ABSTRACT AND CONCRETE MODELS OF RECURSION

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$\boxed{\text{The abstract is concrete.}}$

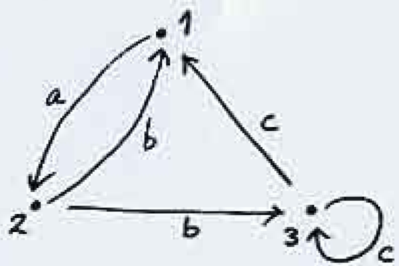
PLAN

1. Background, aims
2. Category theory
3. Feedback, fixed points
4. Feedback, matrices.

Finite Automata

Traversing a diagram

Example



Suppose 1 is initial state and 3 is terminal state

Then

$$a\,(ba)^*\,b\,(c\,(ab)^*)^*$$

is accepted.

Result of Calculation

(Warning. Traditional matrix convention gives words in reverse.)

$$\begin{pmatrix} 0 & b & c \\ a & 0 & 0 \\ 0 & b & c \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} (c^*ba)^* & (c^*ba)^*c^*b & (c^*ba)^*cc^* \\ (ac^*b)^*a & (ac^*b)^* & (ac^*b)^*acc^* \\ ((ba)^*c)^*b(ab)^*a & ((ba)^*c)^*b(ab)^* & ((ba)^*c)^* \end{pmatrix}$$

What is this?

REGULAR LANGUAGES

Σ a finite alphabet

Σ^* set of words in Σ

A language is a subset of Σ^*.

The regular languages are those generated from singleton letters x, y, z by

empty set	0
union	$+$
singleton trivial word	1
concatenation	$\cdot$
star	$(\ )^*$

(where $a^* = 1 + a + a^2 + \ldots$)

KLEENE'S THEOREM

Fix a finite alphabet Σ. The languages recognized by a finite automaton over Σ are exactly the regular languages.

WHY ???

WHY IS IT INTUITIVELY OBVIOUS ?

THEOREM

which encapsulates the characterization direction

There is a traced monoidal functor

$$\underline{Aut} \longrightarrow \underline{Reg}$$

category based on automata

category of matrices of regular languages

AIM To explain this!

ALGEBRA OF REGULAR OPERATIONS

$$0+a = a = 0+a$$
$$a+(b+c) = (a+b)+c$$
$$a+b = b+a$$

$$1.a = a = 1.a$$
$$a.(b.c) = (a.b).c$$

$$0.a = 0 = a.0$$
$$(a+b).c = a.c+b.c \qquad a.(b+c) = a.b+a.c$$

$$(a.b)^* = 1 + a(b.a)^* b$$
$$(a+b)^* = (a^* b)^* a^*$$

$$a^{**} = a^*$$
$$(a^n)^*(1+a+\cdots+a^{n-1}) = a^*$$

Conway Algebras

$$0+a = a = 0+a$$
$$a+(b+c) = (a+b)+c$$
$$a+b = b+a$$

$$1.a = a = 1.a$$
$$a.(b.c) = (a.b).c$$

$$0.a = 0 = a.0$$
$$(a+b).c = a.c+b.c \qquad a.(b+c) = a.b+a.c$$
$$(a.b)^* = 1 + a\,(b.a)^*\,b$$
$$(a+b)^* = (a^* b)^* a^*$$

CLAIM This should be the fundamental concept.
WHY?

Exercises

1. What are the small Conway Algebras?

2. Show that

$$a^{**} = a^*$$
and
$$(a^n)^* (1+a+\cdots+a^{n-1}) = a^*$$

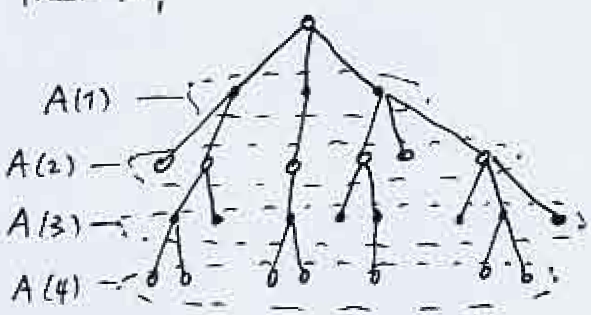
do not follow from the axioms for Conway Algebras.

GAMES

as trees or forests

$A(1) \leftarrow A(2) \leftarrow A(3) \cdots$

that is,

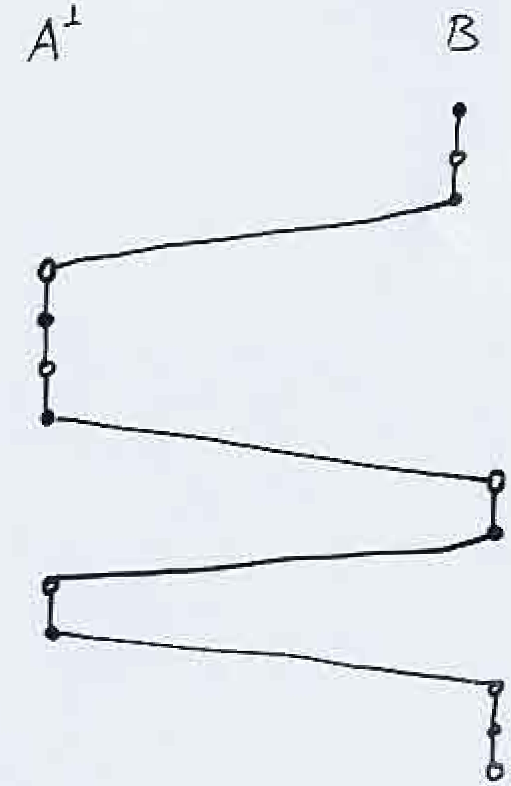


Here the opponent starts and player / opponent alternate.

So Opponent plays the odd
Player plays the even stages.

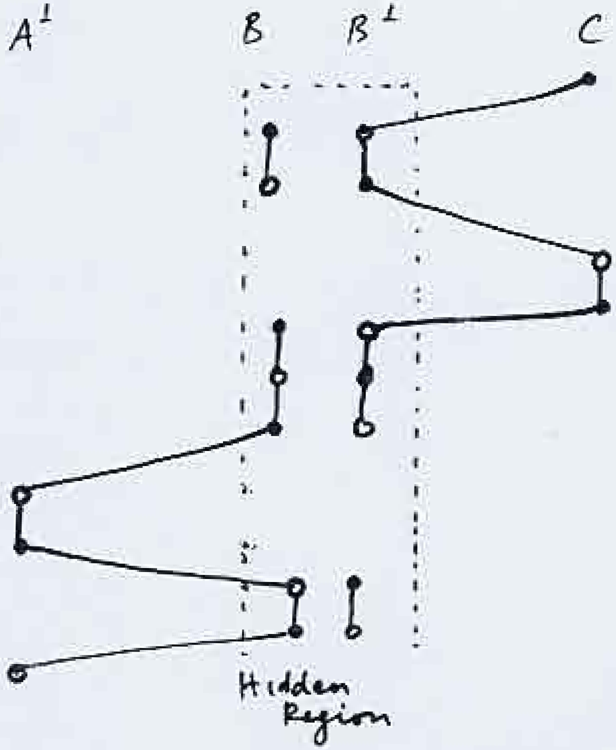
LINEAR FUNCTION SPACE

$A \multimap B$

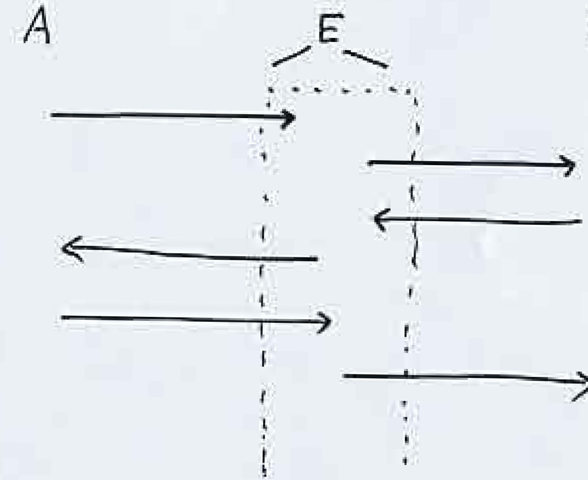


Composition of Strategies
(Player Strategies)

$$\frac{A \xrightarrow{\ \sigma\ } B \qquad B \xrightarrow{\ \tau\ } C}{A \xrightarrow{\ \sigma;\tau\ } C}$$



Security
The shape of an attack



The attack hidden from A and B

Pavlovic : Category of cord processes

Cf papers of Durgin, Mitchell, Pavlovic

History-Free Strategies

A_- opponent tokens in A

A_+ player tokens in A

Strategy is partial function

$$A_- \longrightarrow A_+$$

with good properties

So strategy in $A \multimap B$

is partial function

$$A_+ + B_- \longrightarrow A_- + B_+$$

History-Free Composition

$$\begin{tikzcd}
A_+ \arrow[r, bend right] \arrow[d, bend left] & A_- \\
B_+ \arrow[r, bend right] \arrow[d, bend left] \arrow[ur, phantom, "\sigma"] & B_- \arrow[u, bend left] \arrow[l, bend right] \\
C_+ \arrow[ur, phantom, "\tau"] & C_- \arrow[u, bend left] \arrow[l, bend right]
\end{tikzcd}$$

$$A_+ + B_- \xrightarrow{\sigma} A_- + B_+$$

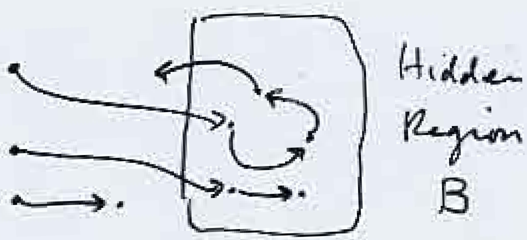
$$B_+ + C_- \xrightarrow{\tau} B_- + C_+$$

CONTRASTS

Automaton

follow all runs of a token through a diagram

Composition



each point goes to a visible point, iterating within the hidden region until it becomes visible

RANGE OF MODELS

Automata

Flow diagrams

Circuits

Interactive systems

Action structures

Proofs

Diagrammatic methods pervade computer science. Is there a unifying point of view?

CATEGORIES

In the style of dependent type theory

A category consists of

- a collection of objects $\mathrm{ob}\,C$
- for each pair a, b of objects a collection $C(a,b)$ of arrows (maps / morphisms)
- for each object a an identity arrow $1_a = a \in C(a,a)$
- for every a, b, c objects a composition map $C(b,c) \times C(a,b) \to C(a,c);\ g, f \mapsto g \circ f$

satisfying identity and associativity axioms

$$1 \circ f = f = f \circ 1 \qquad h \circ (g \circ f) = (h \circ f) \circ f$$

Write $f : a \to b$ for $f \in C(a,b)$.

USUAL EXAMPLES 1

Big examples

Categories of sets: Sets, Boolean Valued Models, Toposes

Categories of algebras: Groups, Rings, a monad $\to$ T-Algebras

Categories of spaces: Top, Simplicial sets, Schemes

Categories for CS: Scott domains, Stable domains, Games

The category of (small) categories.

USUAL EXAMPLES 2

Small examples

Preorders are categories with at most map between any two objects.

Monoids are categories with just one object.

Groupoids are categories in which all maps are invertible.

Groups are one-object groupoids.

OTHER EXAMPLES

Very small examples

0 the category with no objects and so no maps

1 the category with one object and just the identity map

With one object and two maps we have

$$\begin{tikzcd} \bullet 1 \arrow[loop right, "e"] \end{tikzcd} \quad e^2 = e \qquad \begin{tikzcd} \bullet 1 \arrow[loop right, "s"] \end{tikzcd} \quad s^2 = 1$$

(The two 2-element monoids.)

With two objects in addition to disjoint sums of monoids we have

$$\begin{tikzcd} \overset{0}{\bullet} \arrow[r, "a"] & \overset{1}{\bullet} \end{tikzcd} \qquad \begin{tikzcd} \overset{0}{\bullet} \arrow[r, bend left, "a"] & \overset{1}{\bullet} \arrow[l, bend left, "\bar{a}"] \end{tikzcd}$$

a poset $\begin{matrix} \cdot 1 \\ \vee\!/ \\ \cdot 0 \end{matrix}$ $\qquad \bar{a}.a = 1_0 \quad a.\bar{a} = 1_1$

FUNCTORS

Structure preserving maps of categories:

A functor $F : C \longrightarrow D$ consists of $F : \mathrm{ob}\,C \longrightarrow \mathrm{ob}\,D$

indexed families of maps

$$F : C(a,b) \longrightarrow D(Fa, Fb)$$

such that

$$F(1_a) = 1_{Fa}$$

and whenever $f : a \to b$, $g : b \to c$

$$F(g \circ f) = Fg \circ Ff$$

(Type theory good here!)

The category of (small) categories.

NATURAL TRANSFORMATIONS

Suppose $F, G : C \to D$ functors.

A natural transformation

$$\alpha : F \longrightarrow G$$

consists of a family of maps

$$(\alpha_a : Fa \to Ga)_{a \in \mathrm{ob}\,C}$$

in D such that for all $f : a \to b$ we have $Gf \circ \alpha_a = \alpha_b \circ Ff$

Homotopy

$$\begin{tikzcd} Fa \arrow[r, "\alpha_a"] \arrow[d, "Ff"'] & Ga \arrow[d, "Gf"] \\ Fb \arrow[r, "\alpha_b"] & Gb \end{tikzcd}$$

The diagram commutes.

CLOSED STRUCTURE

For (small) categories C, D
we have a category $[C,D]$
of functors (objects of $[C,D]$)
and natural transformations
(arrows)

The category of (small)
categories is cartesian closed
(i.e. a model of typed λ-calculus)

$$\mathrm{Cat}(E,[C,D]) \cong \mathrm{Cat}(E \times C, D)$$

natural isomorphism

FREE CATEGORIES
(over graphs)

$G = \left(\begin{smallmatrix} E \\ \downarrow\downarrow \\ V \end{smallmatrix} \right)$ a directed graph

eg. 

The category C_G generated by
G has
as objects V the vertices
as maps $a \to b$ the paths

$$a = x_n \xrightarrow{e_{n-1}} x_{n-1} \to \cdots \to x_1 \xrightarrow{e_0} x_0 = b$$

between a and b (read backwards)

Identities are empty/null strings
Composition is concatenation

TOY EXAMPLES
(of free categories)

From $\bullet$ we get $\mathbf{1}$

From $\begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd}$ we get $\mathbb{N}$, the natural numbers as a monoid.

From the infinite

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots$$

we get $\mathbb{N}$ as a poset.

FINITE AUTOMATA
(again)

A directed graph with labelling is a directed graph G with a graph map to the graph

$$\Sigma = \begin{tikzcd} \bullet \arrow[loop above, "a"] \arrow[loop right, "b"] \end{tikzcd} \cdots$$

with edges given by Σ.

So we get

$$C_G \longrightarrow C_\Sigma = \Sigma^*$$

and the regular languages can be read off as the images of the $C_G(x,y)$.

CATEGORIES WITH STRUCTURE

'Up to isomorphism'

are suppressed as far
as possible.

Coherence theorems show
this is safe
(when it is !)

MONOIDAL CATEGORIES

(Monoids in Cat)

Categories M with

$$1 \xrightarrow{I} M$$

$$M \times M \xrightarrow{\otimes} M$$

satisfying the monoid laws

$$I \otimes a = a = a \otimes I$$

$$a \otimes (b \otimes c) = (a \otimes b) \otimes c$$

as functors i.e. for
objects and for arrows.

FREE MONOIDAL CATEGORY

Take C a category. The free monoidal category $M(C)$ has

objects finite strings $a_1 \cdots\cdots a_n$ of objects of C

arrows finite strings
$$\begin{array}{ccc} a_1 & \cdots\cdots & a_n \\ {\scriptstyle f_1}\downarrow & \cdots & \downarrow{\scriptstyle f_n} \\ b_1 & \cdots\cdots & b_n \end{array}$$
of arrows of C

identities strings of identities

composition elementwise

identity object I empty string

tensor product $\otimes$ concatenation

FREE MONOIDAL CATEGORY ON 1

This is the category with

objects natural numbers

arrows just identities

(so identity and composition easy)

[i.e. $\mathbb{N}$ as discrete category]

identity object I 0

tensor product $\otimes$ $+$

MONOIDS BY DIAGRAMS

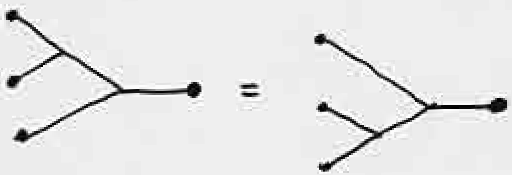
Monoid 

Identity map 

Identity element 

Multiplication 

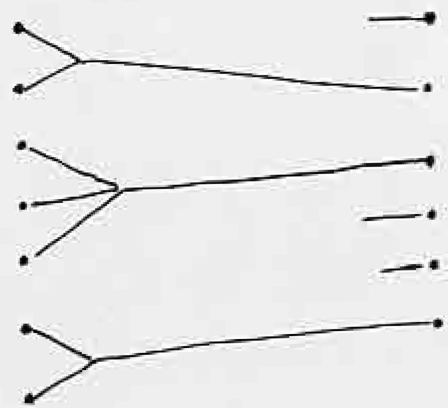
Axioms



Makes sense in a monoidal category!

FREE MONOIDAL CATEGORY ON A MONOID

Has objects and arrows looking like



with pictorial structure.

(AUGMENTED)

SIMPLICIAL CATEGORY

objects finite ordinals

maps order preserving

$$\Delta_+$$

The unique maps in

$\Delta_+(0,1)$ $\Delta_+(2,1)$

give the monoid structure

SYMMETRIC

MONOIDAL CATEGORIES

are equipped with a symmetry

$$s_{a,b} : a \otimes b \longrightarrow b \otimes a$$

with $s^2 = 1$

and

coherence condition

$$\begin{tikzcd} a \otimes b \otimes c \arrow[rr, "s_{a,\, b\otimes c}"] \arrow[dr, "s_{a,b} \otimes c"'] & & b \otimes c \otimes a \\ & b \otimes a \otimes c \arrow[ur, "b \otimes s_{a,c}"'] & \end{tikzcd}$$

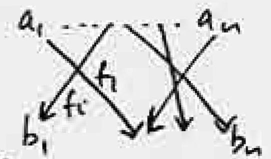
commuting

FREE SYMMETRIC MONOIDAL CATEGORY

Take C a category. The free symmetric monoidal category $\Sigma(C)$ has

objects: finite strings $a_1 \cdots\cdots a_n$ of objects of C

arrows: permutations + finite strings



of arrows of C

identities: strings of identities

composition: elementwise + composition of permutations

identity object I: empty string

tensor product: concatenation of diagrams

symmetry: relevant permutation

FREE SYMMETRIC MONOIDAL CATEGORY ON 1

$\Sigma(1)$ is the category with

objects: natural numbers

arrows:

$$\Sigma(1)(n,m) = \begin{cases} S_n & n = m \\ \emptyset & \text{o.w.} \end{cases}$$

so it is the disjoint sum of the symmetric groups

identity object I: 0

tensor product $\otimes$: $+$ with evident extension to arrows

$\mathbb{P}$ category of finite permutations

FREE SYMMETRIC MONOIDAL CATEGORY ON A COMMUTATIVE MONOID

The picture 'shows' that this is the category $\mathbb{F}$ of finite sets (cardinals) and all maps

(N.B. Commutative monoid says

)

CATEGORY WITH PRODUCTS

For each a, b there is a (choice of) product diagram

$$\begin{tikzcd} & a \times b \arrow[dl, "fst"'] \arrow[dr, "snd"] & \\ a & & b \end{tikzcd}$$

such that (universal property) composition with fst, snd gives a bijection

$$\mathbb{C}(c, a \times b) \cong \mathbb{C}(c, a) \times \mathbb{C}(c, b)$$

$$\begin{tikzcd} & c \arrow[d, "{(f,g)}"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & a \times b \arrow[dl, "fst"] \arrow[dr, "snd"'] & \\ a & & b \end{tikzcd}$$

CATEGORY WITH PRODUCTS CTD

There is a terminal object 1 such that (universal property)

for every a there is a unique

$$a \longrightarrow 1$$

i.e. $C(c,1) \cong 1$ one element set

—

A terminal object and binary products $\sim$ finite products

Then every object has a unique comonoid structure

$$a \longrightarrow 1$$

$$a \longrightarrow a \times a$$

FREE CATEGORY WITH COPRODUCTS

Take C a category. The free category with coproducts has

objects finite strings $a_1 \ldots\ldots a_n$ of objects of C

arrows functions $\phi : n \longrightarrow m$ $\quad a_1 \ldots\ldots a_n$

and arrows $\quad b_1 \ldots\ldots b_m$

$$a_i \longrightarrow b_{\phi(i)}$$

of C.

identities strings of identities

composition composition of functions and in C

symmetric monoidal structure as before

monoid structure from

$$0 \longrightarrow 1$$
$$2 \longrightarrow 1$$

FREE CATEGORY WITH COPRODUCTS ON 1

This by construction is the category $\mathbb{F}$ again

WHY?

(There is a categorical explanation.)

EXERCISES

1. What is the free category with products?

2. Characterize the category of finite sets and partial functions

Biproducts are products and coproducts ($\sim$ direct sums $\oplus$)

3. (Hard?) What is the free category with biproducts

4. Characterize the category of finite sets and relations.

Product of Maps

Given

$$a \xrightarrow{f} c \qquad b \xrightarrow{g} d$$

in a category with finite products

want $\quad a \times b \xrightarrow{f\times g} c \times d$

So sufficient to give $\quad \begin{array}{l} a\times b \longrightarrow c \\ a \times b \longrightarrow d \end{array}$

$$\begin{tikzcd}
a \arrow[d, "f"'] & a \times b \arrow[l, "fst"'] \arrow[r, "snd"] \arrow[d, dotted] & b \arrow[d, "g"] \\
c & c \times d \arrow[l, "fst"] \arrow[r, "snd"'] & d
\end{tikzcd}$$

Thus $f \times g$ is the unique map such that

$$fst \circ (f \times g) = f \circ fst$$
$$snd \circ (f \times g) = g \circ snd$$

Functoriality

We can check

$$1_a \times 1_b = 1_{a\times b}$$

and for $\quad \begin{array}{ll} f : a \longrightarrow c & h : c \longrightarrow c' \\ g : b \longrightarrow d & k : d \longrightarrow d' \end{array}$

$$(h \times k) \circ (f \times g) = h \circ f \times k \circ g$$

which together say that

$\times$ is functorial

(And diagram before says that fst, snd are natural transformations.)

Proofs exercises in the universal property.

JUSTIFICATION BY COMPUTATION

Define $f \times g$ by

$$f \times g\,(z) = \text{let } z \text{ be } (x,y) \text{ in } (f(x), g(y))$$

Then

$$\text{let } z \text{ be } (x,y) \text{ in } (x,y) = z \qquad (\eta\text{-rule})$$

$$\text{let } \big(\text{let } z \text{ be } (x,y) \text{ in } (f(x),g(y))\big) \text{ be } (x',y') \text{ in } (h(x'),k(y'))$$

$$= \text{let } z \text{ be } (x,y) \text{ in } \big(h(f(x)), k(g(y))\big).$$

CLARIFICATION

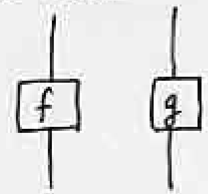
A product gives a symmetric monoidal structure on a category

(In the coherent isomorphism sense.)

DIAGRAMS FOR PRODUCTS

As primitives we take

$f \times g$



(The exact geometry does not matter as $f \times g = (1 \times g) \circ (f \times 1)$ etc.)

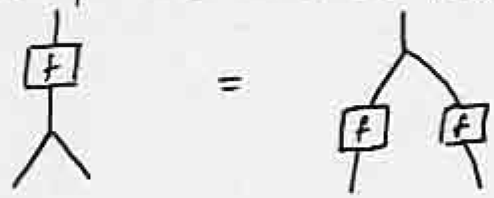
$\Delta : a \to a \times a$

$(\Delta = (1,1))$

or

$t : a \to 1$

Basic computational tool $\Delta \circ f = (f \times f) \circ \Delta$

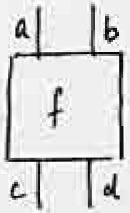


DIAGRAMS IN SYMMETRIC MONOIDAL CATEGORIES

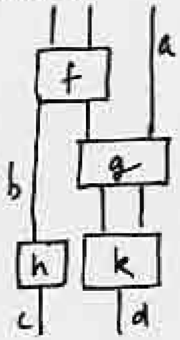
A map

$f : a \otimes b \longrightarrow c \otimes d$

is drawn:



Typical composition



represents

$(h \otimes k) \circ (b \otimes g) \circ (f \otimes a)$

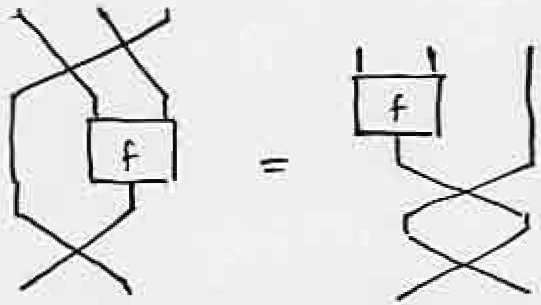
but with a little change in the geometry it could be

$(c \otimes k) \circ (h \otimes g) \circ (f \otimes a)$ or $(h \otimes d) \circ (b \otimes k \circ g) \circ (f \otimes a)$

all equal!

REASONING WITH DIAGRAMS

A matter of geometry



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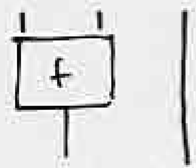
"Unravelling"



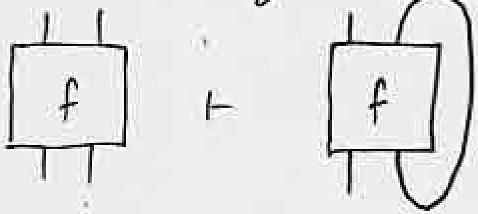
(Naturality of s.)

Traced Monoidal Categories

These are symmetric monoidal categories equipped with an operation

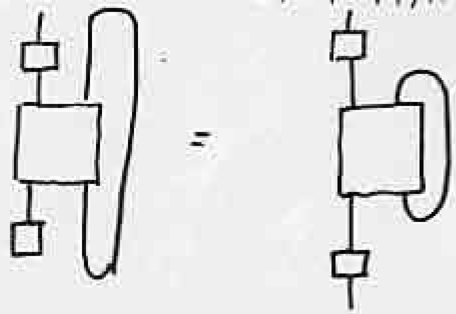
$$\frac{A\otimes U \xrightarrow{f} B\otimes U}{A \xrightarrow[tr_U(f)]{} B}$$

which we represent by a feedback diagram

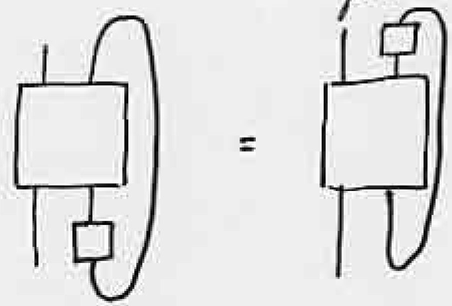


satisfying natural axioms

Naturality Axioms



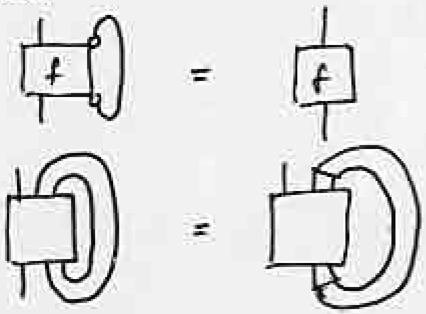
(The domain and codomain naturality are hidden in the geometry.)



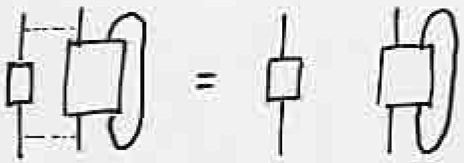
(This is the main tool.)

OTHER AXIOMS

Action



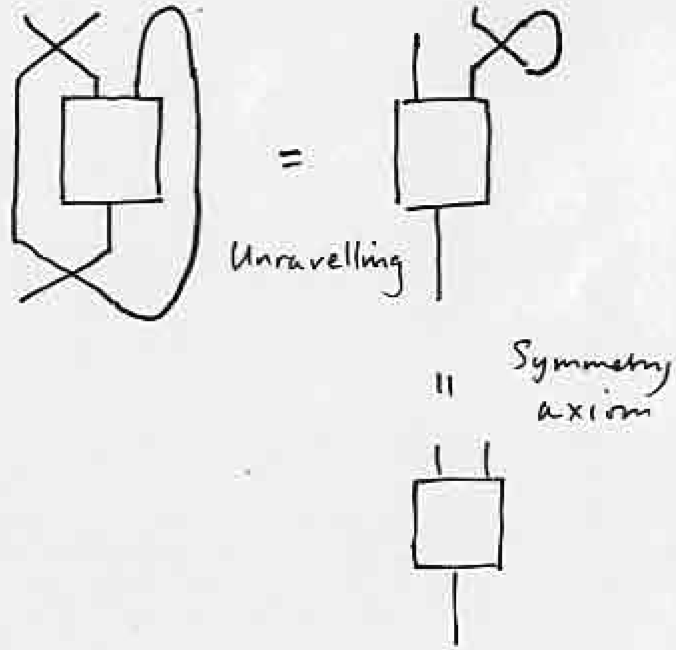
Independence



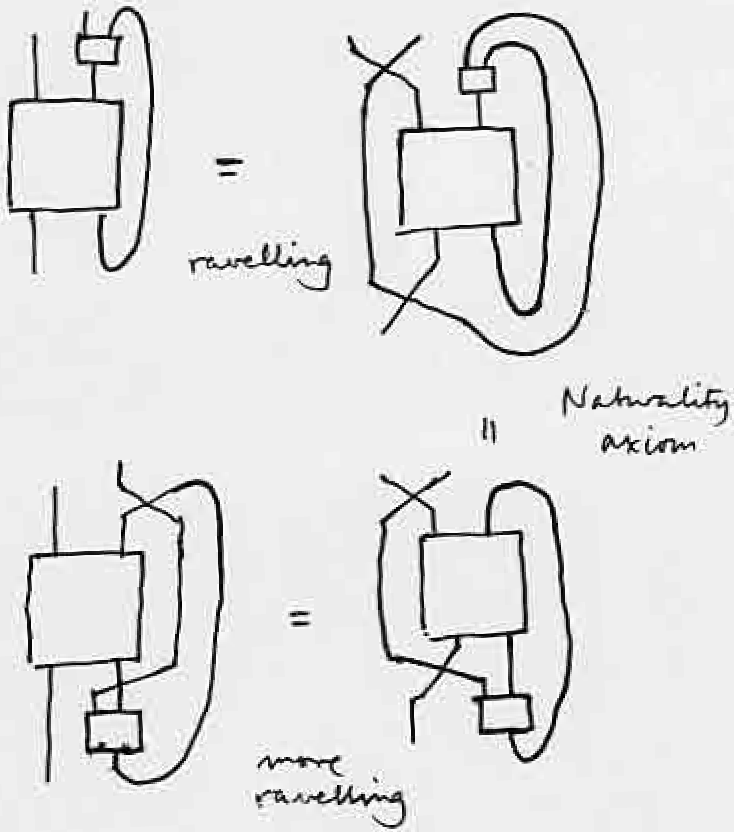
Symmetry

(All hidden in the geometry.)

DIAGRAMS WITH TRACE



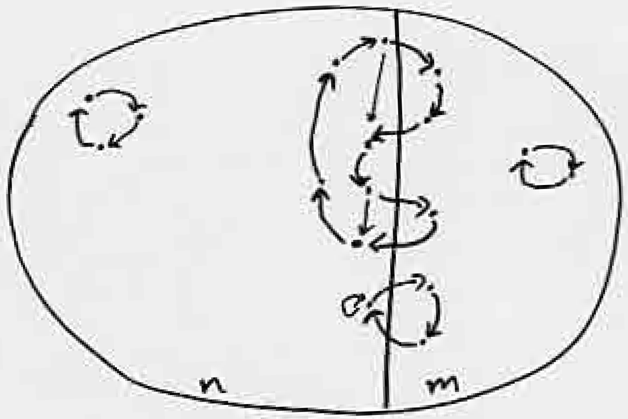
STRENGTHENED NATURALITY



UNEXPECTED EXAMPLE

The category P of finite cardinals and permutations has a trace.

Given $f : n+m \longrightarrow n+m$ we obtain $tr_m f : n \longrightarrow n$ thus



COMPACT CLOSED CATEGORIES

These are symmetric monoidal categories in which every object A is equipped with a dual A^* with $I \xrightarrow[\eta]{} A^* \otimes A$

$$A \otimes A^* \xrightarrow[\varepsilon]{} I$$

such that

$$A \xrightarrow{A\eta} A \otimes A^* \otimes A \xrightarrow{\varepsilon A} A$$

$$A^* \xrightarrow[\eta A^*]{} A^* \otimes A \otimes A^* \xrightarrow[A\varepsilon]{} A^*$$

are identities

(triangle identities)

Examples

Finite dimensional vector spaces

<u>Rel</u>, sets and relations with $\times$ as tensor product

Conway games

CANONICAL TRACE

Given $f : A \otimes U \longrightarrow B \otimes U$ in a compact closed category, define $tr_U f : A \longrightarrow B$ by

$$A \xrightarrow{A\eta} AU^*U \xrightarrow{As} AUU^* \xrightarrow{fU} BUU^* \xrightarrow{B\varepsilon} B$$

This is always a trace.

(And this trace is always unique.)

This gives examples and every traced monoidal category embeds as such in a compact closed category.

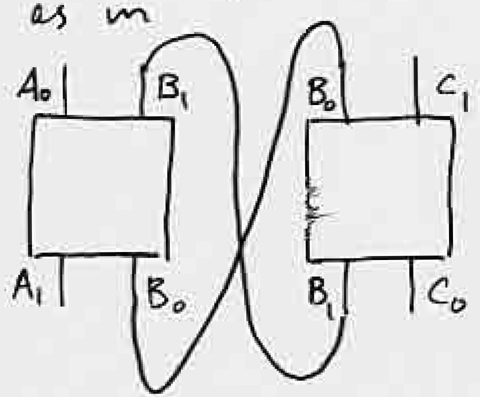
Construction of Integers

If $\mathcal{C}$ a traced monoidal category, then $\mathrm{Int}(\mathcal{C})$ is defined as follows.

Objects pairs (A_0, A_1) of objects of $\mathcal{C}$

Maps $(A_0, A_1) \longrightarrow (B_0, B_1)$ are maps $A_0 \otimes B_1 \longrightarrow A_1 \otimes B_0$

Composition is given by trace as in



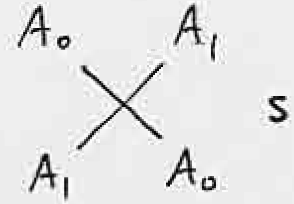
Structure in $\mathrm{Int}(\mathcal{C})$

$I \qquad (I, I)$

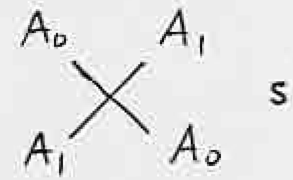
$\otimes \qquad (A_0, A_1) \otimes (B_0, B_1) = (A_0 \otimes B_0, A_1 \otimes B_1)$

Duals $\qquad (A_0, A_1)^* = (A_1, A_0)$

$\eta \qquad (I, I) \longrightarrow (A_0, A_1)^* \otimes (A_0, A_1)$



$\varepsilon \qquad (A_0, A_1) \otimes (A_0, A_1)^* \longrightarrow (I, I)$



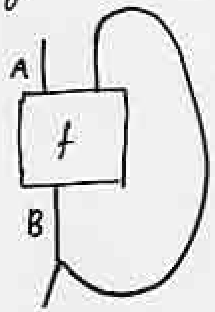
Symmetry axiom $\sim$ Triangle identities

TRACED CATEGORIES WITH PRODUCTS 1

Define an operation

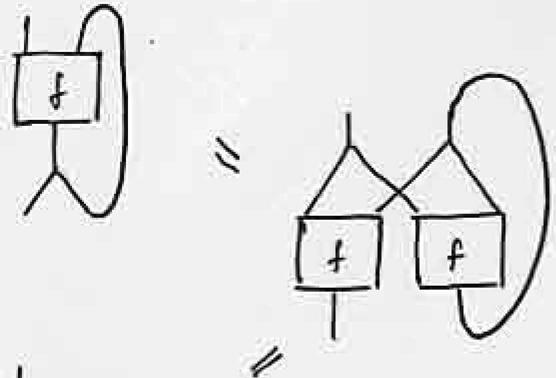
$$\frac{A \times B \xrightarrow{\ f\ } B}{A \xrightarrow[\mu b f]{} B}$$

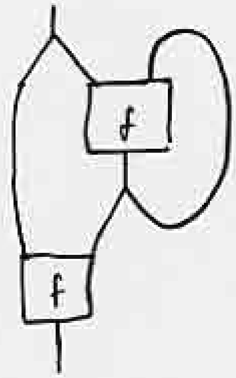
by the diagram



Algebraic notation $\mu b.\, f(a,b)$ is justified by naturality in A

TRACED CATEGORIES WITH PRODUCTS 2



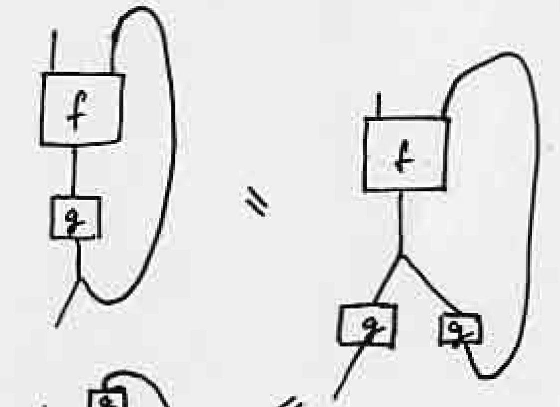


Thus gives

$$\mu b. f(a,b) = f(a, \mu b. f(a,b))$$

so $\mu b. f(a,b)$ is a fixed point of $f: A \times B \longrightarrow B$ parametrized in A.

TRACED CATEGORIES WITH PRODUCTS 3



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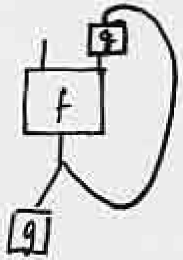


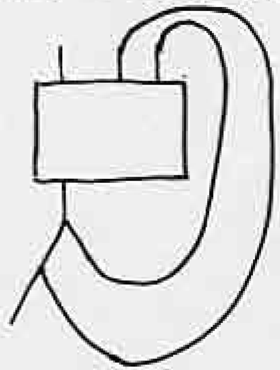
This gives

$\mu b.\, g(f(a,b))$

$= g(\mu c.\, f(a, g(c)))$

The main naturality property of fixed points.

TRACED CATEGORIES WITH PRODUCTS 4



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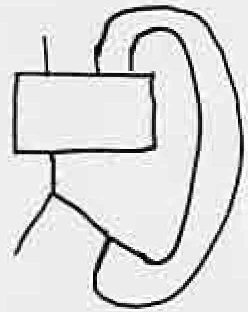

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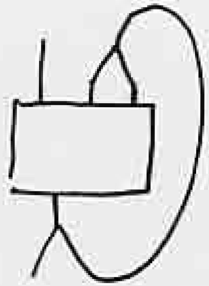


This gives

$\mu b_1\, \mu b_2\, f(a, b_1, b_2)$

$= \mu b\, f(a, b, b)$

A diagonal property of fixed points.

TRACED CATEGORIES WITH PRODUCTS 5

From a trace on a category with products we obtain a fixed point operator

$$\frac{A \times B \xrightarrow{f} B}{A \xrightarrow[\mu b f]{} B}$$

natural in A and satisfying

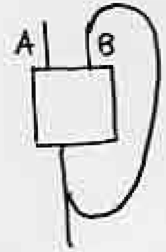
$$\mu b .\, g(f(a,b)) = g(\mu c .\, f(a, g(c)))$$

$$\mu b_1 \mu b_2\, f(a, b_1, b_2) = \mu b\, f(a, b, b)$$

(The Bekič property for simultaneous fixed points follows.)

CATEGORIES WITH FIXED POINTS 1

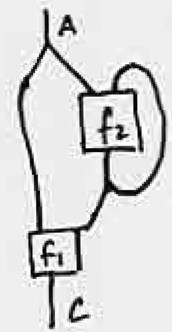
We represent the fixed point operator in diagrams thus:



Define an operation

$$\frac{A \times B \xrightarrow{f} C \times B}{A \xrightarrow[\mathrm{tr} f]{} C}$$

by the diagram



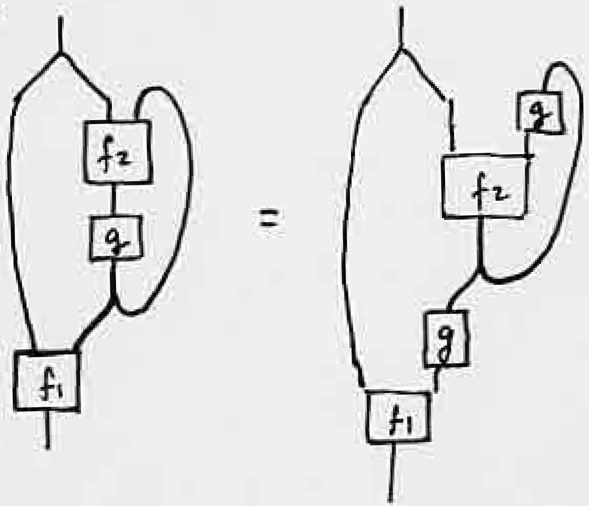
where

$f_1 = \mathrm{fst} \circ f$

$f_2 = \mathrm{snd} \circ f$

CATEGORIES WITH FIXED POINTS 2

Naturality in the output C is obvious and that in A is easy (naturality of Δ). The main naturality for trace is



by an axiom for fixed points

CATEGORIES WITH FIXED POINTS 3

The action axiom

$$tr_B\, tr_C(h) = tr_{B\times C}(h)$$

follows from the diagonal axiom for fixed points (essentially via Bekič)

Other axioms straightforward so we get a trace on the category with products.

THEOREM

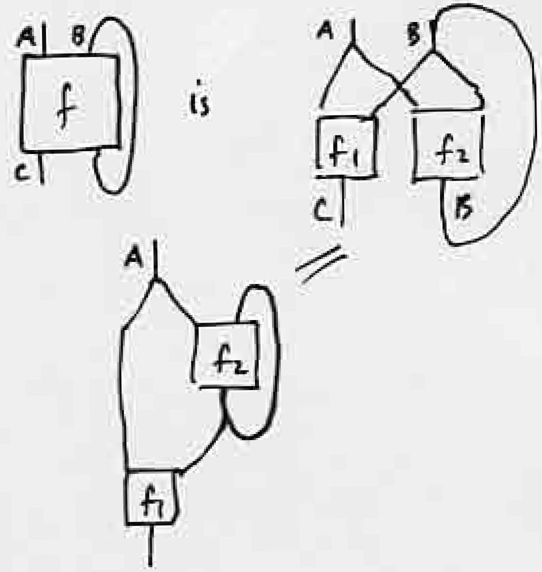
Traces on a category with products correspond exactly to fixed point operators satisfying the basic axioms.

Consequence. Categories of domains are traced categories with products.

(The fixed points have stronger properties but ... .)

PROOF

In one direction



In the other

diagram for fixed point property

BIPRODUCTS

A category with finite biproducts has

(i) a zero object 0 (i.e. initial and terminal)

(ii) a choice for each A, B of a diagram

$$
\begin{tikzcd}
A \arrow[dr, "in_l"'] & & B \arrow[dl, "in_r"] \\
& A \oplus B \arrow[dl, "fst"'] \arrow[dr, "snd"] & \\
A & & B
\end{tikzcd}
$$

which is both product and coproduct and with a good property.

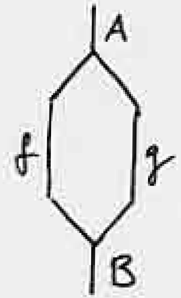
ENRICHMENT 1

In a category with biproducts we have unique maps

$$A \longrightarrow 0 \longrightarrow B$$

and so there is a unique map 0 in each $\mathcal{C}(A,B)$ factoring through 0.

We can add $f, g : A \to B$ by



addition is associative and commutative with 0 as zero.

ENRICHMENT 2

In a category $\mathcal{C}$ with biproducts each $\mathcal{C}(A,B)$ is a commutative monoid (written additively).

Moreover

$$\mathcal{C}(B,C) \times \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$$

is bilinear. So $\mathcal{C}$ is enriched in commutative monoids.

In particular each $\mathcal{C}(A,A)$ is a rig (ring without negatives).

ENRICHMENT 3

In $\mathcal{C}$ with biproducts maps

$$A \oplus B \longrightarrow C \oplus D$$

are given by matrices $\begin{pmatrix} u & v \\ s & t \end{pmatrix}$ with $u \in \mathcal{C}(A,C)$ $\quad v \in \mathcal{C}(B,C)$
$s \in \mathcal{C}(A,D)$ $\quad t \in \mathcal{C}(B,D)$

and composition of maps

$$A \oplus B \longrightarrow C \oplus D \longrightarrow C' \oplus D'$$

is given by the evident matrix multiplication.

This extends from 2×2 to $m \times n$ matrices.

ENRICHMENT 4

Suppose $\mathcal{C}$ is a category with biproducts 'generated by a single object': that is the objects are $0, 1, 2, \ldots$ with $n \oplus m = n + m$.

Then we know that the 'maps' in $\mathcal{C}(n,m)$ correspond to $m \times n$-matrices with entries in the rig $\mathcal{C}(1,1)$.

TRACED CATEGORIES WITH BIPRODUCTS 1

A map $A \oplus B \longrightarrow C \oplus B$ is given by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with d 'square' i.e. from a rig.

Then $\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : A \longrightarrow C$.

Easy naturalities

$$\operatorname{tr}\begin{pmatrix} ar & b \\ cr & d \end{pmatrix} = \operatorname{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}\right)$$
$$= \operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} r$$

$$\operatorname{tr}\begin{pmatrix} sa & sb \\ c & d \end{pmatrix} = \operatorname{tr}\left(\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$
$$= s \operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

TRACED CATEGORIES WITH BIPRODUCTS 2

The strong naturality

$$\mathrm{tr}\begin{pmatrix} a & b \\ ec & ed \end{pmatrix} = \mathrm{tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$$

$$= \mathrm{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}\right)$$

$$= \mathrm{tr}\begin{pmatrix} a & be \\ c & de \end{pmatrix}$$

Independence

$$\mathrm{tr}\left(\begin{array}{c|cc} u & 0 & 0 \\ \hline 0 & a & b \\ 0 & c & d \end{array}\right) = \left(\begin{array}{c|c} u & 0 \\ \hline 0 & \mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}\right)$$

TRACED CATEGORIES WITH BIPRODUCTS 3

By work on products we know that tr is definable in terms of fix where that in its turn is now given by

$$\mathrm{fix}(a\ b) = \mathrm{tr}\begin{pmatrix} a & b \\ a & b \end{pmatrix}$$

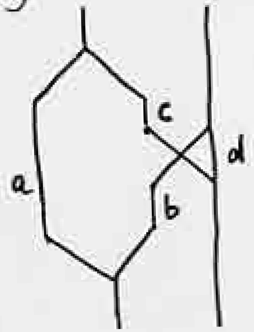
And the product formula gives

$$\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a\ b)\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{fix}(c,d) \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= a + b\,\mathrm{fix}(c,d)$$

$$= a + b\,\mathrm{tr}\begin{pmatrix} c & d \\ c & d \end{pmatrix}$$

Geometrically the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ factorizes



that is,

$$\begin{pmatrix} 1 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{c|cc} a & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & d \end{array}\right) \begin{pmatrix} 1 & 0 \\ c & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

So we have

$$\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + b \operatorname{tr}\begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} c$$

For simplicity now analyze the basic case where the category is generated under $\oplus$s by a single object 1. We look at the rig $R = \mathcal{C}(1,1)$.

On R we define $(\ )^*$ by

$$a^* = \operatorname{tr}\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$$

this really is a 2×2 matrix.

So $\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bd^*c$

(for now just for 2×2 matrices)

TRACED CATEGORIES WITH BIPRODUCTS 6

Consider

$$(ab)^* = tr\begin{pmatrix} 0 & 1 \\ 1 & ab \end{pmatrix}$$

$$= 0 + 1.tr\begin{pmatrix} 1 & ab \\ 1 & ab \end{pmatrix} \quad \text{fix stuff}$$

$$= tr\begin{pmatrix} 1 & ab \\ 1 & ab \end{pmatrix}$$

$$= tr\begin{pmatrix} 1 & a \\ b & ba \end{pmatrix} \quad \text{strong naturality}$$

$$= 1 + a\,(ba)^*\,b \quad \text{trace formula}$$

(Again just 2×2 matrices.)

TRACED CATEGORIES WITH BIPRODUCTS 7

Consider

$$(a+b)^* = tr\begin{pmatrix} 1 & a+b \\ 1 & a+b \end{pmatrix} = tr\begin{pmatrix} 1 & (1\ 1)\begin{pmatrix} a \\ b \end{pmatrix} \\ 1 & (1\ 1)\begin{pmatrix} a \\ b \end{pmatrix} \end{pmatrix}$$

$$= tr_2\begin{pmatrix} 1 & 1 & 1 \\ a & a & a \\ b & b & b \end{pmatrix} \quad \text{strong naturality}$$

$$= tr\left[\begin{pmatrix} 1 & 1 \\ a & a \end{pmatrix} + \begin{pmatrix} 1 \\ a \end{pmatrix} b^* \,(b\ b)\right] \quad \text{action}$$

$$= tr\begin{pmatrix} 1+b^*b & 1+b^*b \\ a(1+b^*b) & a(1+b^*b) \end{pmatrix}$$

$$= tr\begin{pmatrix} b^* & b^* \\ ab^* & ab^* \end{pmatrix} \quad \text{previous result}$$

$$= b^*\, tr\begin{pmatrix} 1 & 1 \\ ab^* & ab^* \end{pmatrix} \quad \text{naturality}$$

$$= b^*\, tr\begin{pmatrix} 1 & ab^* \\ 1 & ab^* \end{pmatrix} \quad \text{strong naturality}$$

$$= b^*\,(ab^*)^*.$$

PROPOSITION

Let $\mathcal{C}$ be traced category with biproducts. Then for every object A the rig

$$\mathcal{C}(A,A)$$

has the structure of a Conway algebra.

OBSERVATION

We know $tr\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + bd^*c$

when $d \in \mathcal{C}(A,A)$ (i.e. a 1×1-matrix).

So the traced monoidal category structure on the subcategory of $A^{\oplus n}$ is determined by the Conway algebra.

THEOREM

If R is a Conway algebra then the category $Mat(R)$ with $Mat(R)(n,m) =$ $m\times n$ matrices in R is traced monoidal.

The essential fact is that $M_2(R)$ (2×2-matrices) is a Conway algebra with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} (a^*bd^*c)^*a^* & d^*c \\ bd^* & d^*(ca^*bd^*)^* \end{pmatrix}$$

CATEGORY OF FINITE AUTOMATA $\mathbb{A}$ 1

Take as

Objects the standard finite sets (cardinals) $0, 1, 2, 3 \ldots$

Arrows in $\mathbb{A}(n,m)$ consist of finite automata (S, etc) plus injective maps

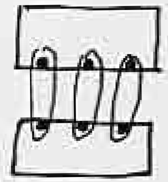
$$n \longrightarrow S$$
$$m \longrightarrow S$$

The states in the image of $n \to S$ are input states those in the image of $m \longrightarrow S$ are output states, and the rest internal states.

CATEGORY OF FINITE AUTOMATA 2

Identities (and symmetries): Trivial automata (no transitions)

Composition $\mathbb{A}(m,p) \times \mathbb{A}(n,m) \longrightarrow \mathbb{A}(n,p)$ is by merging the states indexed by m



Trace $\mathbb{A}(n+p, m+p) \longrightarrow \mathbb{A}(n,m)$ takes the states corresponding to the last p inputs and outputs merges them pairwise (if necessary) and makes them internal.

ASIDE

Composition is controlled by the following category.

Objects finite cardinals $0, 1, 2, \ldots$

Maps $n \longrightarrow m$ are injective cospans

$$\begin{tikzcd} n \arrow[dr] & & m \arrow[dl] \\ & S & \end{tikzcd}$$

Composition is as of cospans

$$\begin{tikzcd} n \arrow[dr] & & m \arrow[dl] \arrow[dr] & & p \arrow[dl] \\ & S \arrow[dr] & & T \arrow[dl] & \\ & & S \underset{m}{+} T & & \end{tikzcd}$$

CATEGORY OF REGULAR LANGUAGES

The category $Mat(Reg)$ whose maps are matrices with entries from the Conway algebra Reg of regular languages is traced.

But we don't need the big theorem!

Reg is a Conway subalgebra of $P(\Sigma)$ and $Mat(P(\Sigma))$ is evidently traced. Because traced structure is determined by the Conway algebra, $Mat(Reg)$ inherits the structure.

THEOREM

(as previously advertized)

There is a traced monoidal functor

$$\mathbb{A} = \text{Aut} \longrightarrow \text{Mat}(\text{Reg}) = \underline{\text{Reg}}$$

Description of functor

If $(S, \text{etc}) : n \longrightarrow m$ in Aut, it gives an $|S| \times |S|$ transition matrix with entries single letters. Take the $(\ )^*$ and reduce to the $m \times n$ submatrix.

Routine to check this works.

BUT

WARNING

$$\underline{\text{Aut}} \longrightarrow \underline{\text{Reg}}$$

is not onto.

Example $\begin{pmatrix} a^* \\ b \end{pmatrix}$

That "gives the reason" why simulating regular languages by finite automata is not trivial.

N.B. $\underline{\text{Aut}}$ is not "inductively defined" as a category, as far as I know.

CONCLUDING REMARKS

Original aims

Explain

$$\underline{\mathrm{Aut}} \longrightarrow \underline{\mathrm{Reg}}$$

and the connection with the characterization direction of Kleene's Theorem.

Exercise in style

An abstract approach to concrete mathematics.

Abstraction

$\sim$ Precision