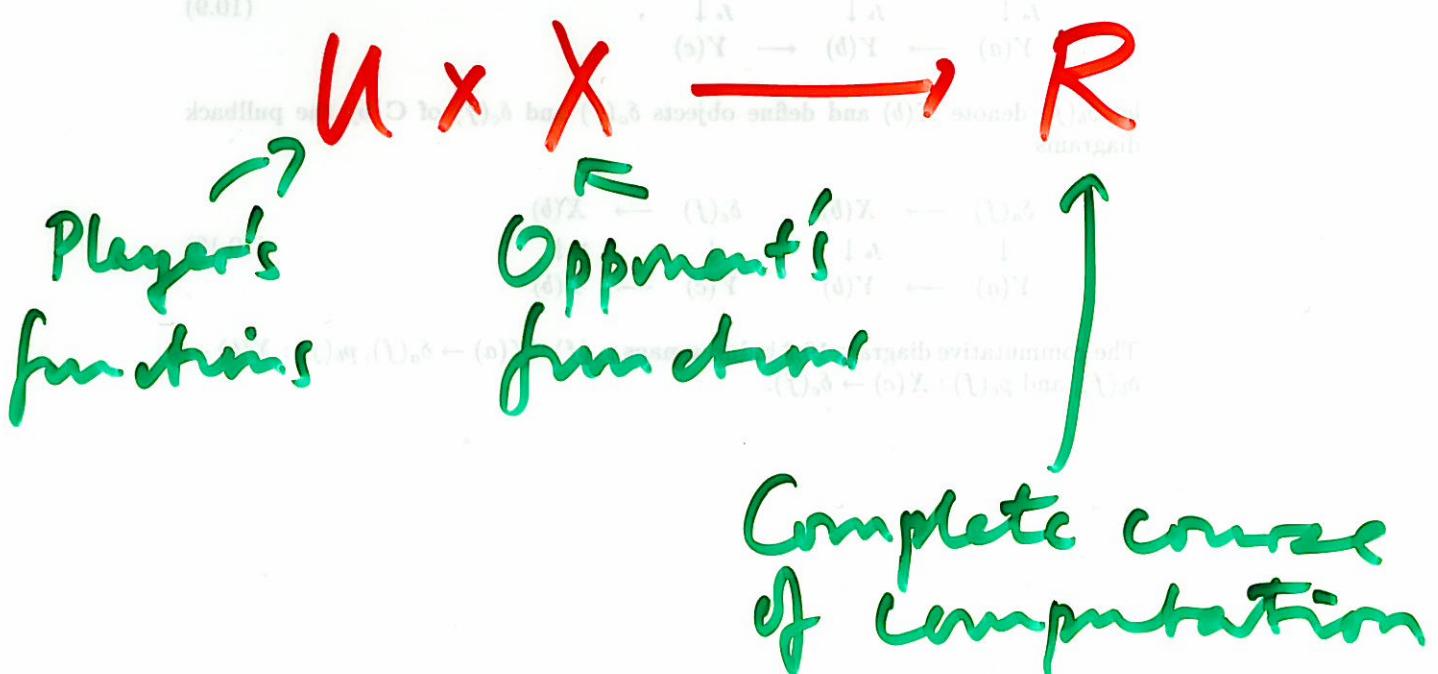
$$\langle u|x \rangle = \langle u'|x \rangle$$

EXAMPLES

Games



Abstract Games (some cases)



CAUSAL MAPS

Chapter 1

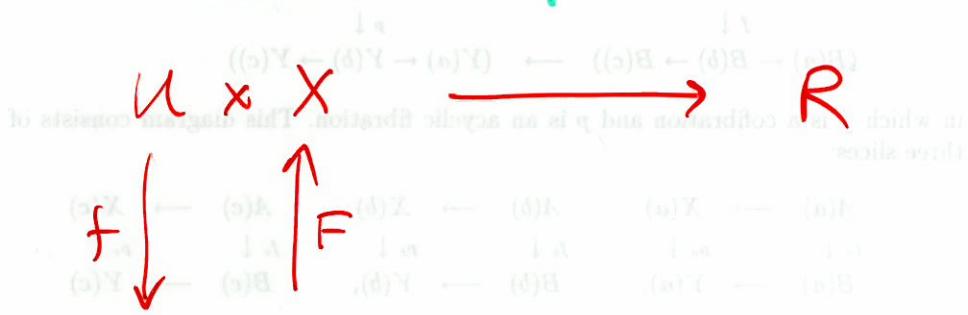
Definitions

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Y-axis: causal maps are configurations in C .
X-axis: causal maps have a base of MG_3 (with weak dependencies and independence) and a base of MG_1 (with causal dependencies).

DEFINITION

A causal map is



such that

$$(u_i \otimes x_j) \rightarrow (u_i \otimes y_k) \quad (u_i \otimes y_k) \rightarrow (u_i \otimes z_l)$$

$$\langle u | F(y) \rangle = \langle u' | F(y) \rangle \Rightarrow \langle f(u) | y \rangle = \langle f(u') | y \rangle$$

$$\langle f(u) | y \rangle = \langle f(u') | y' \rangle \Rightarrow \langle u | F(y) \rangle = \langle u' | F(y') \rangle$$

$$\begin{array}{c} (u_i \otimes x_j) \rightarrow (u_i \otimes y_k) \\ \downarrow \quad \downarrow \quad \downarrow \\ u_i \rightarrow u_i \quad x_j \rightarrow y_k \\ \downarrow \quad \downarrow \\ u_i \otimes y_k \end{array}$$

EXAMPLE

Maps of games as causal maps

A map $\sigma: A \rightarrow B$ of games
is a strategy in $A^\perp \otimes B$.

Then $f: U \rightarrow V$ is just
 $P_{\text{in } A} \quad P_{\text{in } B}$
composition with σ :

$$u: I \rightarrow A \xrightarrow{\sigma} \sigma \cdot u: I \rightarrow B$$

and $F: Y \rightarrow X$ is similar.

Intuition for causal condition!

THEOREM

The category of absolute causality spaces is

* - autonomous.

(In fact models full linear logic.)

INTERNAL FUNCTION SPACE

$$\text{If } A = (U \times X \rightarrow R)$$

$$B = (V \times Y \rightarrow S)$$

$$\text{then } A \rightarrow B =$$

$$\left(\{(f, F) \mid \text{causal maps}\} \times (U \times V) \longrightarrow R \times S \right)$$

$$\langle (f, F) \mid (u, v) \rangle = \left(\langle u \mid F(v) \rangle, \langle f(u) \mid v \rangle \right).$$

FACT

Over Sets, absolute causality spaces have a full \star -autonomous multicategory equivalent to 'weak totality spaces'.
(So not much gain save the conceptual; however quite different uses of the internal logic, so...)

RELATIVE CAUSALITY SPACES

DEFINITION

A relative causality space is

$$\langle | \rangle : U \times X \longrightarrow R$$

with $\leq_m U, \leq_m X$

such that

$$\text{for } u \leq u', x \leq x' \quad \langle u|x \rangle = \langle u'|x' \rangle$$

$$\langle u|x \rangle = \langle u'|x' \rangle \Rightarrow \langle u|x \rangle = \langle u'|x \rangle = \langle u'(x) \rangle$$

MORE RESTRICTED VERSION

$$\leq_m U, \leq_m X, \leq_m R$$

and $\langle | \rangle$ order preserving.

RELATIVELY CAUSAL MAPS

DEFINITION

A relatively causal map is

$C \subseteq E(\mathbb{R})$

$\text{such that } f : X \rightarrow Y \text{ is a map in } C \text{ such that } f \text{ is injective and } f^{-1}(U) \cap f^{-1}(V) = \emptyset \text{ for all } U, V \in \mathcal{F}$

$f^{-1}(U) \times f^{-1}(V) \subseteq C \text{ whenever } U, V \in \mathcal{F}$

$f \text{ is a local homeomorphism}$

$f^{-1}(U) \times f^{-1}(V) \subseteq C \text{ whenever } U, V \in \mathcal{F}$

$f \text{ is continuous}$

$f^{-1} : f(E(X)) \rightarrow E(Y)$

$\text{such that } F : \mathbb{R} \times Y \rightarrow S$

$\text{such that } F \circ f = g : X \rightarrow S$

$\text{such that } F \circ f^{-1} = h : S \rightarrow Y$

such that

for $u \leq u'$, $y \leq y'$

$$\langle u | F(y) \rangle = \langle u' | F(y) \rangle \Rightarrow \langle f(u) | y \rangle = \langle f(u') | y \rangle$$

$$\langle f(u) | y \rangle = \langle f(u) | y' \rangle \Rightarrow \langle u | F(y) \rangle = \langle u' | F(y') \rangle$$

$$C \xrightarrow{f} (f(E)) \xrightarrow{F} S$$

$$(0, 1)$$

such that $F \circ f = g$

$\text{such that } F \circ f = g : X \rightarrow S$

such that $F \circ f = g$

such that $F \circ f = g$

(1)

EXAMPLES

Games with weaker notion of result
(e.g. the set of moves independent of order)

Coherence spaces

$$\begin{array}{ccc} \mathcal{U} \times X & \longrightarrow & R \\ \text{diques in } A & \in & |A|_+ \\ (u, x) & \longmapsto & u \sqcap x \end{array}$$

$$\begin{array}{ccc} B & \xrightarrow{\pi} & B \\ V & \xrightarrow{\pi} & V \\ Y & \xrightarrow{\pi} & Y \end{array} \quad (8.13)$$

topological quotients

THEOREM

The category of relative causality spaces is

* - autonomous. (In fact it models full linear logic.)

CM categories

Бъдещото съдържание и това във времето ю съдържанието

предвидено във времето. Това ю е бъдещото съдържание до този

което CM категория ю е съдържанието между първите и тези, които са

първи. Ако съдържанието ю ю съдържанието между първите и тези, които са

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първи. Ако съдържанието ю ю съдържанието между първите и тези, които са

първи, то CM категория ю ю съдържанието между първите и тези, които са

първи.

съдържанието $\text{Hom}_{\text{CM}}(T, Y)$ ю $\text{Hom}^{\text{opp}}_{\text{CM}}(T^{\text{opp}}, Y^{\text{opp}})$.

този ю ю един CM-категория, която ю изоморфна на $(Y^{\text{opp}}, T^{\text{opp}})$.

изоморфна ю Y^{opp} и този $(Y^{\text{opp}}, T^{\text{opp}})$ ю един CM-категория, която ю изоморфна

на $(Y^{\text{opp}}, T^{\text{opp}})$. Ако $(Y^{\text{opp}}, T^{\text{opp}})$ ю един CM-категория, тогава $(Y^{\text{opp}}, T^{\text{opp}})$ ю

SEQUENTIABILITY

GENERAL APPROACH Take some causality category + add a sequential condition.

DEFINITION

A sequentiability space is a causality space

$$\langle 1 \rangle! : \mathcal{U} \times \mathcal{X} \rightarrow \mathbb{R}$$

$$\leq m \mathcal{U}, \leq m \mathcal{X}$$

such that

$$\text{for } u \leq u' \quad x \leq x'$$

$$\langle u | x \rangle = \langle u | x' \rangle \text{ or } \langle u | x \rangle = \langle u' | x \rangle.$$

projection of required rule for $n = 0$ and the definition of clause for $n > 1$

[32 p. 404] If for each position $x \in \mathcal{X}$ the map $\chi : \mathcal{U}^n(\mathcal{X}, x) \rightarrow \mathcal{U}^n(\mathcal{X}, \chi(x))$ is a 1-1 function χ with $\chi : \mathcal{X} \rightarrow \mathcal{X}$ of abelian groups and monoidal categories

EXPLANATION

GAME INTERPRETATION

8. Двоичные адреса

Suppose $\langle u|x \rangle$ a finite play.

If Player just played then however much faster Player is prepared to go, the game will progress no further without input from Opponent.

So $\langle u|x \rangle = \langle u'|x' \rangle$ all $u \leq u'$

Similarly if Opponent just played then

$\langle u|x \rangle = \langle u|x' \rangle$ all $x \leq x'$.

$$X \rightarrow \text{Geo}(\Sigma, V) \rightarrow Y$$

by the object theorem:

for Σ be the set of maps $(0 \rightarrow D^n)^{\infty}$ we consider the interpretation of X in terms of

the object of MCR(\mathcal{B}) as well as for $\Lambda : X \rightarrow Y$, we can do the precom-

position of Λ with codice of Y

product of codice of Y becomes to the codice from $\text{Geo}(\Sigma, V)$ to which the

$\text{Geo}(\Sigma, V)$ is the construction the choice into of $\text{Geo}(\Sigma, V)$ and a function Λ which

is object of first Σ^{∞} is a construction. This is essentially a condition in each object in

it is interpreted from Σ^{∞} but it is also a condition so that we have

SEQUENTIAL MAPS

DEFINITION

A sequential map

$$u : \mathbb{S}_{n-1} \rightarrow D_0 \text{ for all } n > 0$$

$$(i) \text{ for each } u \in \mathbb{S}_n \text{ there exists a unique } u' \in \mathbb{S}_{n-1} \text{ such that } u = u' \circ f$$

$$(ii) \text{ if } u, v \in \mathbb{S}_n \text{ and } u \leq v \text{ with respect to the order } u \rightarrow v \text{ then }$$

$$f^{-1}(u) \leq f^{-1}(v) \text{ with respect to the order } f^{-1}(u) \rightarrow f^{-1}(v)$$

$$\begin{array}{ccc} U \times X & \xrightarrow{\quad} & R \\ f \downarrow & \uparrow F & \\ V \times Y & \xrightarrow{\quad} & S \end{array}$$

is an (order-preserving) map
of causality spaces such
that

$$\text{for all } u \leq u' \quad y \leq y'$$

$$V \xrightarrow{u} C_p(X^b) \xrightarrow{F} C_{p+1}(X^b) \longrightarrow C_{p+1}(X^b)$$

$$\langle u | F(y) \rangle = \langle u' | F(y) \rangle \text{ or } \langle f(u) | y \rangle = \langle f(u') | y' \rangle.$$

$$\begin{array}{ccc} V & \xrightarrow{u} & C_p(X^b) \\ V' & \xrightarrow{u'} & C_{p+1}(X^b) \end{array}$$

definition:

Yield: Consider a continuous function f and u one of the input functions in

SECTION 3

continuous functions

de

EXPLANATION

GAME INTERPRETATION

We play the P-strategy (f, F) in $A \rightarrow B$ against the O-strategy (u, y) . Suppose the resulting pairs $(\langle u | F(y) \rangle, \langle f(u) | y \rangle)$ are finite. If we stopped in A^\perp ; then further effort by O in B will make no difference, so

$$\langle f(u) | y \rangle = \langle f(u) | y' \rangle \text{ all } y \leq y'.$$

Similarly if we stopped in B then

$$\langle u | F(y) \rangle = \langle u' | F(y) \rangle \text{ all } u \leq u'.$$

SEQUENTIAL ORDER ON MAPS

DEFINITION

Suppose (f, F) and (g, G) are maps

$$(U \times X \rightarrow R) \rightarrow (V \times Y \rightarrow S).$$

Set $(f, F) \leq (g, G)$ if &

only if $[(f, F) \leq (g, G) \text{ pointwise} \&]$

for all $u \leq u' y \leq y'$

$$\begin{cases} \langle u | F(y) \rangle = \langle u' | F(y') \rangle \\ \& \langle f(u) | y \rangle = \langle f(u') | y' \rangle \end{cases}$$

$$\begin{cases} \langle u | F(y) \rangle = \langle u' | G(y) \rangle \\ \& \langle f(u) | y \rangle = \langle g(u') | y' \rangle \end{cases}$$

$$\begin{cases} \langle u | F(y) \rangle = \langle u | G(y') \rangle \\ \& \langle f(u) | y \rangle = \langle g(u) | y' \rangle \end{cases}$$

EXPLANATION

GAME INTERPRETATION

1st case O has just played
So doesn't matter how much
further O goes

2nd case P has just played
in B.

3rd case P has just played in
A^L.

$$\begin{array}{c} B \rightarrow K \\ \uparrow \quad \rightarrow \\ A \rightarrow X \end{array}$$

Observation:

In games this order
coincides with the point-
wise order.

THEOREM

The category of sequentiality spaces is ***-autonomous**.

In fact it models full linear logic.

(iii) for every λ of A -modifies there is a morphism $\lambda^* : A^* \rightarrow M^*$ such that $\lambda^* \circ \lambda = \text{id}_{A^*}$

(ii) for every λ of A -modifies there is a morphism $\lambda^* : M^* \rightarrow A^*$ such that $\lambda \circ \lambda^* = \text{id}_{M^*}$

(i) λ is a direct summand of A -modifies

($\psi \geq 0$): consider first the case where λ is a summand of A -modifies. Then there is a morphism $\lambda' : A \rightarrow A$ of A -modifies such that $\lambda = \lambda' \circ \lambda'$. Now consider the case where λ is a summand of M^* -modifies. Then there is a morphism $\lambda' : M^* \rightarrow M^*$ of M^* -modifies such that $\lambda = \lambda' \circ \lambda'$. In both cases we have $\lambda^* \circ \lambda = \text{id}_{A^*}$ and $\lambda \circ \lambda^* = \text{id}_{M^*}$. This shows that λ is a summand of A -modifies. To show that λ is a summand of M^* -modifies, consider the morphism $\lambda^* \circ \lambda : M^* \rightarrow M^*$. It is a morphism of M^* -modifies since $\lambda^* \circ \lambda \circ \lambda^* = \lambda^* \circ \text{id}_{M^*} = \lambda^*$. It is also a morphism of A -modifies since $\lambda^* \circ \lambda \circ \lambda^* = \lambda^* \circ \lambda = \text{id}_{M^*}$. Therefore $\lambda^* \circ \lambda$ is a summand of M^* -modifies. Since M^* is a summand of M^* -modifies, it follows that λ is a summand of M^* -modifies.

Now consider the case where λ is not a summand of A -modifies.

Since λ is a summand of M^* -modifies, there is a morphism $\lambda' : M^* \rightarrow M^*$ of M^* -modifies such that $\lambda = \lambda' \circ \lambda'$. Consider the morphism $\lambda^* \circ \lambda : M^* \rightarrow M^*$. It is a morphism of M^* -modifies since $\lambda^* \circ \lambda \circ \lambda^* = \lambda^* \circ \text{id}_{M^*} = \lambda^*$. It is also a morphism of A -modifies since $\lambda^* \circ \lambda \circ \lambda^* = \lambda^* \circ \lambda = \text{id}_{M^*}$. Therefore $\lambda^* \circ \lambda$ is a summand of M^* -modifies. Since M^* is a summand of M^* -modifies, it follows that λ is a summand of M^* -modifies.

GLIMPSE OF PROOF

Show eg

$$\frac{A \otimes B}{A} \longrightarrow C^\perp$$

$$A \longrightarrow (B \dashv C^\perp)$$

by showing both equivalent to symmetric $A \otimes B \otimes C \longrightarrow \perp$.

Symmetric conditions.

for all $u \in u' \quad v \in v' \quad w \in w'$

$$\langle u | f(v, w) \rangle = \langle u' | f(v, w) \rangle \& \langle w | g(u, w) \rangle = \langle v' | g(u, w) \rangle$$

OR

$$\langle u | f(v, w) \rangle = \langle u' | f(v, w) \rangle \& \langle w | h(u, v) \rangle = \langle w' | h(u, v) \rangle$$

OR

$$\langle v | g(u, w) \rangle = \langle v' | g(u, w) \rangle \& \langle w | h(u, v) \rangle = \langle w' | h(u, v) \rangle$$

EXAMPLES

QUESTION Which of these embed in the subcategories of sequential spaces and maps?

(1) Usual categories of games

SEQUENTIAL ✓

(2) Typical categories of abstract games

NOT SEQUENTIAL ✓

EXAMPLES

CONTINUED

(3) Coherence spaces

Hypercoherence spaces
SEQUENTIAL!

(4) Games and

non-deterministic
strategies

NOT SEQUENTIAL?

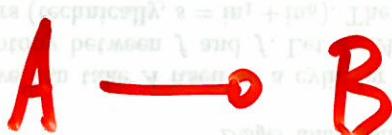
(Non-determinism)
vs
(Concurrency)

Misleading:
can be
fixed with
different
embedding.

TOWARDS GAMES

ISSUES

- Detecting individual moves
- Sequential order v. pointwise order
- Opponent strategies in



'RESTRICTION'

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Homotopy functors

Section 4

it follows from λ that $\lambda \wedge 1$ is a coproduct. Since λ is the composite

Suppose play $\langle u/x \rangle$

it follows from λ that $\lambda \wedge 1$ is a coproduct. Since λ is the composite

Then there should be minimal

it follows from λ that $\lambda \wedge 1$ is a coproduct. Since λ is the composite

$$\bar{u} = u/x \leq u$$

it follows from λ that $\lambda \wedge 1$ is a coproduct. Since λ is the composite

$$\bar{x} = x/u \leq x$$

such that

$$\langle \bar{u}/\bar{x} \rangle = \langle u/x \rangle.$$

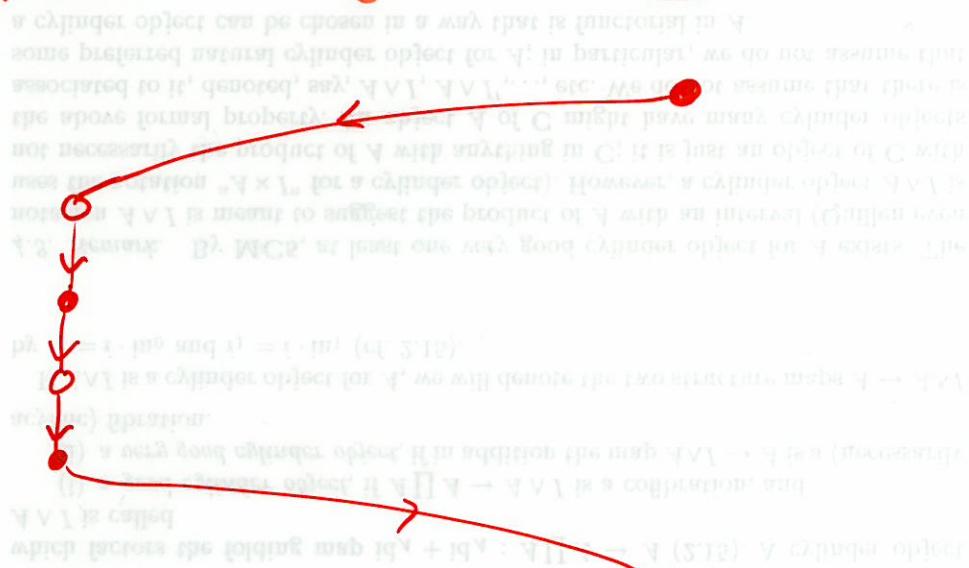
Axioms in these terms

- detect individual moves
- make the sequential order pointwise.

OLD PROBLEM (REVERSED)

When playing

$A^\perp \not\rightarrow B$



O cannot change sides, but
eg. when about to play
in B can use information
about what happened in
 A^\perp . SO MUCH MORE
THAN $U \times Y$

COMPLETING

(DOMAINS OF)

STRATEGIES

ONE SOLUTION (works in examples)

Use absolute sequentiality.

Complete under sequential
 \leq and \checkmark (preserving \perp).

(Not quite as stated: the orthogonality \perp plays a bigger role.)

WHY CAN THIS WORK?

(4. Totality Spaces)

ONWARD

Relation with other
work in H/Ong/Wallen
project.

Nickau/Ong investigate
large #'s of flavours
of game.

Hope to detect these
via abstractly defined
sub categories.