CONSTRUCTING CATEGORIES OF ABSTRACT GAMES

PLAN

Motivation
- Hints towards constructive content of classical proof
- Informal description of abstract games

Mathematical background
- Traced monoidal categories & compact closed categories.
- Glueing for $*$-autonomous categories. (1st examples)
- Orthogonality and totality
- Unresolved difficulties

HINTIKKA GAMES

(Games & classical truth)

M a structure for predicate calculus signature.

ϕ a sentence (in $\forall, \exists, \wedge, \vee$ of basics).

To determine $M \models \phi$ play a game:

Opponent moves

- $\phi \equiv \forall x\, \psi(x)$ O chooses $m \in M$ and consider $\psi(m)$
- $\phi \equiv \psi \wedge \chi$ O chooses one of ψ, χ and consider it.

Player moves

- $\phi \equiv \exists x\, \psi(x)$ P chooses $m \in M$ & consider $\psi(m)$
- $\phi \equiv \psi \vee \chi$ P chooses one of ψ, χ and consider it.

HINTIKKA GAMES CTD.

Player wins if we end at a true basic formula.

$M \models \phi$ iff P has a winning strategy

PROBLEMS

- Truth not proof so highly non-constructive.

  (NB. Herbrand's Theorem)

- No compositional theory of algorithms.

EXAMPLE

(Intuition of Coquand)

Take $f : \mathbb{N} \longrightarrow \mathbb{N}$: consider

Π^0_3 $\quad \forall n\ \exists m \geq n\ \forall k \geq m\ f(k) \geq f(m)$

A natural proof (eg in Novikov Calculus) suggests a procedure

O chooses n

P chooses $m_0 = n$

O chooses $k_0 > m_0$ s.t. $f(k_0) < f(m_0)$

P chooses $m_1 = k_0$

O chooses $k_1 > m_1$ s.t. $f(k_1) < f(m_1)$

etc

HOW TO MAKE THIS PRECISE?

(In a uniform fashion!)

LESSON

- Procedure for $\forall n\, \exists m \geq n\, \forall k \geq m\; f(k) \geq f(m)$ has no familiar type!
- In particular an algorithm associated with a classical proof of ϕ may not 'do what ϕ says':
  (in contrast with the intuitionistic case)
- Procedure is some kind of interactive process.

AIM Extract interesting interactive procedures from classical proofs.

ANALYSIS OF CLASSICAL PROOF

(one possible approach)

Traditional classical system

LK+

$\Downarrow$

LU+

- via a choice of polarities, so **SYSTEMATIC AMBIGUITY**
- we do not have any understanding of this!

$\Downarrow$

CLL+

$\rightsquigarrow$

Computational model of CLL+

<u>NOTE</u> Cut free proofs get computational content "directly". Relation with 'cutting with input data'?

COMPUTATIONAL MODELS

- Games models
  (Interaction of strategies)
- Geometry of interaction
  (Execution formula)
- Abstract games (extended G of I)
  (Interaction)

? Concrete abstract games
  (Interactive data types)

? Bicompletions (JOYAL)
  (? ? ?)

WHY ABSTRACT GAMES?

- Are games so great?
  - Generally do not have a good computational grip on strategies. (Languages of strategies.)
  - Cheap / poor models for CLL
- More general paradigms of interaction?
  - Time irrelevance?
  - True concurrency?
- What are abstract properties of categories of games?

WHAT ARE ABSTRACT GAMES?

Typically given by domains of strategies for P (Player) and O (Opponent) and information on how they interact.

Extended G of I formulation

Have

U domain of data P can give

X domain of data O can give

P-strategy is $X \xrightarrow{r} U$

O-strategy is $U \xrightarrow[R]{} X$

Then $r \wr R : UX \longrightarrow UX$ and this endomap on the data space iterates to give the interaction.

TRACED MONOIDAL CATS.

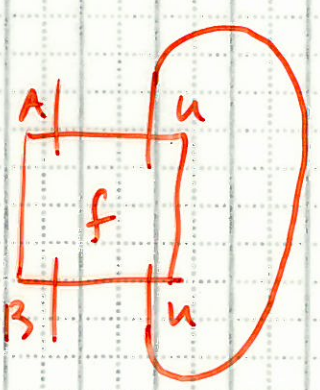
$\mathbb{C}$, I, $\otimes$, etc symmetric monoidal category

with operation

$$\frac{A \otimes U \xrightarrow{f} B \otimes U}{A \xrightarrow[tr_U(f)]{} B}$$

with natural properties!

Feedback loop



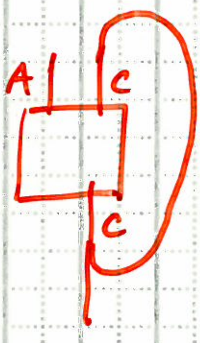
$= tr_U(f)$

Idea:- Maps in such are our basic computational algorithms.

TRACED PRODUCT CATY
= CATY WITH FIXED POINTS

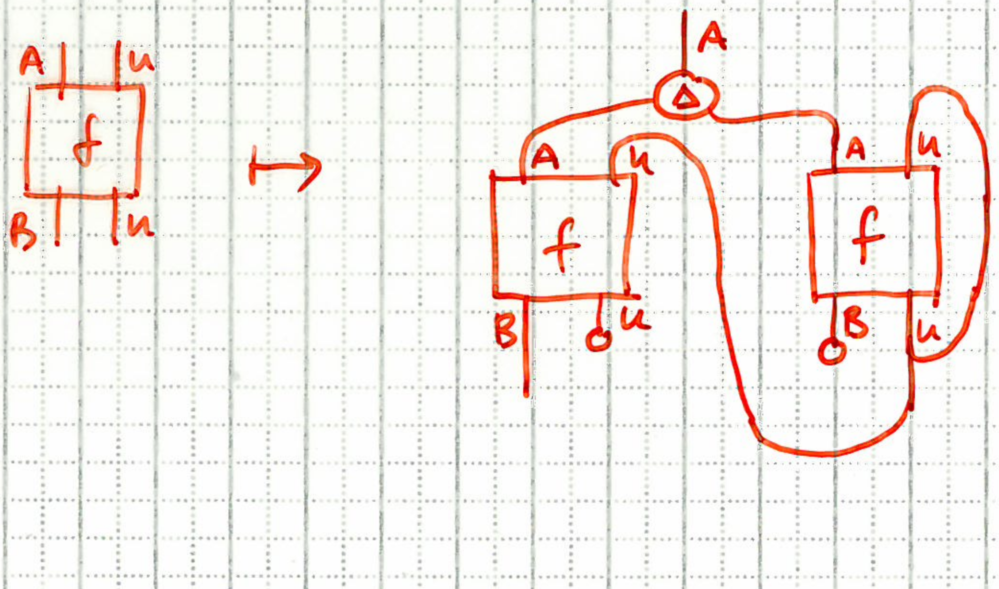
FIXED POINTS

$$\frac{A \times C \xrightarrow{f} C}{A \xrightarrow[\mathrm{fix}_C(f)]{} C}$$



Trace $\rightsquigarrow$ Fixed points

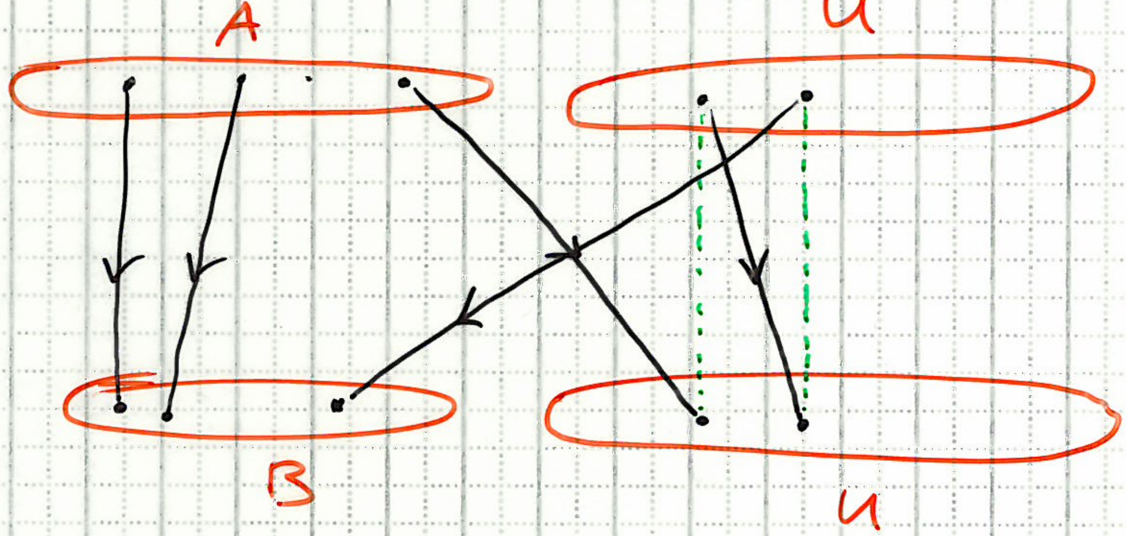
Fixed points $\rightsquigarrow$ Trace



EXAMPLE

Sets and partial functions, with + as the tensor product.

TRACE



[Similarly eg sets and finite automata.]

(N.B. These opposites of categories with ×.)

COMPACT CLOSED CATEGORY

is $\mathbb{C}$, I, $\otimes$, etc symmetric monoidal

equipped with duals (adjoints)

i.e. $U \rightsquigarrow U^*$

with $I \xrightarrow{\eta} U^* \otimes U$ unit

$U \otimes U^* \xrightarrow{\varepsilon} I$ counit

satisfying Δ identities (aliter

eg "all that is true in fin. dim. vector spaces").

Associated trace

$$\frac{A \otimes U \xrightarrow{f} B \otimes U}{A = A \otimes I \xrightarrow[1 \otimes \eta]{} A \otimes U^* \otimes U \cong A \otimes U \otimes U^* \xrightarrow[f \otimes 1]{} B \otimes U \otimes U^* \xrightarrow[1 \otimes \varepsilon]{} B \otimes I = B}$$

TRACED ⟶ COMPACT CLOSED

From $\mathbb{C}$ traced monoidal, form $\tilde{\mathbb{C}}$:-

<u>objects</u> pairs (U, X) $U, X \in \mathbb{C}$

(intuitively $(U,X) \equiv U \otimes X^*$)

<u>maps $(U,X) \longrightarrow (V,Y)$</u>

maps $U \otimes Y \longrightarrow V \otimes X$ in $\mathbb{C}$

(for in compact closed

$$\frac{U \otimes X^* \longrightarrow V \otimes Y^*}{U \otimes Y \longrightarrow V \otimes X}$$
)

<u>composition</u> by trace = feedback

FACT $\tilde{\mathbb{C}}$ is compact closed

(freely generated by traced $\mathbb{C}$)

EXAMPLE

Compact closed category generated by sets and partial functions

objects (U, X)

output input sets

maps

$$\begin{tikzcd} U \arrow[r, bend left] \arrow[d] & X \\ V & Y \arrow[u] \arrow[l, bend right] \end{tikzcd}$$

partial functions

$U + Y \longrightarrow V + X$

composition

$$\begin{tikzcd} U \arrow[r, bend left] \arrow[d] & X \\ V \arrow[r, bend left] \arrow[d] & Y \arrow[u] \arrow[l, bend right] \\ W & Z \arrow[u] \arrow[l, bend right] \end{tikzcd}$$

iterate the feedback

OBSERVATION

"Proofs" of (U,X) in $\tilde{\mathbb{C}}$

$$\begin{array}{ll} \tilde{\mathbb{C}} & (I,I) \longrightarrow (U,X) \\ \hline \mathbb{C} & X \longrightarrow U \end{array}$$

"Refutations" of (U,X) in $\tilde{\mathbb{C}}$
($\equiv$ proofs of $(X,U) = (U,X)^*$)

$$\begin{array}{ll} \tilde{\mathbb{C}} & (I,I) \longrightarrow (X,U) \\ \hline \mathbb{C} & U \longrightarrow X \end{array}$$

So an abstract game consists of (U,X) plus some collection of

$\begin{cases} \text{proofs \& of refutations} \\ \text{strategies \& counterstrategies} \end{cases}$

GLUEING

Given $L : \mathbb{E} \longrightarrow \mathbb{F}$ we "glue" to form the arrow category

objects / maps

$$\begin{tikzcd}
(E \in \mathbb{E}, \arrow[d, "f"] & F \in \mathbb{F}, \arrow[d, "g"] & F \arrow[r, "u"] \arrow[d, "g"'] & L(E) \in \mathbb{F}) \arrow[d, "L(f)"] \\
(E' & F' & F' \arrow[r, "u'"'] & L(E'))
\end{tikzcd}$$

FREYD — Syntactic proof-theoretic results derived by glueing free cats along suitable functors.

E.g. Existence property for intuitionistic logic $\equiv$ 1 projective in free caty $\mathbb{E}$

Glue along $\Gamma : \mathbb{E} \longrightarrow \mathbf{Sets}$

E.g. (cf Lafont's thesis)

Higher order functions conservatively extend 1st order data types

GLUEING

(for $*$-autonomous cats / cats with duality)

Data $\mathbb{C}$ $*$-autonomous $\quad ()^{\perp} : \mathbb{C}^{op} \to \mathbb{C}$.

$\mathcal{V}$ smc, complete cocomplete

$\Phi : \mathbb{C} \longrightarrow \mathcal{V}$ monoidal

$(\Phi^{\perp} = \Phi \circ ()^{\perp} : \mathbb{C}^{op} \longrightarrow \mathcal{V}.)$

Output $\mathcal{G}$ $*$-autonomous

$F : \mathcal{G} \longrightarrow \mathbb{C}$ $*$-autonomous

$G_0 : \mathcal{G} \longrightarrow \mathcal{V} \qquad G_1 : \mathcal{G}^{op} \longrightarrow \mathcal{V}$

$$\begin{tikzcd} & \mathcal{G} \arrow[dl, "F"'] \arrow[dr, "G_0"] \arrow[d, phantom, "\swarrow \alpha_0"] & \\ \mathbb{C} \arrow[rr, "\Phi"'] & {} & \mathcal{V} \end{tikzcd} \quad \& \quad \begin{tikzcd} & \mathcal{G}^{op} \arrow[dl, "F^{op}"'] \arrow[dr, "G_1"] \arrow[d, phantom, "\swarrow \alpha_1"] & \\ \mathbb{C}^{op} \arrow[rr, "\Phi^{\perp}"'] & {} & \mathcal{V} \end{tikzcd}$$

with suitable universal property.

(For simplicity consider only case α_i mono.)

APPLICATION

Take $\widetilde{\mathbb{C}}$ associated with traced monoidal $\mathbb{C}$

glue along

$$\Gamma : \widetilde{\mathbb{C}} \longrightarrow \mathrm{Sets} = \widetilde{\mathbb{C}}(I, -) : \widetilde{\mathbb{C}} \longrightarrow \mathrm{Sets}$$

Then $\mathcal{G}$ has objects

$$(U, X \,;\, A_0 \subseteq \mathbb{C}(X, U) \,,\, A_1 \subseteq \mathbb{C}(U, X))$$

data sets   P-strategies   O-strategies

So is a $*$-autonomous category of (very?) abstract games.

ADDITIONAL PROPERTIES

- If compact closed $\tilde{\mathcal{C}}$ has (weak) products and coproducts then so has $\mathcal{G}$.

(No computationally compelling examples with strong products. Why?)

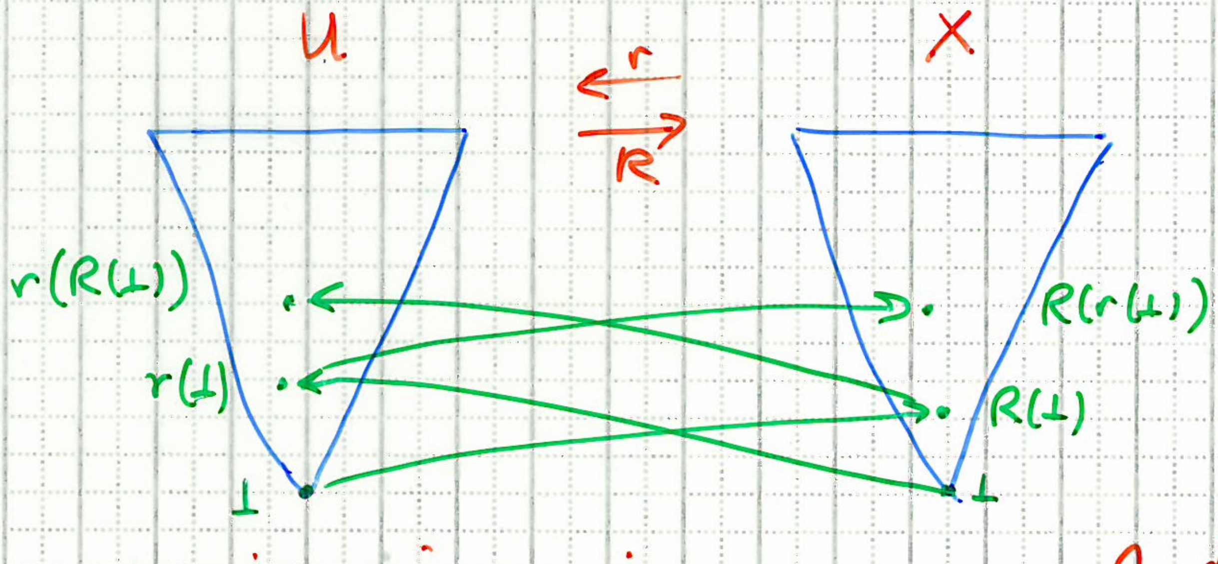
- If compact closed $\tilde{\mathcal{G}}$ has an exponential then so has $\mathcal{G}$.

(Plenty such cases.)

ABSTRACT GAMES / DOMAINS
(NOT classical domain theory)

Take $\mathbb{C}$ (simple) category of domains closed under $\times$ (& ....).
$\mathbb{C}$ has fixed points so trace.
So abstract category of games .....

Given an object $(U, X; A_0, A_1)$;
a P-strategy $r : X \longrightarrow U$
& O-strategy $R : U \longrightarrow X$
interact in time thus:-



producing increasing sequence of data.

NATURAL TO CONSTRAIN THESE.