SORTS OF ALGEBRAIC THEORIES

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What is a general notion of algebraic theory?

How to account for special kinds of theories?

Project joint with

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THEORIES AS MONADS

Traditional view

A theory is a monad T on a category $\mathbb{C}$

This gives T-algebras, free T-algebras etc.

Special classes of theory determined by dense $\mathbb{C}_0 \hookrightarrow \mathbb{C}$ (category with arities)

Then T special if determined by $T|_{\mathbb{C}_0}$

(Berger CT 2010, Weber ...)

Doesn't cover cases of interest.

AGAINST MONADS

Hyland–Power: A Categorical Understanding of Universal Algebra: Lawvere Theories and Monads.

One issue: an algebraic theory has models in many categories – not just in the one.

This talk ~ Beyond Lawvere Theories.

LAWVERE PERSPECTIVE

A many sorted algebraic theory is a category $\mathbb{T}$ with products.

The category of models in $\mathbf{C}$ tacitly with products is

$$\mathrm{Prod}(\mathbb{T}, \mathbf{C})$$

A single sorted theory is

$$\mathbb{F}^{op} \longrightarrow \mathbb{T}$$

identity on objects

(Where does that come from!)

2-DIM PERSPECTIVE

Let S be the 2-monad on $\underline{Cat}$ for categories with products.

An algebraic theory is an S-Algebra:

$$SA \longrightarrow A$$

A model in another such $\mathcal{C}$ is an object of $T\text{-}Alg(A, \mathcal{C})$ the category of pseudo maps

(See Blackwell - Kelly - Power Two dimensional monad theory.)

that is maps preserving products up to isomorphism.

[So models involve irrelevant choices of products.]

FALSE STEP

But fun and instructive

For what other 2-monads

S does the recipe

theory = S-algebra

model = pseudo map

amount to a form of algebra?

COPRODUCTS

$$\mathrm{Coprod}(A, \mathcal{C}) = \mathrm{Prod}(A^{op}, \mathcal{C}^{op})$$

is misleading as e.g. Set is not self-dual.

Does Coprod (A, Set) make sense?

Case A opposite of associative algebra (without unit)

$$x \longrightarrow x + x$$

is a partition-retraction. YES?

Case A opposite of a theory with constant

$$x \longrightarrow 0$$

only trivial model. No?

$\geq$ Comonads, coalgebras are something else!

BIPRODUCTS

- Set is not a category with biproducts

- Biproducts are products in an additive setting

BUT

they do not produce the algebraic theories typically of interest.

What does?

Z Something to understand about finitary monads Kelly – Power etc.

SYMMETRIC MONOIDAL

Useful theory of

Hyland - Power. Symmetric monoidal sketches and categories of wirings

especially in an additive setting

- concurrency
- modern algebra

But is it algebra?

Monoids (Lie algebras, Poisson algebras)	YES!
Comonoids	NO!
Frobenius algebras Hopf algebras	NO!

SYMMETRIC MONOIDAL
(CONTINUED)

For **Set** with $\times$ (why?)

- Monoids make good sense
- Objects have unique comonoid structure
- So e.g.

  Monoids always bialgebras

  Only the trivial monoid is a Frobenius algebra.

SYMMETRIC MONOIDAL

(CONTINUED^2)

In modern algebra

- 'single-sorted' symmetric monoidal category ~ PROP
- corresponding sketch ~ PROPERAD
- algebraic theories of special type ~ OPERADS

How to explain?

EXTENDING 2-MONADS

The bicategory **Prof** has

1-cells $A \overset{M}{\nrightarrow} B$ i.e. $\dfrac{A \to P(B)}{A \times B^{op} \to Set}$

so is the Kleisli for the presheaf

restricted monad on **Cat**

Consider 2-monads S on

Cat which extend to (pseudo monads on) **Prof**

$$\begin{array}{c} \mathbf{Prof} \circlearrowleft S \\ \uparrow \\ \mathbf{Cat} \circlearrowleft S \end{array}$$

(Distributive law.)

EXAMPLES

Monads which extend

Terminal object
Products
Finite limits
Monoidal
Symmetric monoidal
Equip with endofunctor
Equip with factorization

Monads which don't

Initial object
Coproducts
Biproducts
! Equalizers !

KLEISLI BICATEGORY

(Fiore, Gambino, Hyland, Winskel. Kleisli Bicategories. Still in preparation!)

The Kleisli for S on $Prof$ has

1-cells $\quad A \xrightarrow{M} SB$

with composition given by

generalized substitution

(formally
$$\frac{A \xrightarrow{M} SB \qquad B \xrightarrow{N} SC}{A \xrightarrow{M} SB \xrightarrow{SN} S^2C \xrightarrow{\mu} SC}$$
)

An extended monad is a notion of substitution!

GENERALISED ALGEBRA

Let S be an extended monad.

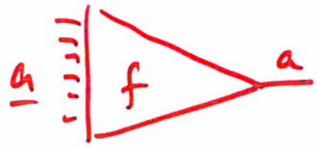
An S-Algebraic theory or S-Multicategory is a monad

$$A \overset{M}{\longrightarrow\!\!\!\!\!\!\!+\,\,} SA$$

in the Kleisli bicategory.

Idea: $f \in M(\underline{a}, a)$ has input arity $\underline{a} \in SA$ output arity $a \in A$.

i.e.



N.B. There is a notion of normal such which I ignore here.

STANDARD CASES

- S Products

  We get cartesian multicategory
  ~ standard algebraic theory

- S Symmetric monoidal

  We get (multicoloured) operads
  ~ symmetric multicategories

But other interesting S!

MONAD PICTURE

Take $A \overset{M}{\longrightarrow\!\!\!\!\!\!+\;\;} SA$ a theory.

In case $SO = 1$, M acts on

$$Kl(S)(A, SO) = [A, Set]$$

as a monad.

(Version of standard thought that a usual many sorted algebraic theory gives a monad on $Set/Sorts$.)

In case $A = 1$ we get the usual monads on the standard case.

LAWVERE PICTURE

Each S-Algebra $SA \xrightarrow{a} A$
induces an S-Algebraic theory
$$A \overset{a^*}{\longrightarrow\!\!\!\!\!\!\!+\;\;} SA$$

There is a left (bi) adjoint:
there is a free S-Algebra
generated by an S-theory
$$A \overset{M}{\longrightarrow\!\!\!\!\!\!\!+\;\;} SA$$

- S Products This gives the Lawvere theory as category with products.

- S Symmetric monoidal This gives the PROP corresponding to an operad.

CONSTRUCTION OF FREE S-ALGEBRA

Given $A \xrightarrow{M} SA$ we consider

$$SA \xrightarrow{SM} S^2A \xrightarrow{M} SA$$

as a monad in

$$S\text{-}\mathrm{Prof} \subseteq S\text{-}\mathrm{Prof}_{lax}$$

and there we take the Kleisli object

Z $S\text{-}\mathrm{Prof}_{lax}$ as a strict form of equipment

That means we get something up to isomorphism

FREE S-ALGEBRA: SIMPLE CASES

In general (as with usual free constructions) we need an iteration.

BUT if S preserves bijective on object functors (boos) in <u>Cat</u>

then it preserves Kleisli in <u>Prof</u>

and so the Kleisli of

$$SA \xrightarrow{S\mu} S^2A \xrightarrow{\mu} SA$$

in Prof is already the required S-Algebra.

(Happens in standard example.)

THEOREM

Let $SA \xrightarrow{a} A$ and $SB \xrightarrow{b} B$ be S-Algebras. Then to give a Kleisli cone

$$\begin{tikzcd}
SA \arrow[d, "Sa^*"'] \arrow[drr] & & \\
S^2A \arrow[d, "\mu"'] \arrow[r, phantom, "\Nearrow"] & & B \\
SA \arrow[urr] & &
\end{tikzcd}$$

in S-Prof

is to give a lax algebra map

$$\begin{tikzcd}
SA \arrow[r] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow"] & SB \arrow[d, "b"] \\
A \arrow[r] & B
\end{tikzcd}$$

SOME 2-DIM MONAD THEORY

Blackwell – Kelly – Power

$$\begin{tikzcd} T\text{-}Alg_{str} \arrow[rr, ""] \arrow[rr, phantom, bend right=30, "\perp"] & & T\text{-}Alg_{lax} \arrow[ll, bend right=40, "(\ )^+"'] \end{tikzcd}$$

giving a comonad on $T\text{-}Alg_{str}$

. Corollary, For extended S we obtain A^+ as the Kleisli in

$$\begin{tikzcd} SA \arrow[d, "Sa^*"'] \arrow[drr] & & \\ S^2A \arrow[d, "\mu"'] \arrow[rr, phantom, "\Uparrow"] & & A^+ \\ SA \arrow[urr] & & \end{tikzcd}$$

$(\ )^+$ -COALGEBRAS

The free S-Algebra generated by a $A \xrightarrow{M} SA$ is naturally a $(\ )^+$ - coalgebra

Corollary

$$\begin{array}{ccccc} S\text{-}\mathrm{Alg}_{lex} & & S\text{-}\mathrm{Alg}^{ic} & & \\ \wr & \hookrightarrow & \text{Themes} & \hookrightarrow & (\ )^+\text{-}\mathrm{Coalg} \\ \text{Free } (\ )^+\text{-}\mathrm{Coalg} & & & & \end{array}$$

Z (S-algebraic theory determined by an S-Algebra)

Folklore: S symmetric monoidal then S-Algebraic themes = $(\ )^+$ - Coalgebras

For this sufficient that free S-Alg (via Kleisli) preserve (certain) equalizers.

[New take on old category theory.]

CASE OF PRODUCTS

Then S is co-lax idempotent.

So lax S-algebra maps are pseudo and $(\ )^+$ is $(\ )'$ (notation of Blackwell-Kelly-Power)

But then each S-Algebra A is equivalent to A' which is a free $(\ )'$ coalgebra.

I.e. Up to equivalence categories with products $\sim$ product theories

CASE OF SYMMETRIC MONOIDAL

Then S is not co-lax idempotent

S-Algebraic theories are

$(\ )^+$ - coalgebras

BUT it is not the case

(even up to equivalence)

that every S-algebra

has a $(\ )^+$ - coalgebra

structure.

i.e. There are symmetric

monoidal categories not

arising from symmetric

multicategories.

RANDOM THOUGHTS

(Obviously!) This is only a glimpse. There are theories of change of base comparing notions of theory etc.

(Not so obviously!) There are interesting examples beyond the standard ones.

(Disturbingly!) $(\ )^+$ makes sense for 2-monads which do not extend. Can that give an approach to notions not yet captured?