LOGIC

MATHEMATICAL PERSPECTIVE

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LOGIC

The study of REGULARITIES
in (correct) REASONING

Aristotle's Prior Analytics

We must first state the subject of our inquiry and the faculty to which it belongs: its subject is demonstration and the faculty which carries it out demonstrative science.

MATHEMATICAL LOGIC

Logic from a mathematical point of view

- could mean the use of logic (informally or formally) within mathematics ×
- could be the mathematical study of logic - formal logic ✓

So logic as a part of mathematics:

compare

School Mathematics: Quantity, Space (Chance)

Formal Logic: demonstration, reasoning

PLAN

1. Aristotle's Syllogisms (6)
2. Propositional Calculus (12)
3. Predicate Calculus (6)
4. Paradoxes (4)
5. Syntax (4)
6. Incompleteness Theorems (8)

SYLLOGISMS 1

Examples

$$\frac{\text{All men are mortal} \quad \text{Socrates is a man}}{\text{Socrates is mortal}}$$

All men are mortal
Socrates is a man

Socrates is mortal

All animals are mortal
All men are animals

All men are mortal

Structure

$$\frac{A \subseteq M \qquad H \subseteq A}{H \subseteq M}$$

Compare algebra

$$A/M \cdot H/A \vdash H/M$$

SYLLOGISMS 2

(TRADITIONAL VIEW)

Forms of statement

All A are B	$A \subseteq B$	(A)
No A is B	$A \cap B = \emptyset$	(E)
Some A are B	$A \cap B \neq \emptyset$	(I)
Some A are not B	$A \not\subseteq B$	(O)

Write $\Phi(A,B)$ etc for one of these forms.

Then a syllogism is a correct rule of inference of the form

$$\frac{\Phi(A,B)\ ,\ \Psi(B,C)}{\Theta(A,C)}$$

(Usually $\Psi(B,C)$ major premiss
$\Phi(A,B)$ minor premiss
$\Theta(A,C)$ conclusion.)

EXISTENCE

(A hidden issue.)

Consider

No animals are vegetables
Some men are animals

Some men are not vegetables

No animals are vegetables
All men are animals

Some men are not vegetables

Are these (correct) syllogisms?

HINT: Replace men by unicorns.

CLASSIFYING SYLLOGISMS

How many syllogisms can we find?

- Depends on what we allow: simplest to be strict about existence.
- Depends on how we count: warning for web cheats

Some A are B ~ Some B are A

No A is B ~ No B is A

SORITES

(syllogisms combined to give deductions)

Exercise

Some civilised people are not logicians

All who appreciate poetry love Leopardi's Canti

All civilised people appreciate poetry

?

Exercise

No liars are honest

Some politicians are well-intentioned

All politicians are liars

?

NON - EXAMPLE

No civilised person adds parmesan to seafood pasta

Some English people add parmesan to spaghetti alle vongole

Spaghetti alle vongole is a seafood pasta

?

Why is this not an example?

(CLASSICAL) PROPOSITIONAL CALCULUS

Study of propositions $A, B, C, \ldots$ built up from primitive propositions $p, q, r, \ldots$ by means of connectives:

$A \wedge B$	A and B
$A \vee B$	A or B
$A \to B$	if A then B
$\neg A$	not A

Take $\qquad \perp \qquad$ false

and then can identify

$$\neg A \quad \text{as} \quad A \to \perp$$

(Could also identify

$\top \qquad$ true

with

$\perp \to \perp$)

TRUTH TABLES

Propositions are either true $\top = T$
or false $\bot = F$

If we know the values for the primitives

then we know the values for all propositions

using truth tables

$A \wedge B$	B: T	F
A: T	T	F
F	F	F

$A \vee B$	B: T	F
A: T	T	T
F	T	F

$A \to B$	B: T	F
A: T	T	F
F	T	T

$\neg A$	
A: T	F
F	T

- Mathematics: compare multiplication tables

- Mathematics: de Morgan duality

PROOFS

(Prior Analytics : Demonstration)

Given by proof trees formed from

Rules of Inference

$\wedge$

$$\frac{A \qquad B}{A \wedge B} \qquad\qquad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$\vee$

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B} \qquad\qquad \frac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}$$

$\rightarrow$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \rightarrow B} \qquad\qquad \frac{A \rightarrow B \qquad A}{B}$$

$\neg / \bot$

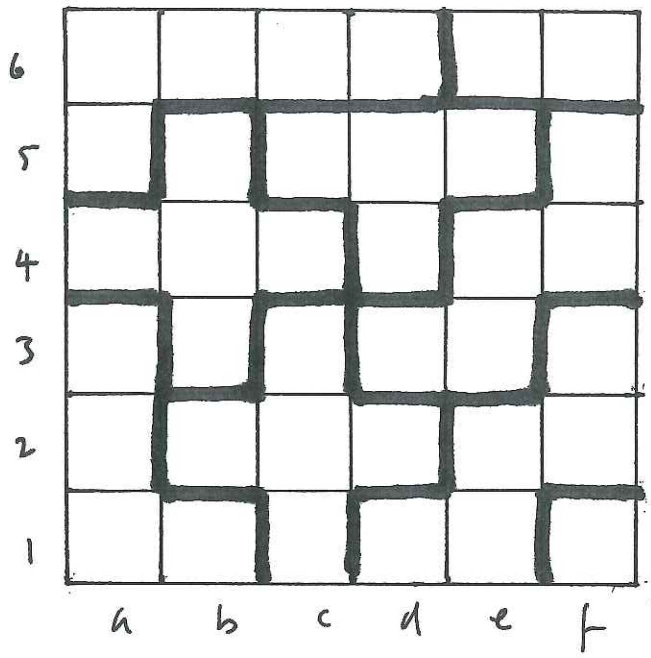
$$\frac{\begin{array}{c}[\neg A = A \rightarrow \bot]\\ \vdots \\ \bot\end{array}}{A}$$

Diagrams define $\Gamma \vdash A$ i.e.

hypotheses $\Gamma = C_1, \dots, C_n$ entail

the conclusion A

SUGURU 1



Each square to get a number

1, 2, 3, 4, 5.

(Square can be named as in chess.)

Primitive propositions $\rho = 2,\ \rho = 3,\ \sigma = 1,\ \tau = 5, \ldots,$

where ρ, σ, τ denote squares.

[But NB. No internal structure.]

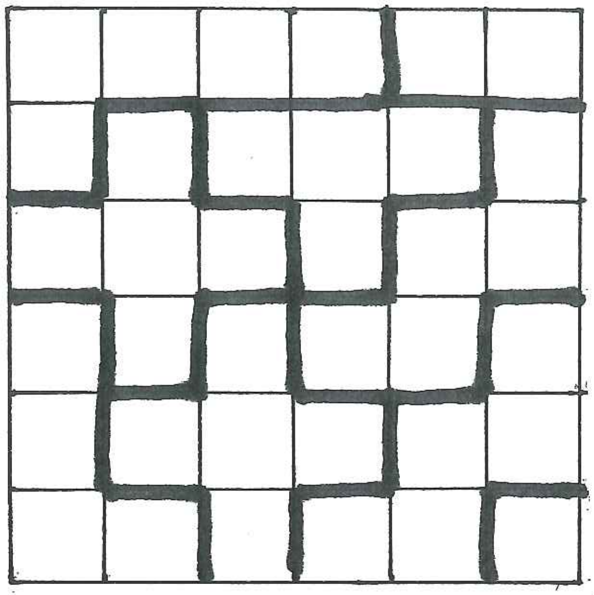
Background rules (axioms)

$$\rho = 1 \wedge \rho = 2 \rightarrow \perp \quad \text{etc etc}$$

and for ρ, σ adjacent

$$\rho = 1 \wedge \sigma = 1 \rightarrow \perp$$

SUGURU 2



Region requirements (axioms)

E.g. suppose region with squares ρ, σ, τ (so size three) then

$$\rho = 1 \vee \rho = 2 \vee \rho = 3 \quad \text{etc}$$

and

$$\rho = 1 \wedge \sigma = 1 \rightarrow \perp \quad \text{etc.}$$

Initial markings (axioms)

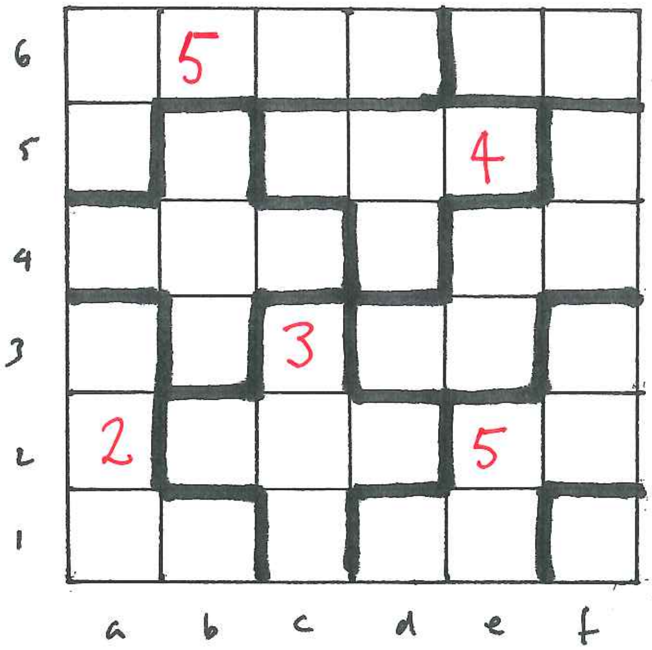
$$\rho = 1$$

$$\sigma = 2 \quad \text{etc}$$

(for some specific choice of ρ, σ)

AIM Determine truth value of all propositions.

EXAMPLE



SQUARES OF SPECIAL INTEREST

f1

b2

c3

REGIONAL DEDUCTION

Region of two squares ρ, σ

We have

$$\rho=1 \vee \rho=2 \qquad\qquad \sigma=1 \vee \sigma=2$$

$$\rho=1 \wedge \sigma=1 \to \bot \qquad\qquad \rho=2 \wedge \sigma=2 \to \bot$$

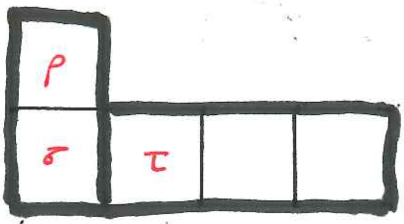
Intuitively clear that e.g.

$$\rho=1 \vee \sigma=1$$

PROOF

$$\dfrac{\rho=1\vee\rho=2 \quad \dfrac{\cancel{\rho=1}^{(2)}}{\rho=1\vee\sigma=1} \quad \dfrac{\dfrac{\sigma=1\vee\sigma=2 \quad \cancel{\sigma=1}^{(1)} \quad \dfrac{\dfrac{\rho=2\wedge\sigma=2\to\bot \quad \dfrac{\cancel{\rho=2}^{(2)} \quad \cancel{\sigma=2}^{(1)}}{\rho=2\wedge\sigma=2}}{\bot}}{\sigma=1}}{\sigma=1}\,(1)}{\rho=1\vee\sigma=1}}{\rho=1\vee\sigma=1}\,(2)$$

INTERACTION DEDUCTIONS



Intuitively clear that $\tau = 3$

For a proof first prove $\tau = 1 \to \perp$ i.e. $\neg(\tau = 1)$.

$$\dfrac{\dfrac{\rho=1\vee\sigma=1 \qquad \dfrac{\rho=1\wedge\tau=1\to\perp \quad \dfrac{\cancel{\rho=1}^{1}\quad\cancel{\tau=1}^{2}}{\rho=1\wedge\tau=1}}{\perp} \qquad \dfrac{\sigma=1\wedge\tau=1\to\perp\quad \dfrac{\cancel{\sigma=1}^{1}\quad\cancel{\tau=1}^{2}}{\sigma=1\wedge\tau=1}}{\perp}}{\perp}\;(1)}{\tau=1\to\perp}\;(2)$$

Similarly we get
$$\tau = 2 \to \perp .$$

Now

$$\dfrac{\dfrac{\cancel{\tau=1\vee\tau=2}^{2} \qquad \dfrac{\cancel{\tau=1}^{1}\quad \tau=1\to\perp}{\perp} \qquad \dfrac{\cancel{\tau=2}^{1}\quad \tau=2\to\perp}{\perp}}{\perp}\;(1)}{\tau=1\vee\tau=2\to\perp}\;(2)$$

FINAL STEP

$$\dfrac{(\tau=1 \lor \tau=2) \lor \tau=3 \qquad \dfrac{\dfrac{\tau=1 \lor \tau=2 \to \bot \qquad [\tau=1 \lor \tau=2]^{1}}{\bot}}{\tau=3} \qquad [\tau \neq 3]^{1}}{\tau=3}\ (1)$$

I ask you to conclude that all the arguments in SUGURU can be obtained by the rules of inference of the propositional calculus.

COMPLEXITY (ASIDE)

Can we solve SUGURU by trying all possibilities?

How many squares?

36

How many propositions?

180

How many possibilities for truth values?

$$2^{180} \doteqdot 10^{54}$$

EXAMPLE CTD

	5				
				4	
3			2		2
		3	1	4	1
2	5	4	2	5	2
		1	3	4	1

WHERE NEXT?

COMPLETENESS THEOREM (CRUDE VERSION)

For any propositional formula A

EITHER there is a proof of A (i.e. $\vdash A$)

OR there are values of the primitive propositions making A false.

EXERCISE Investigate the following:

$$(p \to q) \to (q \to p)$$

Pierce $((p \to q) \to p) \to p$

Kripke $((((p \to q) \to p) \to p) \to q) \to q$

PREDICATE CALCULUS 1

Propositions with more
complex structure

About INDIVIDUALS so in the language

NAMES Trump, Merkel, Macron, Putin

Argentina, Tanzania, France, USA

Everest, Mt Blanc, Kilimanjaro

FUNCTIONS (SYMBOLS)

The father of —

The mother of —

The largest mountain in —

The nation of —

The favourite horse of —

ITERATION The mother of the father of —

The sire of the favourite horse of —

VARIABLES — or $x, y, z \ldots$

PREDICATE CALCULUS 2

Symbols for PREDICATES

Syllogisms ~ unary predicates	— is mortal
	— is an animal
	— is a man
Binary predicates	x is a child of y
	x is taller than y
Ternary predicates	x lies between y and z on the railway
Quaternary predicates	A, B, C, D are the vertices of a square
Equality or Identity	=
	is

PREDICATE CALCULUS 3

QUANTIFIERS

Everything	Everybody	$\forall x\,(\quad x\quad)$
		forall x (x)
Something	Somebody	$\exists x\,(\quad x\quad)$
		there exists x (x)
Nothing	Nobody	$\forall x\,\neg(\quad x\quad)$
		forall x not (x).

Syllogism forms

All A are B	$\forall x\,(A(x) \longrightarrow B(x))$
No A is B	$\forall x\,(A(x) \wedge B(x) \longrightarrow \perp)$
Some A are B	$\exists x\,(A(x) \wedge B(x))$
Some A are not B	$\exists x\,(A(x) \wedge \neg B(x))$.

EXERCISES 1

Traditional $L(x,y)$ for x loves y

$\forall x\, \exists y\, L(x,y)$ ~ everybody loves somebody

What do

$\exists y\, \forall x\, L(x,y)$

$\forall y\, \exists x\, L(x,y)$

$\exists x\, \forall y\, L(x,y)$ mean?

Drinker's Formula

The context is a bar in which individuals may be drinking: $D(x)$ for x is drinking

$$\exists x\, (D(x) \longrightarrow \forall y\, D(y))$$

Is this valid ?!

EXERCISES 2

Consider

$$\forall x.\ \ Human(x) \longrightarrow \exists y.\ \ x \text{ child of } y$$

$$\forall x\, \forall y.\ \ x \text{ child } y \wedge Human(x) \longrightarrow Human(y).$$

What is the issue?

Formulate in the predicate calculus:

Every year somebody comes top in all the exams.

Can you formulate in the predicate calculus

Some critics admire only one another.

COMPLETENESS THEOREM (CRUDE VERSION)

(There are rules of inference for the predicate calculus giving a notion of $\Gamma \vdash \phi$, there is a proof of ϕ from assumptions Γ.)

For any formula of the predicate calculus

$$\phi(x_1, \ldots, x_n)$$

EITHER there is a proof of ϕ

OR there is a structure (and interpretations $a_1, \ldots, a_n$ of $x_1, \ldots, x_n$ in it) making

$$\phi(a_1, \ldots, a_n) \quad \text{false.}$$

WARNING Truth tables give us a way to decide whether or not a proposition A is provable in the propositional calculus.

There is no such possibility for the predicate calculus.

PARADOXES

OF SELF-REFERENCE

EPIMENIDES (A Cretan)

All Cretans are liars.

What follows?

ABSOLUTE VERSION

This sentence is false.

True or false?

VARIATION

This sentence is true.

True or false

FABLE

A King issues the proclamation

All who come to the city must state their business:
if they speak falsely they will be hanged;
if they speak truely they will be admitted.

So what if ... .

PARADOX / THEOREM

RUSSELL'S PARADOX

Let $R = \{x \mid x \notin x\}$.

Is $R \in R$ or is $R \notin R$?

What about $\{x \mid x \in x\}$?

CANTOR'S THEOREM

There is no surjective map

$$f : X \longrightarrow P(X) = \{A \mid A \subseteq X\}.$$

Proof:- Consider $\{x \mid x \notin f(x)\}$.

If we have $f : X \to P(X)$ then can

$$\{x \mid x \in f(x)\}$$

be the value $f(a)$ for some $a \in X$?

THE UNPREDICTABLE EXAM

Teacher — There will be an unpredicable exam next week.

Students
- No exam Monday – Thursday
  $\longrightarrow$ Exam Friday predictable $\rightarrow \perp$

  So exam Monday – Thursday
- Now no exam Monday – Wednesday
  $\longrightarrow$ Exam Thursday predictable $\rightarrow \perp$

  So exam Monday – Wednesday
- Now no exam Monday, Tuesday
  $\longrightarrow$ Exam Wednesday predictable $\rightarrow \perp$

  So exam Monday, Tuesday
- Now no exam Monday
  $\longrightarrow$ Exam Tuesday predictable $\rightarrow \perp$

  So exam Monday.
- Now exam Monday predictable $\rightarrow \perp$

THUS Exam cannot take place.

?

MODAL LOGIC
(OF PREDICTABILITY)

If A is a proposition write $\square A$ for the proposition A is predictable.

Primitive propositions

MON, TUES, WED, THURS, FRI.

TEACHER

$$(\mathrm{MON} \wedge \neg \square \mathrm{MON}) \vee (\mathrm{TUES} \wedge \neg \square \mathrm{TUES}) \vee (\mathrm{WED} \wedge \neg \square \mathrm{WED})$$
$$\vee (\mathrm{THURS} \wedge \neg \square \mathrm{THURS}) \vee (\mathrm{FRI} \wedge \neg \square \mathrm{FRI})$$

STUDENTS deduce from that and

$$\neg \mathrm{MON} \wedge \neg \mathrm{TUES} \wedge \neg \mathrm{WED} \wedge \neg \mathrm{THURS}$$

the conclusion

$$\square \mathrm{FRI}$$

Are they right to do so?

Why?

What is correct here?

LANGUAGE TALKING LANGUAGE

Compare

Man has two legs

Man has three letters

Quine's suggestion

"Man" has three letters

We shall consider formal languages in which we can talk about the syntax of the language. Use $\# \phi$ for the quotation of a formula ϕ within the language.

SOME LOGIC OF LOGIC

Distinguish

$$A \vdash B$$

the statement that A entails B (in some system of logic)

from

$$A \to B$$

the proposition within the logic (which may or may not be provable)

Confusion arises because of the special status of $A \to B$: it "internalises" the "external" statement $A \vdash B$ in the sense that

$$\Gamma \vdash A \to B$$

exactly when

$$\Gamma, A \vdash B$$

Prove this!

It is the Deduction Theorem.

WHAT THE TORTOISE SAID TO ACHILLES

Let $A = B \to Z$. Then $A, B \vdash Z$ is an instance

$$B \to Z, B \vdash Z$$

of rule of inference Modus Ponens.

$A \wedge B \to Z$ is another sentence in the language (and in fact $\vdash A \wedge B \to Z$).

Then $A, B, A \wedge B \to Z \vdash Z$ is a correct entailment – in which the structure of A is no longer relevant.

Then let $C = A \wedge B \to Z$ and continue ....

IS THERE A MORAL HERE?

INTERNAL SYNTAX

We take a logical system T in which we can describe the syntax of the system.

So

if ϕ is a formula it has a description $\#\phi$ in T;

if π is a proof it has a description $\#\pi$ in T;

there is a formula $Prf(x,y)$ which says

"x is a proof of y"

in the following sense.

For all formulae ϕ and proofs π

$$T \overset{\pi}{\vdash} \phi$$

if and only if

$$T \vdash Prf(\#\pi, \#\phi)$$

PROVABILITY

From the proof formula $Prf(x,y)$ we get by existential quantification a formula

$$Pble(y) = \exists x.\, Prf(x,y)$$

such that

(1) $T \vdash \phi$ implies $T \vdash Pble(\#\phi)$

Moreover

assuming T behaves well (ω-consistent)

we get

$T \vdash Pble(\#\phi)$ implies $T \vdash \phi$

Without any assumptions we have also

(2) $T \vdash Pble(\#(\phi \to \psi)) \to (Pble(\#\phi) \to Pble(\#\psi))$

i.e. T proves the rule of Modus Ponens

and

(3) $T \vdash Pble(\#\phi) \to Pble(\#\, Pble(\#\phi))$

i.e. T proves the rule (1) above.

FIXED POINTS

If $\psi(y)$ is a formula – think of it as talking about the syntax of T – then there is a sentence ϕ with

$$T \vdash \phi \leftrightarrow \psi(\#\phi)$$

so ϕ says "I satisfy ψ".

The proof is by internalising substitution: guess ϕ is of the form $\chi(\#\chi)$ where χ is $\psi(sub(x,x))$. Here we require the defined function $sub(x,y)$ to have the property

$$T \vdash sub(\#\alpha, \#\beta) = \#\alpha(\#\beta).$$

Then ϕ is $\chi(\#\chi)$ so is $\psi(sub(\#\chi,\#\chi))$.

Thus

$$T \vdash \phi \leftrightarrow \psi(\#\chi(\#\chi)) = \psi(\#\phi).$$

PROVABILITY LOGIC

The logic of propositions of T now written p, q, r ; A, B, C ; with propositional connectives $\to, \bot,$ and with the provability operator now written $\Box A$.

We take the propositional calculus (i.e. system of proofs already given) plus

Rule of inference:

$$\text{if } \vdash A \text{ then } \vdash \Box A$$

and axioms

$$\vdash \Box(A \to B) \to (\Box A \to \Box B)$$

$$\vdash \Box A \to \Box\Box A$$

plus maybe a missing axiom!

ASIDE

More general
Rule of inference:

$$\text{if } \Box\Gamma \vdash A \text{ then } \Box\Gamma \vdash \Box A$$

Prove that plus $\vdash \Box(A \to B) \to (\Box A \to \Box B)$ equivalent to the above system.

GÖDEL'S FIXED POINT

There is a proposition G with the property

$$\vdash G \longleftrightarrow \neg \Box G$$

"I am not provable."

Suppose that $\vdash G$: what follows?

Suppose that $\vdash \neg G$: what follows?

(What do you need to assume about our original T?)

Is G true or false?

FIRST INCOMPLETENESS THEOREM

Assume T is well-behaved. Then there is a sentence G such that

$$T \nvdash G \text{ and } T \nvdash \neg G.$$

SECOND INCOMPLETENESS THEOREM

Assume T is consistent (i.e. $T \nvdash \bot$).

Then $T \nvdash \neg \mathrm{Pble}(\bot)$ i.e. T does not prove that T is consistent.

In fact

$$G \longleftrightarrow \neg \Box G \vdash G \longleftrightarrow \neg \Box \bot$$

so the second theorem is an internalisation of one direction of the first.

HENKIN'S FIXED POINT

There is a proposition H with the property

$$\vdash H \longleftrightarrow \Box H$$

"I am provable"

Do we have $\vdash H$ or not?

HINT: By the general fixed point theorem one can assume

either there is an L with

$$\vdash L \longleftrightarrow (\Box L \to H)$$

or there is an M with

$$\vdash M \longleftrightarrow \Box(M \to H).$$

LÖB'S THEOREM

Suppose $\vdash \Box C \to C$: then $\vdash C$.

Moreover this can be proved so

$$\Box(\Box C \to C) \vdash \Box C .$$

Immediate special case

$\vdash H$

and so

H is true

Further axiom

$$\vdash \Box(\Box A \to A) \to \Box A .$$

N.B. Setting $A = \bot$ gives what?

EXERCISE

Suppose the proposition U (undecidable) is such that

$$\vdash U \leftrightarrow (\neg \Box U \wedge \neg \Box \neg U)$$

What can you say about U ?

Suppose the proposition D (decidable) is such that

$$\vdash D \leftrightarrow (\Box D \vee \Box \neg D)$$

What can you say about D ?