SOME MATHEMATICAL FOUNDATIONS OF GAME SEMANTICS

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GAMES + STRATEGIES

Game trees

$$1 = A(0) \xleftarrow{\pi} A(1) \xleftarrow{\pi} A(2) \quad \cdots$$

Strategy trees

$$\begin{tikzcd}
1 = S(0) & S(1) \arrow[l] \arrow[d] & S(2) \arrow[l] \arrow[d] & \cdots \\
1 = A(0) & A(1) \arrow[l] & A(2) \arrow[l] & \cdots
\arrow[from=1-1, to=2-1]
\end{tikzcd}$$

PLAYER STRATEGIES

Opponent can play any move (just once).

$$
\begin{tikzcd}
S(2n) \arrow[d] & S(2n+1) \arrow[l] \arrow[d] \\
A(2n) & A(2n+1) \arrow[l]
\end{tikzcd}
$$

Player may be multi non-deterministic

Simply non-deterministic

$$
\begin{tikzcd}
S(n) \arrow[r, tail] & A(n)
\end{tikzcd}
$$

Deterministic

$$
\begin{tikzcd}
S(2n+1) \arrow[d, tail] & S(2n+2) \arrow[l, tail] \\
A(2n+1) &
\end{tikzcd}
$$

TENSOR + FUNCTION SPACE

$\left(I \qquad 1 \leftarrow 0 \leftarrow 0 \qquad \right)$

$$A \otimes B(n) = \sum_{\phi : p, q \Rightarrow n} A(p) \times B(q)$$

$$A \multimap B(n) = \sum_{\phi : p \vdash q \Rightarrow n} A(p) \times B(q)$$

Notation: $(A \otimes B)_\phi$

$(A \multimap B)_\phi$

$$\begin{array}{ccc} \pi : A(p) \times B(q) & \longrightarrow & A(p-1) \times B(q) \\ \| & & \text{or} \\ (\quad)_\phi & & A(p) \times B(q-1) \end{array}$$

according as $\pi\phi$ maps

MONOIDAL CLOSURE

$$A \otimes B \multimap C \,(n)$$

$$= \sum_{\phi : m \vdash r \Rightarrow n} \left(\sum_{\psi : p, q \Rightarrow m} A(p) \times B(q) \right) \times C(r)$$

$$= \sum_{\chi : p, q, r \Rightarrow n} A(p) \times B(q) \times C(r)$$

$$= \sum_{\sigma : p \vdash \ell \Rightarrow n} A(p) \times \sum_{\rho : q \vdash r \Rightarrow \ell} B(q) \times C(r)$$

$$= A \multimap (B \multimap C)(n)$$

COMPOSITION

$$\frac{S : A \multimap B \qquad T : B \multimap C}{T \circ S : A \multimap C}$$

$$T \circ S \text{ @ } \chi : p \vdash r \Rightarrow \cdot$$

$$\sum_{\substack{\phi : p \vdash q \Rightarrow \cdot \\ \psi : q \vdash r \Rightarrow \cdot}} T_\psi \times S_\phi \qquad \psi \circ \phi = \chi$$

IDENTITY

$$I : A \multimap A$$

$$I(2n) = A(n) \longrightarrow A(n) \times A(n)$$

$$\text{@ } id : n \vdash n \Rightarrow 2n$$

<u>Proposition</u> Monoidal closed category of simple games.

SPANS

$$A \longmapsto |A| = \sum_n A(n)$$

$$S : A \multimap B$$

$$\longmapsto \quad \begin{tikzcd} & {|S|} \arrow[dl] \arrow[dr] & \\ {|A|} & & {|B|} \end{tikzcd}$$

with $$|S| = \sum_m S(2m)$$

(i.e. Player positions).

<u>Proposition</u> $|\;| : \text{GAMES} \longrightarrow \text{SPAN}$

colax monoidal homomorphism

RELATIONS

If $S : A \multimap B$ simply non-det^{c}

then

$$\begin{tikzcd} & {|S|} \arrow[dl] \arrow[dr] & \\ {|A|} & & {|B|} \end{tikzcd}$$

jointly monic.

So GAMES $\longrightarrow$ REL

<u>Proposition</u> Suppose S deterministic.

If $(a, b) \in S_{\phi : p \vdash q}$

and $(a, b) \in S_{\psi : p \vdash q}$

then $\phi = \psi$

<u>Consequence</u> GAMES $\rightarrow$ REL is faithful for deterministic games

[H-Schalk: Abstract Games for Linear Logic]

SHUFFLES

$$n_1, \dots, n_k \xRightarrow{\phi} n \quad (= n_1 + \cdots + n_k)$$

$\phi : \sum n_i \xrightarrow{\cong} n$ with each

$\phi|_{n_i}$ order preserving

The multimaps of a symmetric multicategory.

DECOMPOSITION PROPERTY

E.g.

For $\phi : p, q, r \Rightarrow n$ there is a unique

$$\psi : p, q \Rightarrow m \quad \chi : m, r \Rightarrow n$$

with $\chi\psi = \phi$.

PARITY SHUFFLES

$n_1, \ldots, n_k \xRightarrow{\phi} n$ such that

also each $\phi|_{n_i}$ preserves parity

$n_1, \ldots, n_k \vdash m \xRightarrow{\phi} n$ such that

$\phi|_m$ preserves parity

while each $\phi|_{n_i}$ reverses parity

Compositions more intricate: a non-standard multicategory?

NATURAL DECOMPOSITION PROPERTY

$\sim$ Monoidal Closure of GAMES

COMPOSITION

(Interactive composition)

Simple case

$$\frac{\phi : p \vdash q \Rightarrow \cdot \qquad \psi : q \vdash r \Rightarrow \cdot}{\psi \circ \phi : p \vdash r \Rightarrow \cdot}$$

More generally another multicategory.
(WHAT IS THIS?)

Explanation

- Picture on board?
- McCusker – Power – Wingfield
- Trace / GoI composition

(Cf. H-Schalk. Games on Graphs and Sequentially Realizable Functionals. (Appendix: Merging.)).

INNOCENT STRATEGIES

Old view H - Ong

Innocent strategy is played in a 'large game' generated by an arena or 'small game'

New take Harmer - H - Melliès

Innocent strategy is

$$S : ?(\underbrace{A \boxplus B}_{\text{arena}}) = !A \multimap ?B$$

the old view representation.

Question

Needs of higher order model checking ?

KLEISLI COMPOSITION

$$!A \xrightarrow{S} ?B \qquad !B \xrightarrow{T} ?C$$

$$!A \to !!A \xrightarrow{!S} !?B \to ?!B \xrightarrow{?T} ??C \to ?C$$

"distributive law"

Combinatorics of copying:

$$\begin{tikzcd}
!A \arrow[r] \arrow[d] & !!A \arrow[d] \\
!!A \arrow[r] & !!!A
\end{tikzcd}
\qquad
\begin{tikzcd}
???A \arrow[r] \arrow[d] & ??A \arrow[d] \\
??A \arrow[r] & ?A
\end{tikzcd}$$

$$\begin{tikzcd}
!?A \arrow[rr] \arrow[d] & & ?!A \arrow[d] \\
!!?A \arrow[r] & !?A \arrow[r] & ?!!A
\end{tikzcd}$$

$$\begin{tikzcd}
!??A \arrow[r] \arrow[d] & ?!?A \arrow[r] & ??!A \arrow[d] \\
!?A \arrow[rr] & & ?!A
\end{tikzcd}$$

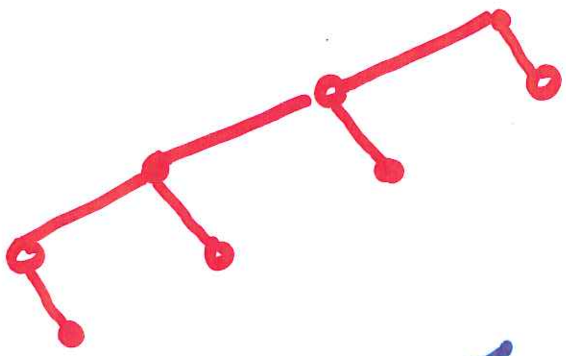
EXAMPLE

$$\lambda F.\, F(\lambda x.\, F(\lambda y.\, \genfrac{}{}{0pt}{}{x}{y}))$$

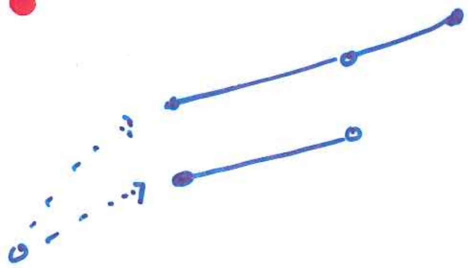
(Tease :- On which elements of $(\mathbb{B} \Rightarrow \mathbb{B}) \Rightarrow \mathbb{B}$ do they differ?)

Arena

$$((A \multimap A) \multimap A) \multimap A$$



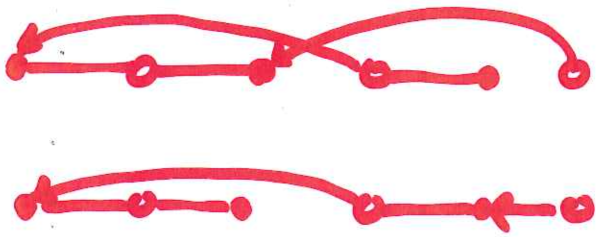
Play:



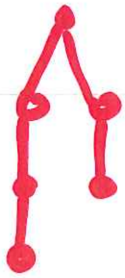
HEAPS

(discarded by H-Ong)

Compare



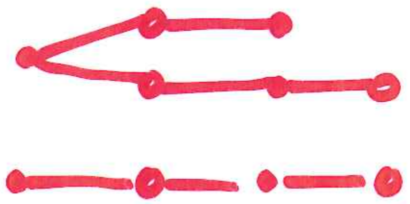
'Same' tree



different sequentializations.

VIEWS

What is



?

They are finite $\begin{array}{c} S \\ \downarrow \\ A \end{array}$ which are deterministic Opponent strategies vs a free Player i.e.

$$S(2n) \leftarrowtail S(2n+1)$$

A Player view (after Opponent has played) is a parity preserving sequentialization of such.

FINAL THOUGHTS

Mathematical foundations

- Type theoretic style
- Categorical combinatorics
- Category theory

(but not the only tools).

Issue of Identity

- Apparent in the use of SPAN
- Other talks addressed this

(but no magic wand).