HIGHER DIMENSIONAL CATEGORIES

- Categories to strict higher categories
- Strict monoidal categories I
- Δ and simplicial sets
- Strict monoidal categories II
- Bicategories and the pentagon
- Higher dimensional category theory

Higher Dimensional Categories

Martin Hyland

According to Eilenberg and Mac Lane, founders of Category Theory, categories were needed to explain functors and functors were needed to explain natural transformations. Thus from the beginning it was clear that the study of 1-dimensional categories required a 2-dimensional structure, a 2-category; and so on. In the simplest versions of the resulting higher dimensional categories, one takes composition to be strictly associative. I shall outline the theory and explain some results.

Unfortunately, for compelling applications the associativity of composition fails, and one needs a theory which treats ever more subtle notions of equivalence. One approach to this borrows ideas from homotopy theory. I shall try both to explain some concrete motivations and to convey the flavour of the subject as it appears to be developing.

CATEGORIES TO STRICT HIGHER CATEGORIES

CATEGORIES

A category $\mathbb{C}$ has

objects
points $\quad A, B, C, \ldots$
0-cells

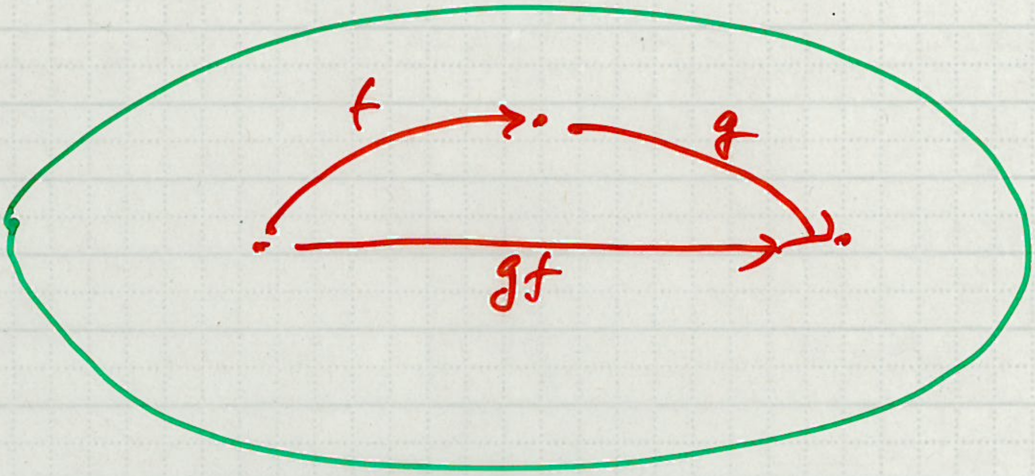
arrows
maps morphisms $\quad A \xrightarrow{f} B$
1-cells

identities $\quad A \xrightarrow{1_A} A$

composition $\quad \dfrac{A \xrightarrow{f} B \quad B \xrightarrow{g} C}{A \xrightarrow{gf} C}$

with evident axioms.

PICTURE



Given $\mathbb{C}$, we have $\mathbb{C}^{op}$ with same object but direction of arrows reversed.

$$(\mathbb{C}^{op})^{op} = \mathbb{C}$$

USUAL EXAMPLES

OBJECTS	ARROWS
Sets	Functions
Sets	Relations
Groups	Homomorphisms
Topological spaces	Continuous maps
Topological groups	Continuous homomorphisms
Functors $(\mathbb{C}, \mathbb{D})$	Natural transformations

SMALL CATEGORIES

A partially ordered set $(P, \leq)$
gives a category with

objects : points of P

maps : unique such
$p \longrightarrow q$ when $p \leq q$

A group G gives a category
with

objects : just one $= *$ say

maps : elements of G,
$g : * \longrightarrow *$

(Composition = Multiplication).

FUNCTOR & NATURAL TRANSFORMATIONS

A structure preserving map

$$F : \mathbb{C} \longrightarrow \mathbb{D}$$

between categories is a functor.

Given two functors

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathbb{D} \end{tikzcd}$$

a natural transformation

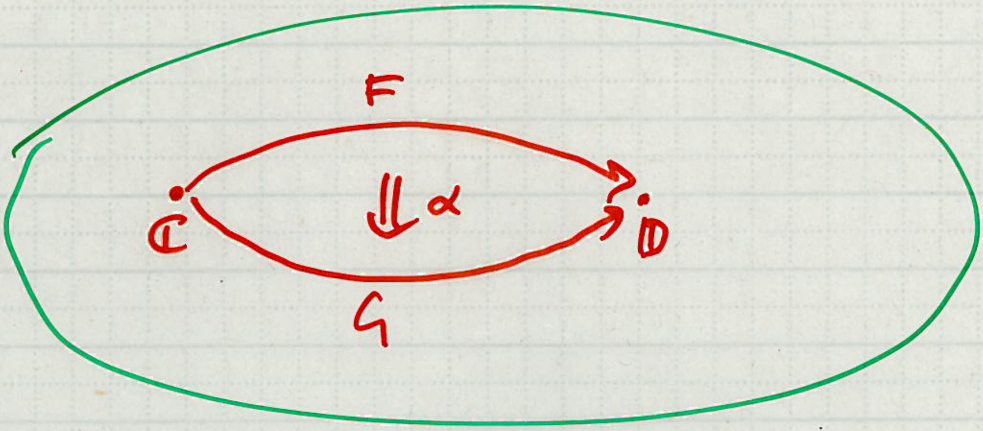
$$\alpha : F \longrightarrow G$$

is given by $\alpha_C : F(C) \longrightarrow G(C)$ in $\mathbb{D}$

with $G(u) . \alpha_C = \alpha_D . F(u)$ all $u : C \to D$ in $\mathbb{C}$

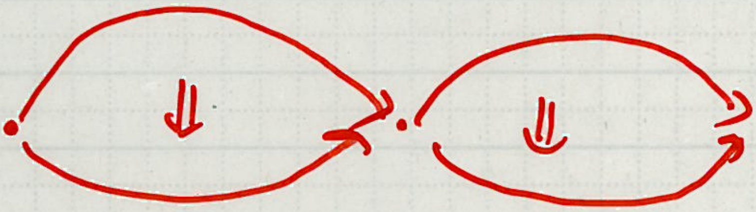
(Secret direction issue!)

PICTURES



with two compositions

$*_0$



$*_1$

2 - CATEGORIES

A 2-category $\mathcal{K}$ has

0-cells $\quad A, B, C \dots$

1-cells $\quad A \xrightarrow{f} B, \; B \xrightarrow{g} C, \; \dots$

2-cells
$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

with identities and compositions

(the two compositions of 2-cells suffice)

MIDDLE 4 INTERCHANGE

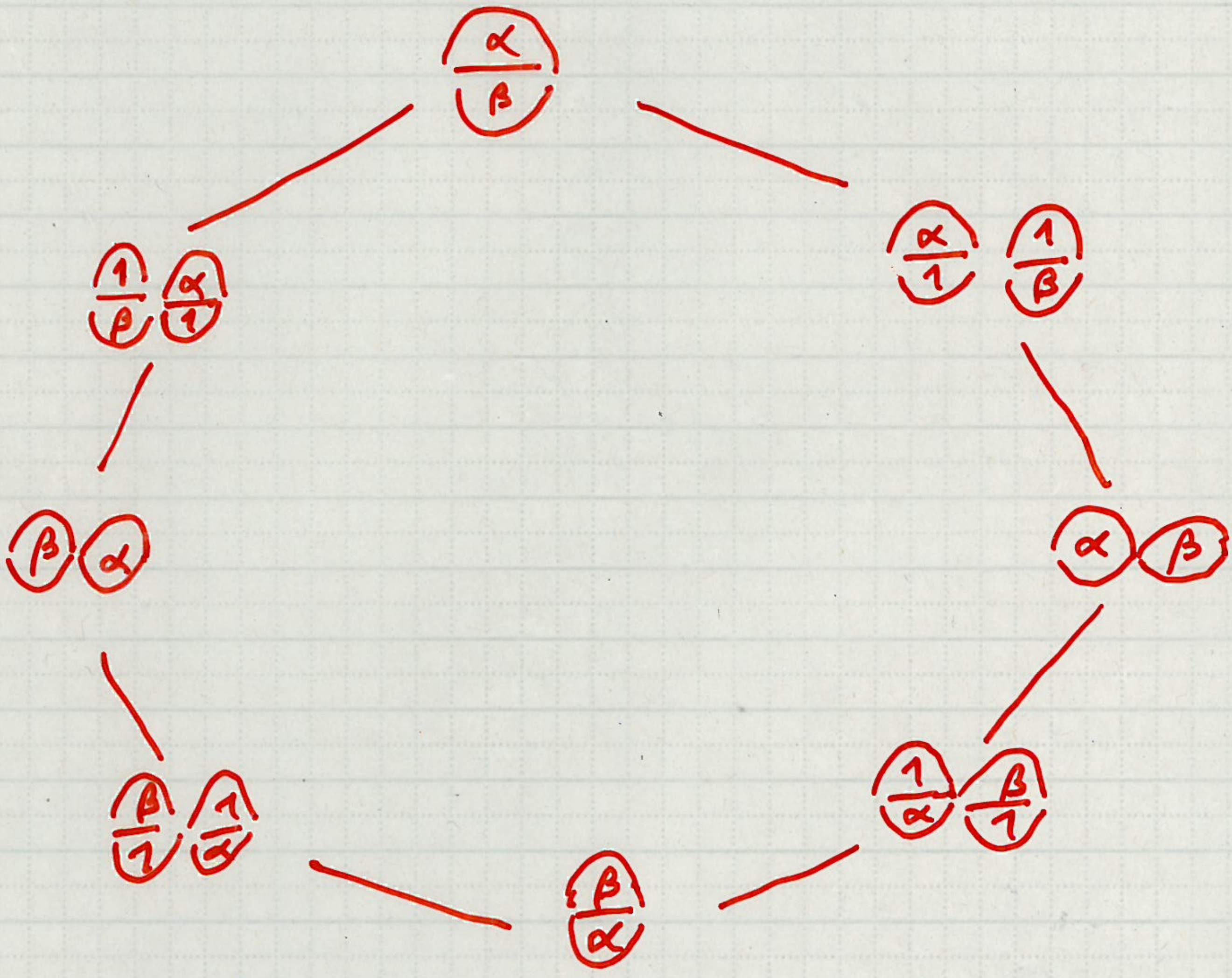
$$\begin{tikzcd}
\bullet \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=2ex, "\alpha\Downarrow"] \arrow[r, phantom, shift right=2ex, "\gamma\Downarrow"] & \bullet \arrow[r, bend left=70] \arrow[r] \arrow[r, bend right=70] \arrow[r, phantom, shift left=2ex, "\Downarrow\beta"] \arrow[r, phantom, shift right=2ex, "\Downarrow\delta"] & \bullet
\end{tikzcd}$$

$$(\delta *_0 \gamma) *_1 (\beta *_0 \alpha) = (\delta *_1 \beta) *_0 (\gamma *_1 \alpha)$$

This says the compositions commute

ECKMANN – HILTON

Commuting compositions with 1 are equal,



and <u>the</u> composition is commutative.

TWO CONSEQUENCES OF E-H ARGUMENT

Let A be a 0-cell (object) in a 2-category (or bicategory) $\mathcal{K}$. Then

$$\mathrm{End}(1_A)$$

is a <u>commutative</u> monoid.

The second homotopy group $\pi_2(X, x_0)$ of a pointed space is commutative (as are the higher homotopy groups).

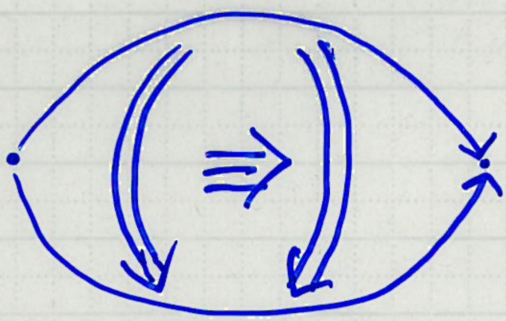
STRICT HIGHER CATEGORIES

$$\cdots\ C_3 \underset{t}{\overset{s}{\rightrightarrows}} C_2 \underset{t}{\overset{s}{\rightrightarrows}} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

with

$ss = st$ (2nd order s)

$ts = tt$ (2nd order t)



globular sets

with identities and compositions at all levels.

STRICT MONOIDAL CATEGORIES I

(STRICT) MONOIDAL CATEGORIES

These are categories $\mathbb{C}$ equipped with

$$I \in \mathbb{C} \qquad \mathbb{C} \times \mathbb{C} \xrightarrow{\otimes} \mathbb{C}$$

such that

$$I \otimes A = A = A \otimes I$$

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C$$

(These do not occur in the usual examples: there we have $\cong$ for $=$.)

These are strict 2-categories with 1 object.

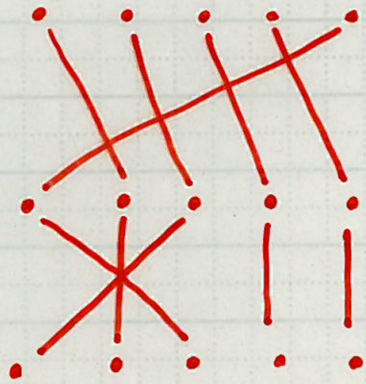
SYMMETRIC GROUPS

objects : natural numbers n

maps : bijections $n \longrightarrow n$

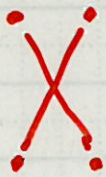
$\otimes$: sum (concatenation)

PICTURE



" $(13)(12345) = (12)(345)$ "

Generated as a monoidal category by



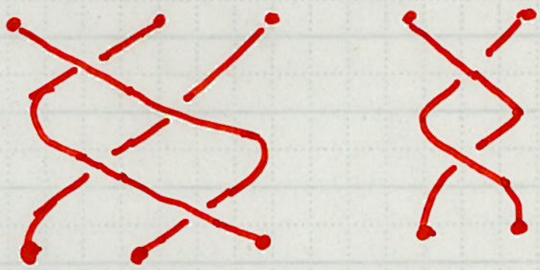
BRAID GROUPS

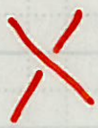
objects $n, \ldots$

maps braids $n \to n$

tensor $+$

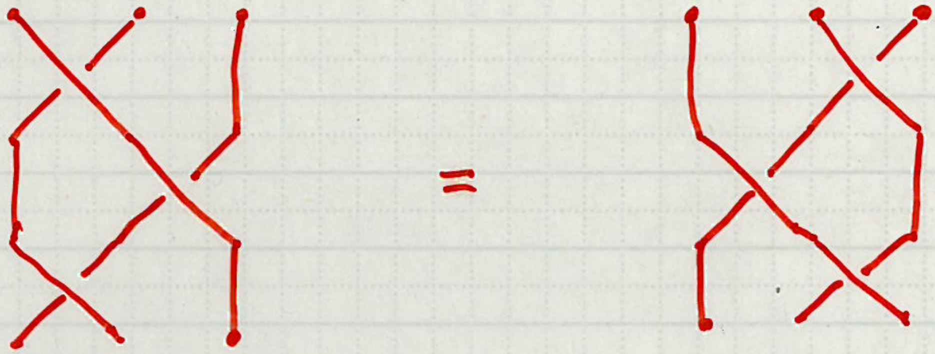
PICTURE



Generated by  and its inverse 

as a monoidal category.

THE BRAID RELATION



$$(s_{12} \otimes 1)(1 \otimes s_{23})(s_{12} \otimes 1) = (1 \otimes s_{23})(s_{12} \otimes 1)(1 \otimes s_{23})$$

$$\underbrace{R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}}_{\text{cf Yang-Baxter equation}}$$

For the symmetric group add

$$s^2 = 1.$$

Δ AND SIMPLICIAL SETS

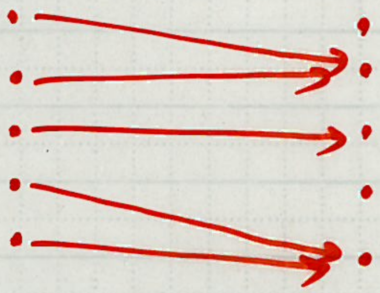
FREE MONOID

Δ_+ has

objects $n = \Delta[n-1]$ (as an ordered set)

maps $n \longrightarrow m$ are order preserving maps

PICTURE



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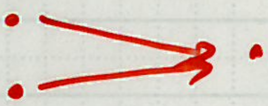
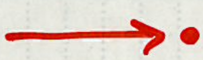
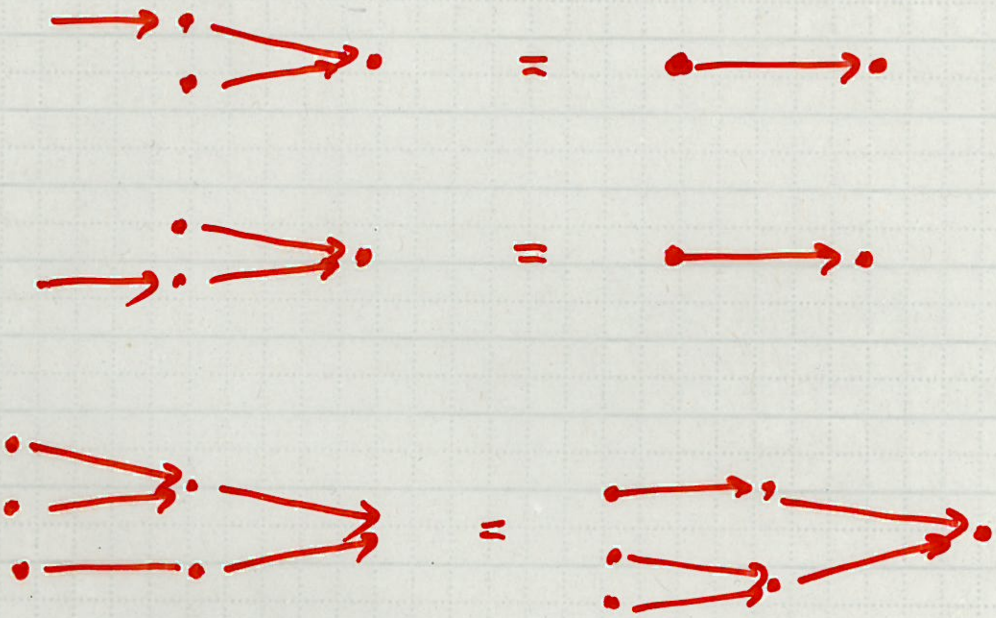
 and 

AXIOMS FOR Δ_+



The unit and associativity laws for a monoid.

SIMPLICIAL SETS

Let Δ be as Δ_+ but omitting

$$0 = \Delta[-1].$$

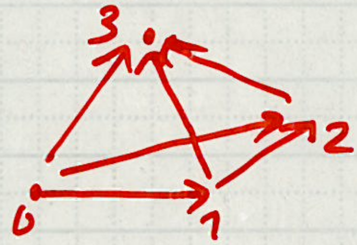
So Δ has objects

$$\Delta[0] = \bullet_0$$

$$\Delta[1] = 0 \to 1$$

$$\Delta[2] = \begin{tikzcd} & 2 & \\ 0 \arrow[ur] \arrow[rr] & & 1 \arrow[ul] \end{tikzcd}$$

$$\Delta[3] =$$



(i.e. formal simplices)

The category of simplicial sets is the functor category $[\Delta^{op}, \mathrm{Sets}]$ i.e presheaves on Δ

(The consequence of formally glueing simplices together.)

$$X \in [\Delta^{op}, \mathrm{Set}] \; : \; X_n = X(\Delta[n])$$

THE NERVE OF A CATEGORY

Regard the ordered sets $\Delta[n]$ as categories: order preserving maps are functors.

So if $\mathbb{C}$ is a category then

$$\Delta(n) \longmapsto [\Delta[n], \mathbb{C}]$$ gives

a simplicial set $N(\mathbb{C})$.

$$N(\mathbb{C})_n = \text{composable } n\text{-paths in } \mathbb{C}$$

E.g.

$N(\mathbb{C})_3$ is the set of diagrams

$$\bullet \to \bullet \to \bullet \to \bullet$$

in $\mathbb{C}$

($N(-)$ has a left adjoint, the fundamental category of a simplicial set.)

STRICT MONOIDAL CATEGORIES

II

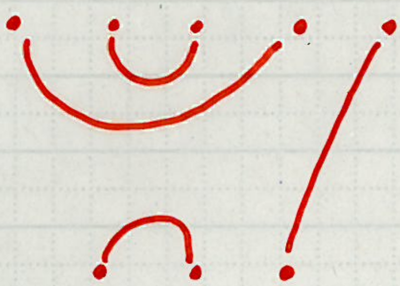
TEMPERLEY-LIEB CATEGORY

The free monoidal category on a self dual object of dimension 1.

objects $n, \ldots$

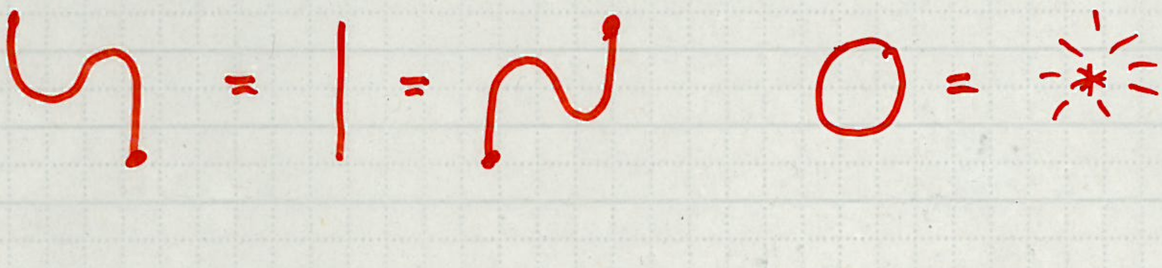
maps $n \longrightarrow m$ are chord diagrams

PICTURE



Generated as a monoidal category by

subject to



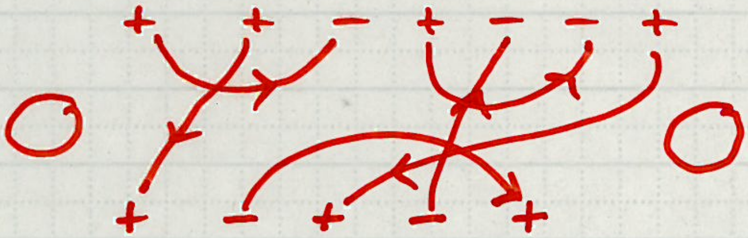
DUALS

Free category on object with involutive duals.

objects sequences of signed points (closed zero dimensional manifolds)

maps bijections respecting signs plus dummy circles (1-dimensional manifolds with appropriate ∂)

PICTURE

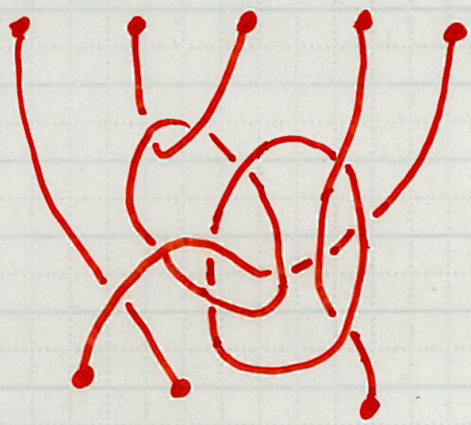


TANGLES

objects $n, m, \ldots$

maps $n \longrightarrow m$ tangles with n, m endpoints

PICTURE



Free braided monoidal on a self dual.

APPLICATION

We have "structure preserving functors" to "naturally occuring" braided monoidal categories. (Determined by where the object • goes.)

Eg to categories of representations of quantum groups, level k representations of loop groups.

A knot or link $\longmapsto$ Map I to I

and

$$\mathrm{End}(I) = \mathbb{C}[q, q^{-1}]$$

So knot invariants.

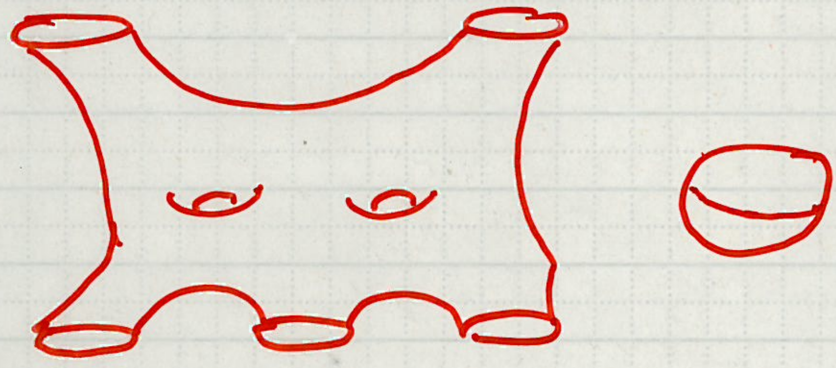
NAIVE STRING THEORY

(Free on a Frobenius algebra)

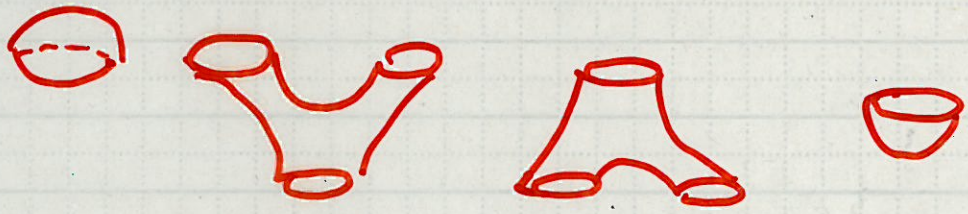
objects formal finite sums of circles
(~ compact closed 1-manifolds)

maps (isomorphism classes of) topological Riemann surfaces with appropriate ∂ (convention).

PICTURE



Generators



BICATEGORIES AND THE PENTAGON

BICATEGORIES

Weak 2-categories

1-cell composition associative with unit

UPTO ISOMORPHISM

$$1f \xRightarrow{\lambda} f \qquad g1 \xRightarrow{\rho} g \qquad (hg)f \xRightarrow{\alpha} h(gf)$$

with COHERENCE CONDITIONS

$$\begin{tikzcd}
((kh)g)f \arrow[rr, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\alpha f"'] & & (kh)(gf) \arrow[d, Rightarrow, "\alpha"] \\
(k(hg))f \arrow[r, Rightarrow, "\alpha"'] & k((hg)f) \arrow[r, Rightarrow, "k\alpha"'] & k(h(gf))
\end{tikzcd}$$

$$\begin{tikzcd}
(g1)f \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\rho f"'] & & g(1f) \arrow[dl, Rightarrow, "g\lambda"] \\
& gf &
\end{tikzcd}$$

(Two more are natural but Kelly showed they are redundant.)

MAC LANE COHERENCE

- All diagrams constructed from

λ, ρ, α

commute. (A property of term rewriting!)
(So λ, ρ, α produce no essential ambiguity.)

- Any bicategory is 'equivalent' to
a 2-category.

SO WHY BOTHER?

WARNING Not true at dim 3 and above; nor with extra structure.

$\boxed{\text{Also } \alpha \text{ has meaning.}}$

DUAL GROUP ALGEBRA

Let $FG = \mathbb{C}^G$ be the algebra of $\mathbb{C}$-valued functions on the finite group G with pointwise multiplication.

(Dual to group algebra $\mathbb{C}[G]$.)

Sufficient to consider 1-dimensional representations of the form

$$\mathbb{C}g \quad \text{for} \quad g \in G$$

with, for $\phi \in FG$

$$\phi . g = \phi(g)\, g .$$

TENSORING FG-MODULES

comes from comultiplication on FG

that is from multiplication on $\mathbb{C}[G]$.

So we have

$$\mathbb{C}g \otimes \mathbb{C}h \cong \mathbb{C}gh.$$

And we might as well take equality.

So choosing an associativity constraint

is choosing

$$\mathbb{C}abc = (\mathbb{C}a \otimes \mathbb{C}b) \otimes \mathbb{C}c \xrightarrow{\alpha_{abc}} \mathbb{C}a \otimes (\mathbb{C}b \otimes \mathbb{C}c) = \mathbb{C}abc$$

such that ....

MAC LANE PENTAGON IN FG-MODULES

$$
\begin{tikzcd}
((a\otimes b)\otimes c)\otimes d \arrow[rr, "\alpha_{ab,c,d}"] \arrow[d, "\alpha_{a,b,c}"'] & & (a\otimes b)\otimes(c\otimes d) \arrow[d, "\alpha_{a,b,cd}"] \\
(a\otimes(b\otimes c))\otimes d \arrow[dr, "\alpha_{a,bc,d}"'] & & a\otimes(b\otimes(c\otimes d)) \\
& a\otimes((b\otimes c)\otimes d) \arrow[ur, "\alpha_{b,c,d}"'] &
\end{tikzcd}
$$

$$\alpha_{a,b,cd}\cdot\alpha_{ab,c,d}=\alpha_{b,c,d}\cdot\alpha_{a,bc,d}\cdot\alpha_{a,b,c}$$

Take logs and rearrange!

COHOMOLOGY OF GROUPS

Let G a group.

Given an abelian group A set

$$C_n = \mathrm{Maps}(G^n, A)$$

and define

$$d : C_n \longrightarrow C_{n+1}$$

by

$$d\phi(g_0, \ldots, g_n)$$

$$= \phi(g_0, \ldots, g_{n-1}) - \phi(g_0, \ldots, g_{n-1}g_n) + \phi(g_0, \ldots, g_{n-2}g_{n-1}, g_n)$$

$$+ \cdots + (-1)^n \phi(g_0 g_1, \ldots, g_n) + (-1)^{n+1} \phi(g_1, \ldots, g_n)$$

FIRST APPROXIMATION

An associativity constraint is a

3 co-cycle i.e. a

member of $Z_3(G; \mathbb{C}^\times)$.

But do the boundaries mean

anything?

$B_3(G; \mathbb{C}^\times)$ gives monoidal isomorphism

with the trivial identity constraint.

SECOND APPROXIMATION

An up-to-isomorphism associativity

constraint is an element of

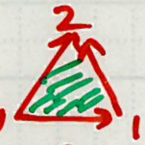
$H_3(G, \mathbb{C}^\times)$.

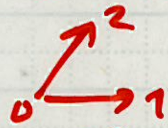
HIGHER DIMENSIONAL CATEGORY THEORY

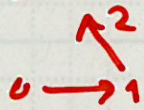
HORNS

The n-simplex $\Delta[n]$ has $n+1$ vertices and opposite each vertex k with $0 \leq k \leq n$ there is an $n-1$ dimensional face. The k'th horn $\Lambda^k[n]$ is the union of the $(n-1)$-dimensional faces other than the k'th.

Example

$\Delta[2]$ is 

$\Lambda^0[2]$ is 

$\Lambda^1[2]$ is 

$\Lambda^2[2]$ is 

KAN COMPLEXES

A simplicial set is a Kan complex

if and only if

every horn has a filler

$$
\begin{tikzcd}
\Lambda^k[n] \arrow[r] \arrow[d, hook] & X \\
\Delta[n] \arrow[ur, dotted] &
\end{tikzcd}
$$

(The fundamental category (i.e. via left adjoint) of a Kan

complex is already a groupoid.)

QUASI-CATEGORIES

$\Lambda^k[n]$ is an inner horn when $0<k<n$.

A simplicial set X is a quasi-category if and only if all inner horns have fillers.

$$\begin{tikzcd} \text{inner} \ \Lambda^k[n] \arrow[r] \arrow[d, hook] & X \\ \Delta[n] \arrow[ur, dotted] & \end{tikzcd}$$

(In Boardman + Vogt: Joyal has systematically developed category theory for quasi categories.)

ONE DIMENSIONAL MEANING

$\Delta[2]$ in $X \in [\Delta^{op}, Set]$ is
$$\begin{tikzcd} & \bullet & \\ \bullet \arrow[ur, "gf"] \arrow[rr, "f"'] & {} & \bullet \arrow[ul, "g"'] \end{tikzcd}$$

where the one-cell is a 'reason' why

$$g \cdot f \simeq gf.$$

$\Lambda^1[2]$ in X is
$$\begin{tikzcd} & & \bullet \\ \bullet \arrow[rr, "f"'] & & \bullet \arrow[u, "g"'] \end{tikzcd}$$
and to fill it is to find a composite and reason why.

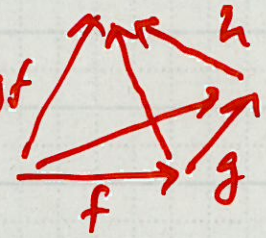
There is no need to try to fill

$\Lambda^0[2]$ in X
$$\begin{tikzcd} & \bullet \\ \bullet \arrow[ur] \arrow[r] & \bullet \end{tikzcd}$$

$\Lambda^2[2]$ in X
$$\begin{tikzcd} & \bullet & \\ \bullet \arrow[ur] & & \bullet \arrow[ul] \end{tikzcd}$$
.

TWO DIMENSIONAL MEANING

$\Delta[3]$ in X is



which is reasons why

$$g.f \simeq gf$$
$$h.g \simeq hg$$
$$h.gf \simeq hgf$$
$$hg.f \simeq hgf$$

plus a reason why the different ways to compose are equivalent.

$\Lambda^1[3]$ in X misses $h.gf \simeq hgf$ and one expects to compose to get that.

$\Lambda^0[3]$ in X misses $h.g \simeq hg$ and there's no evident way to reconstruct it.

AND SO ON !?

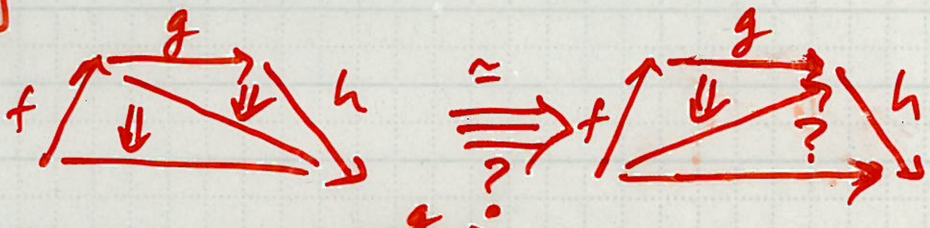
WEAK HIGHER CATEGORIES

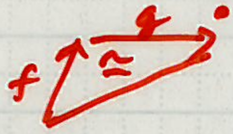
(style of Complicial Sets (Roberts: Street, Verity)

Quasi-categories are weak higher categories with all higher cells ≥ 2 equivalences.

IDEA (parity discipline.)

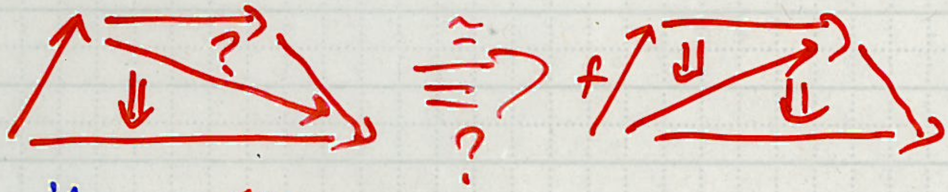
$\Lambda^1[3]$

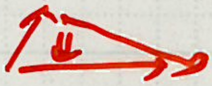


This OK if 

is an equivalence.

$\Lambda^0[3]$



This OK if  is an equivalence

<u>and</u> $f \nearrow$ is an equivalence

WEAK ALGEBRA

Operations given up to equivalences which are themselves up to equivalence and so on.

(Grothendieck's idea!)

But under what circumstances should you be able to "fill in" i.e. solve equations? And what is the combinatorial information implicit?