

# CONSTRUCTING HYPERDOCTRINES

Towards a Model Theory  
of Type Theory

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CT 2014

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# GÖDEL'S DIALECTICA INTERPRETATION

Gödel 1942 : Princeton Lectures

Proof theory:

functional interpretation

Consistency of number theory  
analysis

Proof mining (Kohlenbach)

# SIGMA - PI

Focus: quantifier combination

$$\exists_{u \in U} \forall_{x \in X} \square$$

so for type theory

$$\sum_{u \in U} \prod_{x \in X} \square$$

i.e. SUM - PRODUCT

# DIALECTICA CATEGORIES

1980s (de Paiva's thesis: Rubin from  
Rosolini)

Abstract view  $\eta \rightarrow$

$$U \xleftrightarrow{A} X$$

$$\downarrow \swarrow$$

$$V \leftrightarrow Y$$

$$f: U \Rightarrow V$$

$$F: U \times Y \Rightarrow X$$

$$\phi: \prod_{u,y} A(u, F(u, y))$$

$$\Rightarrow B(f(u), y)$$

# VARIATIONS

1980s Birkhäuser-Nahm (1974) and  
Linear logic

2000s Copenhagen (Biering, Rikkeldt,  
H. Roskam, Schneider, van Oorten)

2010s Exploration of interest

In CS

Plotkin et al The Computer  
Firewall

Flor, Oliver, Powell,  
Robinson

(And now French students -)

# LOGIC TO DEPENDENT TYPES

Is there a Dialectica-style  
interpretation of Dependent  
Type Theory?

YES!

Moreover Category Theory  
is key to the understanding.

# AIM OF TALK

Explains

Given a categorical model  $\mathbb{E}$   
of type theory There is a model  
 $\Sigma\Gamma(\mathbb{E})$ , a variant of Dialectica.

Method

From  $\mathbb{E}$  construct a new model

$\text{Poly}(\mathbb{E})$  and  $\Sigma\Gamma(\mathbb{E}) \doteq \text{Poly}^2(\mathbb{E})$ .

# PLAN OF TALK

- Overview of categorical models
- Sketch of the polynomial model (von Glehn)
- Concrete details in the baby example.
- Remarks on Dialectica and beyond.

# CATEGORICAL SETTING

(but see Awodey's talk)

Category  $\mathcal{C}$  with collection of  
exceptional maps  $\mathbb{E}$  (display maps,  
fibrations ... ) with

$\mathbb{E}$  cartesian  
(and with all maps  $C \rightarrow 1$  in  $\mathbb{E}$ )

So a fibration  $\mathbb{E} \rightarrow \mathcal{C}$  of restricted  
kind. (Coherence suppressed.)

# SUMS

$\text{IE} \rightarrow \mathbb{C}$  has strong sums in  
the sense that  $\text{IE}$  is closed  
under composition

So left adjoints to pb along  
maps in  $\text{IE}$ . (Berk-Chenney  
axiomatis.)

Joyal's notion of trike  
(Already theory at this level.)

# PRODUCTS

$\mathbb{E} \rightarrow \mathbb{C}$  has products in the  
sense that we have right  
adjoints to the ps of  $\mathbb{E}$  maps  
along  $\mathbb{E}$  maps.

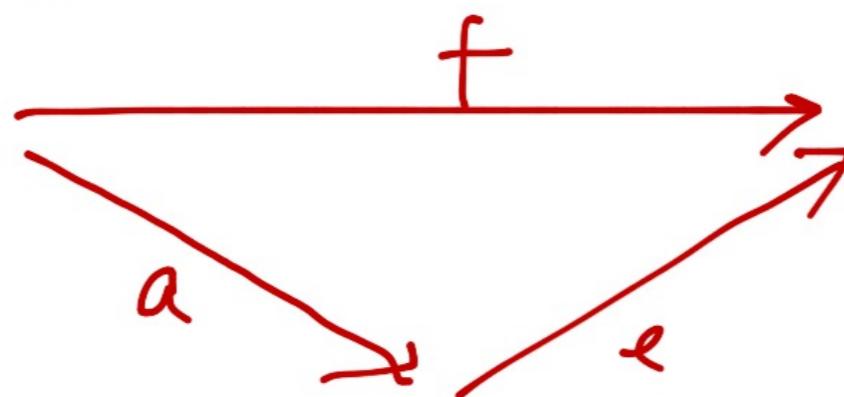
(Beck - Chevalley inherited  
from sums.)

Toyal's notion of  $\pi$ -triple

# FACTORIZATION

(for Identity Types & whatever)

We have a functional factorization  
of maps  $\xrightarrow{f}$  in  $\mathbb{C}$  as



with a moddyne i.e. has  $\mathrm{Itp}$   
with respect to  $\mathbb{E}$  maps; and  
moreover  $\mathbb{E}$  pbs of moddyne are  
anodyne  
~ Joyal's notion of h-trise.

# COPRODUCTS

Special property amounting  
to strong rules for (some)  
finite inductive types.

$E \rightarrow C$  has coproducts in  
each fibre, stable under  $\mathsf{pb}$ .

(Possible weakenings of  
this but not for now.)

# MAXIMAL CASE

$$\mathbb{E} = \mathbb{C}$$

Fatmization is trivial

Result is a locally cartesian closed category with exponentials

[This goes back to Seely.]

# MINIMAL CASE

$\mathbb{E}$  is the collection of product projections.

The factorization is

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow (\text{id}_A, f) & \nearrow \text{and} \\ & A \times B & \end{array}$$

The result is a cartesian closed category with coproducts. (Folklore.)

# MODEL THEORY OF TYPE THEORY

~ Relative Interpretation

Cf 40+ years experience with  
Topos Theory

- Construction of new models  
from old
- Study of the properties of  
models

# POLYNOMIALS / CONTAINERS

(In the fibers over  $\mathbb{I}$ )

objects

$$U \leftarrow X$$

maps

$$\begin{array}{ccccc} & V & \leftarrow & Y & \xleftarrow{F} f^* X \\ & f \downarrow & & & \downarrow \\ U & \leftarrow & & & X \end{array}$$

Huge literature: Gambino - Kock  
Kock<sup>2</sup>

Abawajy, Altenkirch, Ghani

## FUNDAMENTAL OBSERVATION

Altenkirch, Levy, Staton : Over Sets

The category of polynomials is  
cartesian closed.

Hypernet The proof can be  
written in type theory so as  
to apply to any weakly cartesian  
closed category with coproducts.

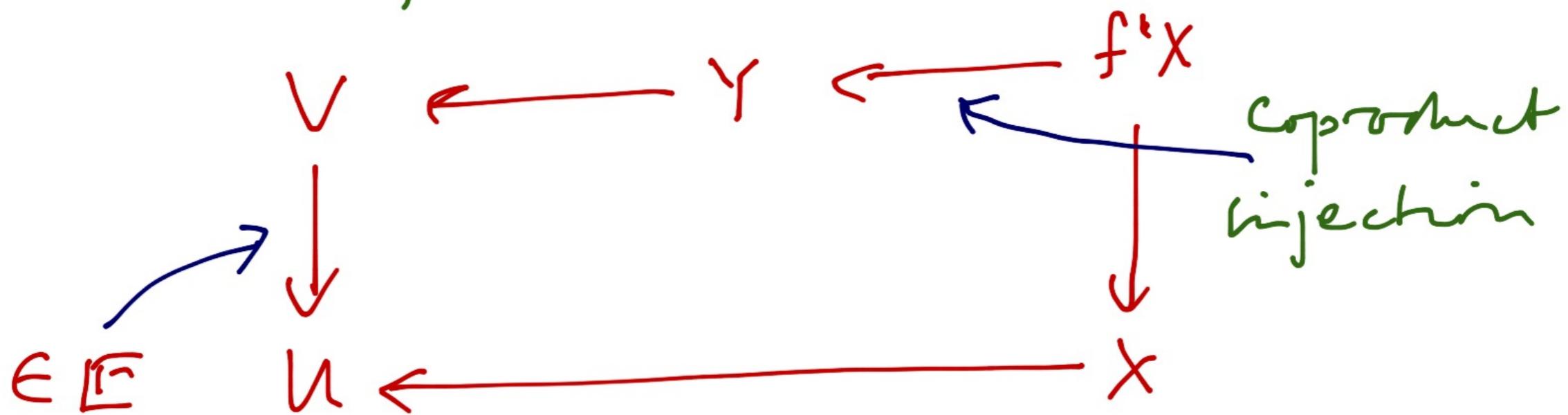
# MAIN THEOREM

(von Glehn) For any model  $\mathbb{E} \rightarrow \mathbb{C}$  of type theory, there is a model

$$\text{Poly}(\mathbb{E}) = \text{Poly}(\mathbb{E} \rightarrow \mathbb{C})$$

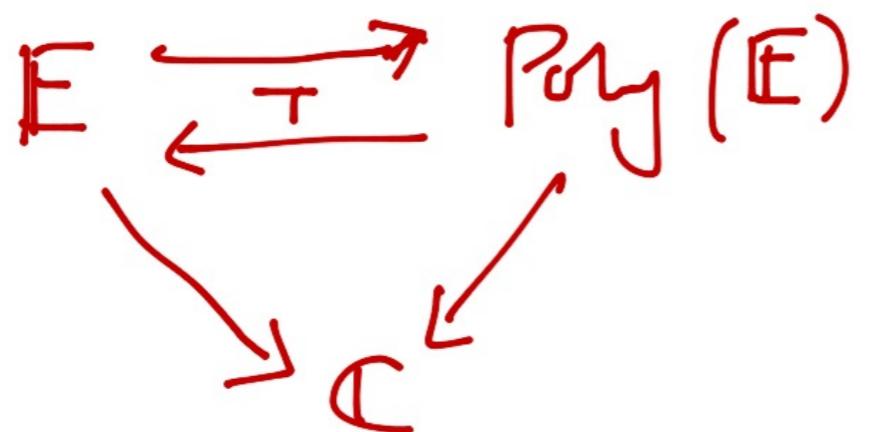
with underlying category poly nomials

$U \leftarrow X$  from  $\mathbb{E}$  and with displays

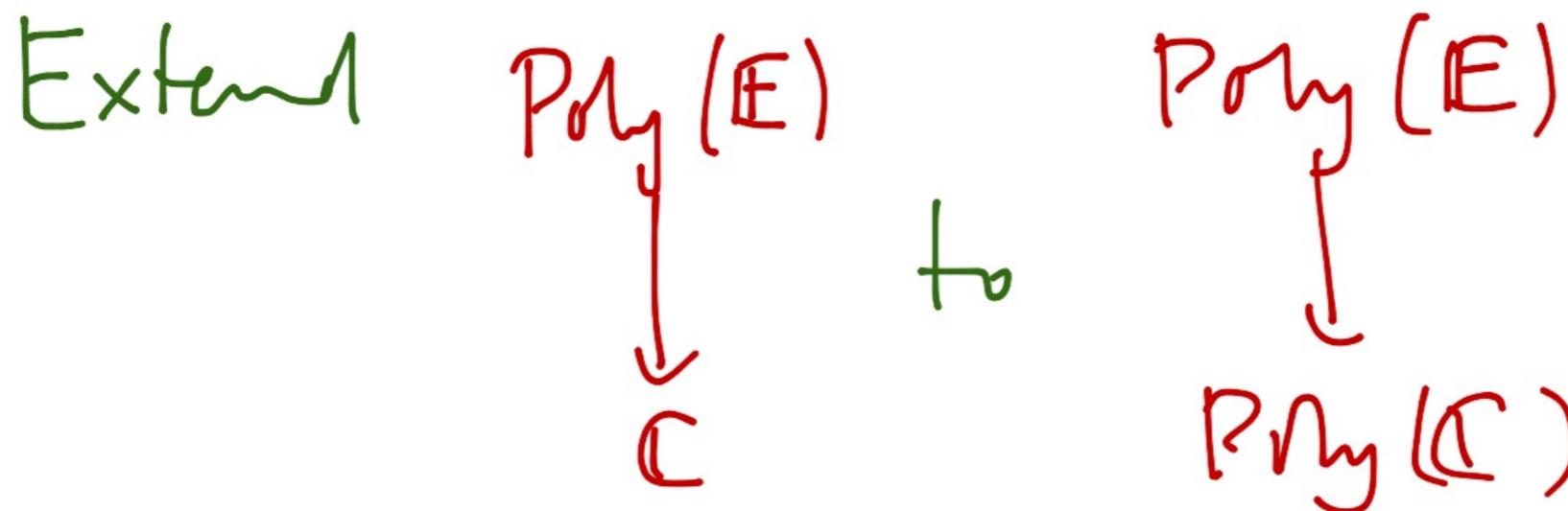


# OUTLINE

By A-L-S



Day situation  
for the cases,  
and with  $\Sigma_s$ ,  $\Pi_s$



defining factorization and then  
extending  $\Sigma$  and  $\Pi$

# FUNCTIONAL EXTENSIONALITY

Characterizes the identity on  
function spaces

$$\prod_{a \in A} Id_{B(a)}(f(a), g(a))$$

$$\rightarrow Id_{\prod_{a \in A} B(a)}(f, g)$$

Joyal:  $\prod$  preserves the homotopy  
relation

Univalence  $\Rightarrow$  Functional  
extensionality

# OBSERVATION

(von Glehn)

Functional extensivity fails  
in non-trivial polynomial models.

Conclusion

A model theory of type theory  
should NOT focus on  
univalent models.

# UNIVERSAL PROPERTY

$$\text{“} \operatorname{Poly}(E) = \sum (E^{\otimes p}) \text{”}$$



sums along  $E$  maps

(Folklore but see recent papers

Hofstra

Gambino - Koch

Kock - Kock

and more)

## EXAMPLE : FINITE SETS

What is

$$\text{Poly}(F) = \sum(F^{\text{op}})$$

↑  
finite sets

As a category it is finite sums  
of representables in the object  
classifier

$[F, \text{Set}]$ .

# THE GENERIC OBJECT

$$U = F(1, -) : k \longrightarrow k$$

$$U^m = F(m, -) : k \longmapsto k^m$$

$\Sigma_0$   $Pry(F)$  is the full subcategory  
on objects of the form

$$\sum_i F(n_i, -) = \sum_i U^{n_i}$$

# ATOMS (or TINY OBJECTS)

Objects  $A$  such that the right adjoint  $A \Rightarrow -$  to  $A^{\times -}$  has itself a right adjoint.

In  $[C^{\text{op}}, \text{Cat}]$  a representable  $\mathbb{C}(-, d)$

is an atom iff

$\mathbb{C}(-, d) \Rightarrow -$  preserves colimits  
if

for all  $c \in \mathbb{C}$  the product  $c \times d$   
is in the idempotent completion.

## CARTESIAN CLOSURE

Closure under finite products is evident.

Further spaces follows from

BASIC OBSERVATION (Lawvere)

$$U \Rightarrow U \cong 1 + U$$

thus atomicity.

# DISPLAYS / FIBRATIONS

Maps  $\sum_{j \in J} F(m_j, -)$  and for each  $j$   
 $p: J \rightarrow I$   $\downarrow$   $n_{p(j)} \xrightarrow{\phi_j} m_j$   
 $\sum_{i \in I} F(n_i, -)$

where each  $\phi_j$  is a coproduct injection.

i.e.  $\sum_{j \in J} U^{m_j}$   
 $p: J \rightarrow I$   $\downarrow$   $\pi_j: U^{m_j} \rightarrow U^{n_{p(j)}}$   
 $\sum_{i \in I} U^{n_i}$  a projection.

# FACTORIZATION

$$\sum_{j \in J} u^j \xrightarrow{(p, \phi)} \sum_{i \in I} u^i$$

$\downarrow$

$$\sum_{j \in J} u^j \times u^{p(j)}$$

is given by summing the minimal factorization.

## IDENTITY

For the atom  $U$  we have

$$\prod_{u,v \in U} Id_u(u, v)$$

is true.

So for general  $\sum_i U^i$  we have

$$\sum_i U^i \times U^i \longrightarrow \sum_{i,i'} U^i \times U^{i'}$$

# EXTENSIONALITY

For any  $A \in \text{Poly}(\mathbb{F})$  we must have

$\prod f, g : A \Rightarrow u . \prod a \in A . \text{Id}_u(f(u), g(a))$   
is true.

However

$$\text{Id}_{u \Rightarrow u} = \text{Id}_{1+u}$$

is not universally true.

Failure of function extensionality

# A DIALECTICA INTERPRETATION

Given  $E = (E \rightarrow C)$  a model for type theory we have  $\text{Poly}(E)$  a model for type theory: so iterate  $\text{Poly}^2(E)$ ?

Almost

$$\begin{aligned}\text{Poly}^2(E) &= \sum \left( \sum E^n \right)^P \\ &= \sum \prod E \quad \text{a Dialectica} \\ &\qquad\qquad\qquad \text{Interpretation}\end{aligned}$$

(Correct if we stay into fibrations over  $C$  until the last step.)

# FUNCTORS

(Polynomial case)

①  $E \rightarrow \text{Poly}(E) ; u \mapsto (u \leftarrow 0)$

analogue of the constant functor

$$\text{Set} \rightarrow [C^P, \text{Set}]$$

②  $E^P \rightarrow \text{Poly}(E) ; X \mapsto (1 \leftarrow X)$

analogue of the Yoneda

$$I \rightarrow [C^P, \text{Set}]$$

# FUNCTORS

(Dialectica case)

We get two functions

$$E \longrightarrow \Sigma\prod E$$

①

$$E \longrightarrow \text{Poly}(E) \longrightarrow \text{Poly}(\text{Poly } E)$$

②

$$E \longrightarrow (\text{Poly}(E))^{\text{op}} \longrightarrow \text{Poly}(\text{Poly } E)$$

# AXIOM OF CHOICE

In models for type theory we have

$$\prod_{a \in A} \sum_{b \in B(a)} C(a, b)$$



$$\sum_{f \in \prod_{a \in A} B(a)} \prod_{a \in A} C(a, f(a)) .$$

an isomorphism (more to the  
obvious map)

So hereditarily everything in type  
theory has  $\sum \prod$  form.

# INTERPRETATION FUNCTOR

AC provides a distributive law

$$\Pi\Sigma \longrightarrow \Sigma\Pi$$

and so  $\Sigma\Pi$  is a premodel.

Each model  $\mathbb{E} \rightarrow \mathbb{P}$  is an algebra  
for  $\Sigma\Pi$  i.e. we get an interpretation

$$\Sigma\Pi\mathbb{E} \longrightarrow \mathbb{E}$$

Both functors  $\mathbb{E} \rightarrow \Sigma\Pi\mathbb{E}$  are  
sections.

(Glimpse of a theory.)

# INDEPENDENCE OF PREMIS

Gödel's Interpretation validates

$$(\forall n \phi(n) \rightarrow \exists y \psi(y)) \rightarrow \exists y (\forall n. \phi(n) \rightarrow \psi(y))$$

where  $\phi$  is QF.

In  $\Sigma\pi\kappa$

$$\prod_{x \in X} A(x) \Rightarrow \sum_{v \in V} \prod_{y \in Y(v)} B(y)$$

$$\subseteq \sum_{v \in V} (\prod_{n \in X} A(n) \Rightarrow \prod_{y \in Y(v)} B(y)).$$

What is the connection?

# CAUTION

Much to understand:

- model theory & type theory is in its infancy
- furnished interpretation gives many more models
- There are parallels (extensibility, IP) between  $\Sigma\text{ITTE}$  and Gödel's interpretation but hard to make precise.