PROOF AS PROCESS

Generalized

Henkin Quantifiers

and

Linear Logic

HINTIKKA GAMES

Recall the game theoretic semantics for classical logic

We Us $\exists$ $\vee$ ($\exists$loise)

play against

They Them $\forall$ $\wedge$ ($\forall$belard)

(assume negation only applied to atomics)

A formula $\phi \equiv \forall x \exists y \forall w \exists z \ldots$ is valid if and only if we win the obvious corresponding game.

A strategy for Us is a suitable collection of Skolem functions.

[Maybe this does not say much]

THE NOVIKOV CALCULUS

(an analysis of classical validity)

- $\top$, $\bot$ are formulae.
- If ϕ_i formulae then $\Pi_i \phi_i$ + $\Sigma_i \phi_i$ are formulae.

Assume like quantifiers merge.

Definition of validity

- $\top$ is valid
- $\Pi_i \phi_i$ is valid if and only if ϕ_i is valid for all i
- $\Sigma_i \phi_i$ is valid if and only if

<u>either</u> some ϕ_{i_0} is $\top$

<u>or</u> some ϕ_{i_0} is $\Pi_j \phi_{i_0 j}$ and for all j $\phi_{i_0 j} \vee \Sigma_i \phi_i$ is valid

Classical validity is the smallest set satisfying these closure conditions.

CONSTRUCTIVE CONTENT AS A STRATEGY

An intuition of Coquand:-

Let $f : \mathbb{N} \to \mathbb{N}$ be any function and consider the classically valid

$$\forall n\ \exists m \geq n\ \forall k \geq m\quad f(m) \leq f(k).$$

A proof procedure for this in the Novikov calculus suggests the following strategy.

They give us an n. We return $m_0 = n$. They give us a $k_0 \geq m_0$. If $f(m_0) \leq f(k_0)$, we win; else $f(k_0) < f(m_0)$ and we return $m_1 = k_0$. They give us a $k_1 \geq m_1$ etc.

AIM

- What is this a strategy for? (Looks like linear logic.)
- In what sense is it the constructive content of a classical proof?

BLASS : DEGREES OF GAMES

(Fund. Math. 1972)

Games as usual in descriptive set theory etc. Blass defined a $\otimes$, $⅋$ and so a degree ordering

$A \vdash B$ iff We win $A \multimap B = A^{\perp} ⅋ B$

swap roles

Essentially linear logic but from point of view of provability only.

BLASS'S LINEAR STRUCTURE

$(\ )^\perp$ is easy (swap the players)
and $\invamp$ defined by duality

$$A \invamp B = (A^\perp \otimes B^\perp)^\perp$$

so enough to define $A \otimes B$

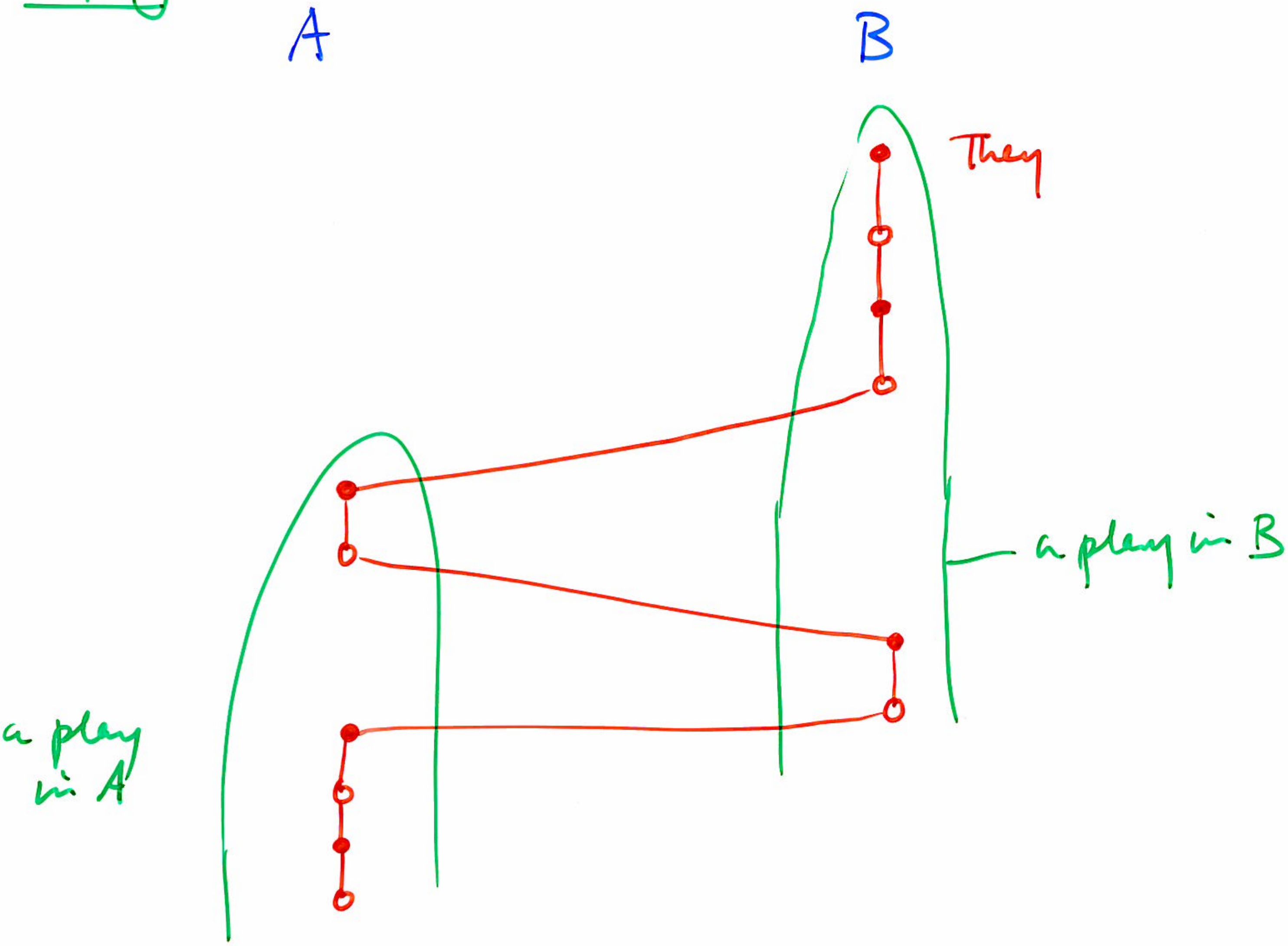
INTUITION Generally we play the games in parallel; but we set things up so that They can swap between games while we cannot.

Work to do as – They may start Π
– We may start Σ

$A \otimes B$

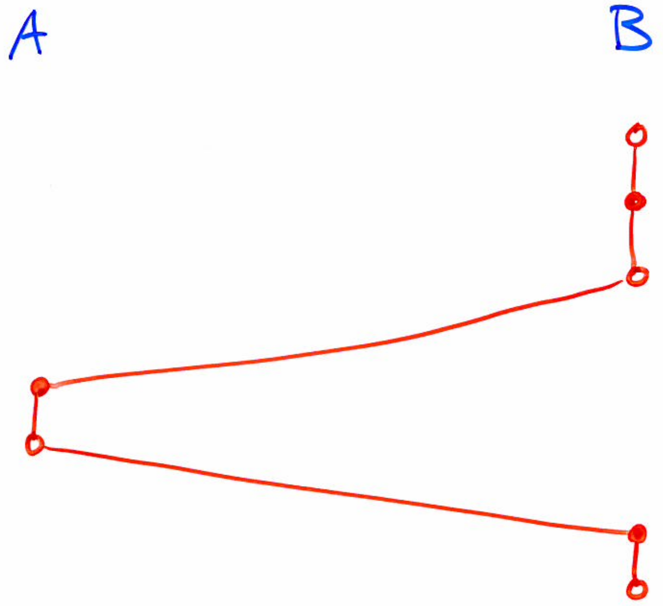
(Π, Π) case

A play:

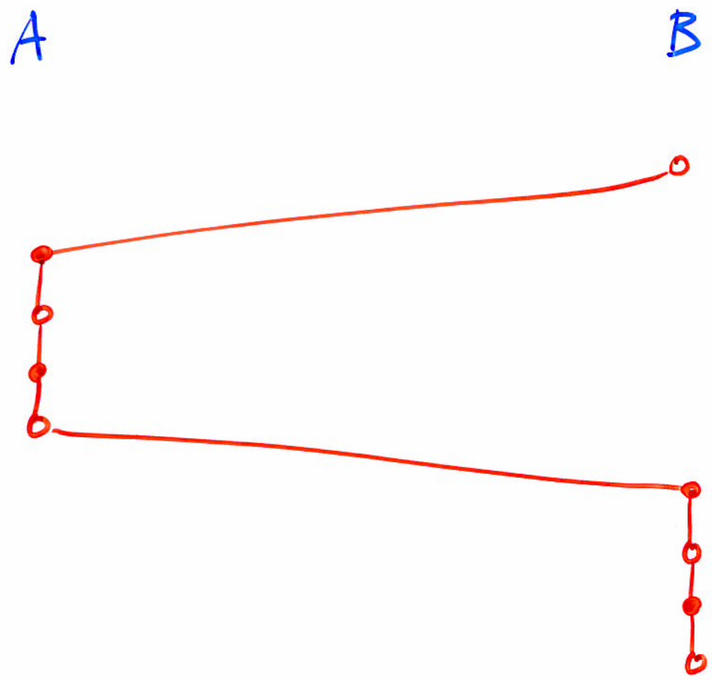


$A \otimes B$ (Π, Σ) case

A play:

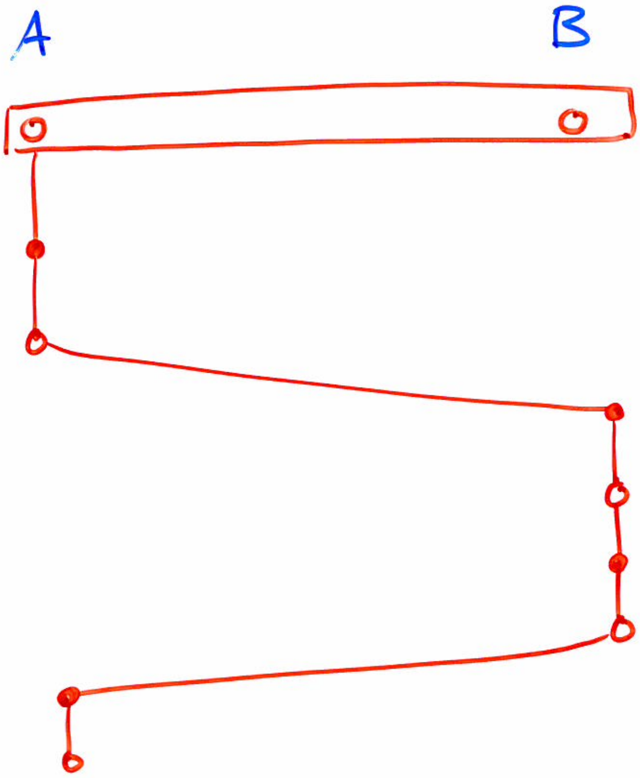


Another play:



$A \otimes B$ (Σ, Σ) case

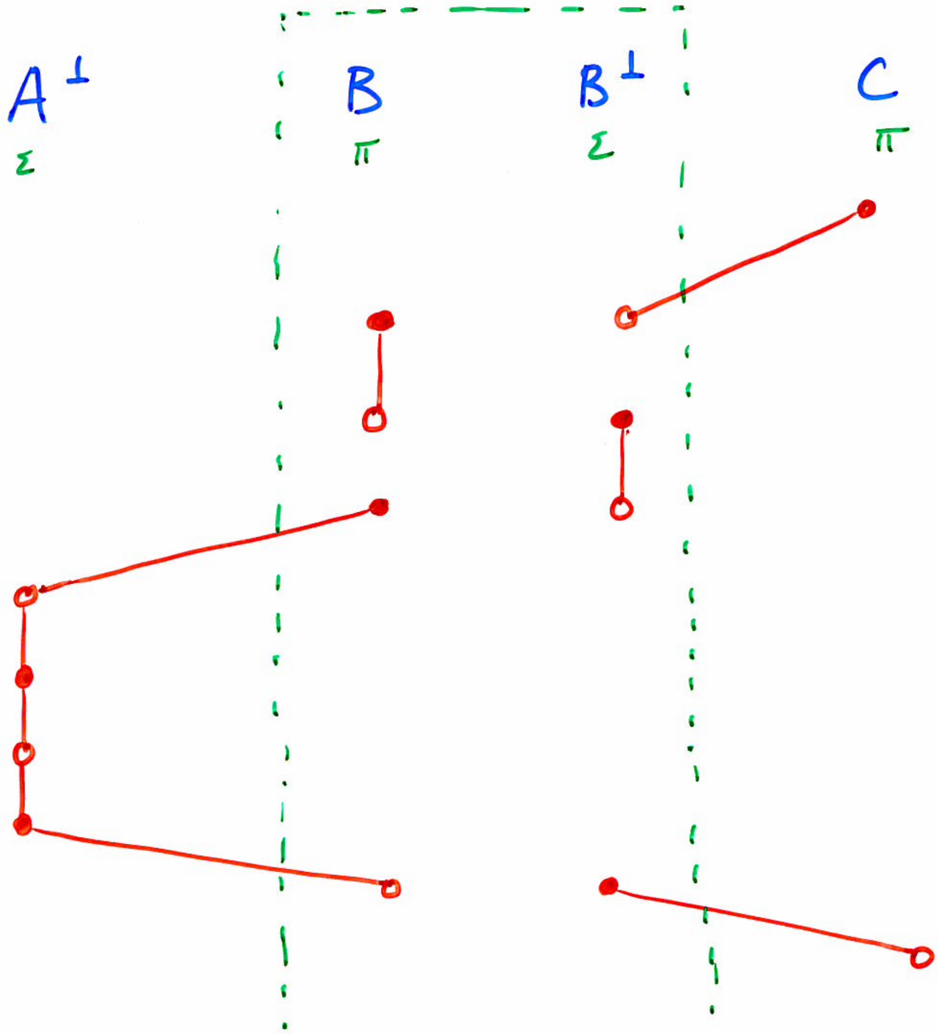
Aplay:-



two moves played simultaneously

COMPOSITION

Case $\Pi \to \Pi \to \Pi$



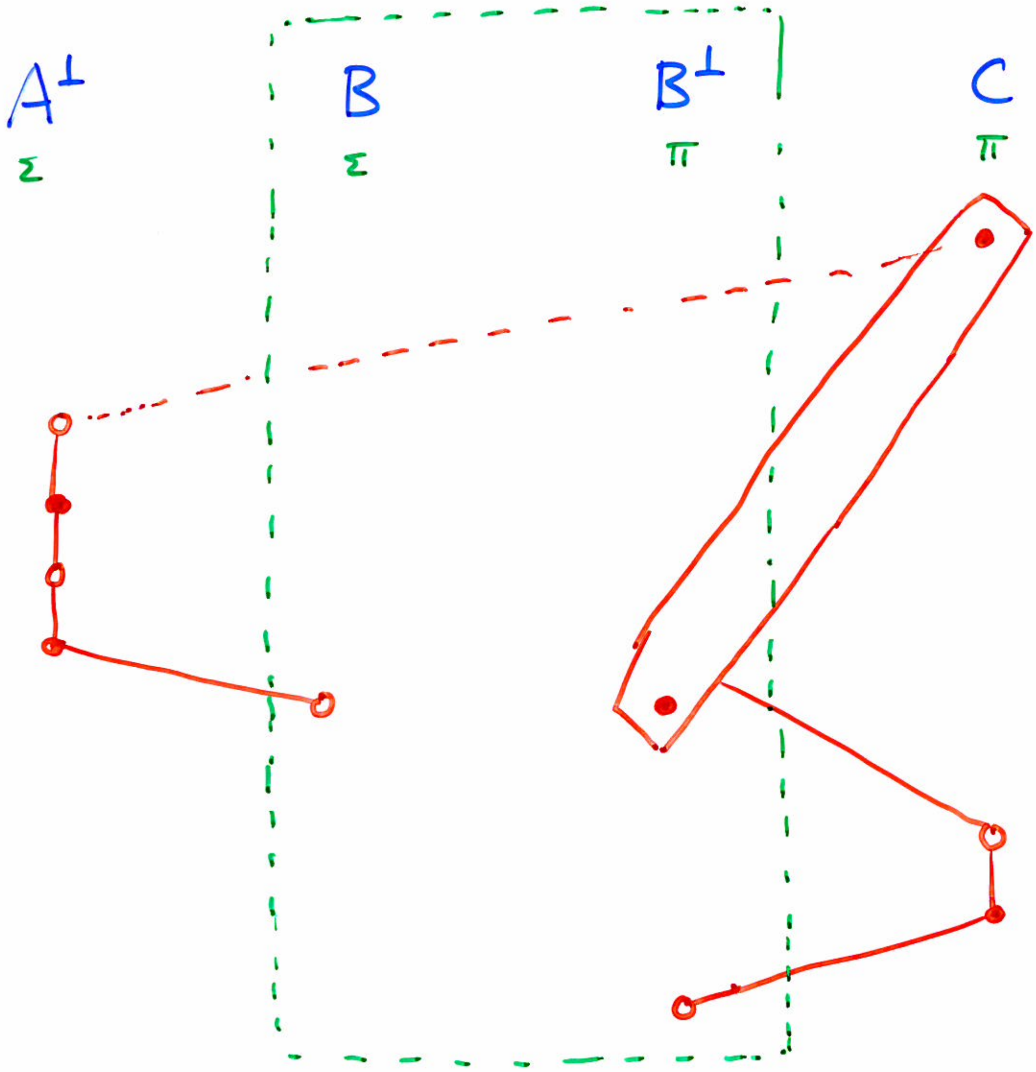
Case $\Pi \to \Pi \to \Sigma$

Not substantially different

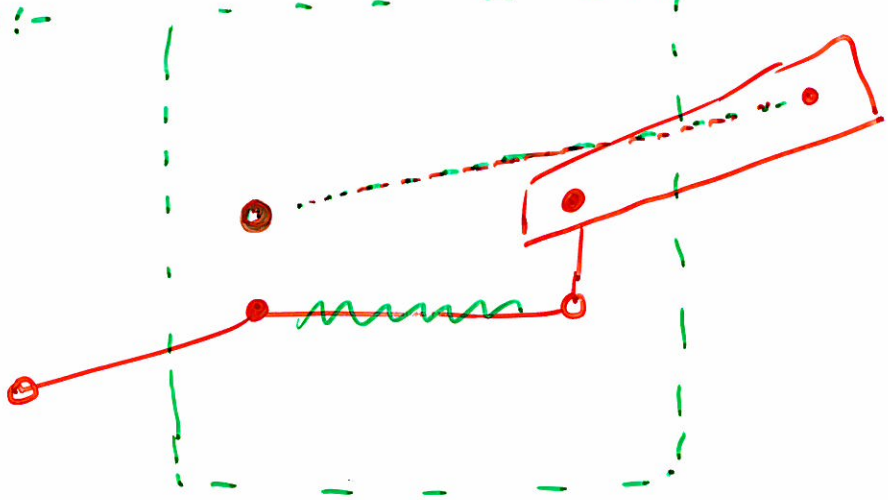
COMPOSITION

Case $\Pi \to \Sigma \to \Pi$

A play:-

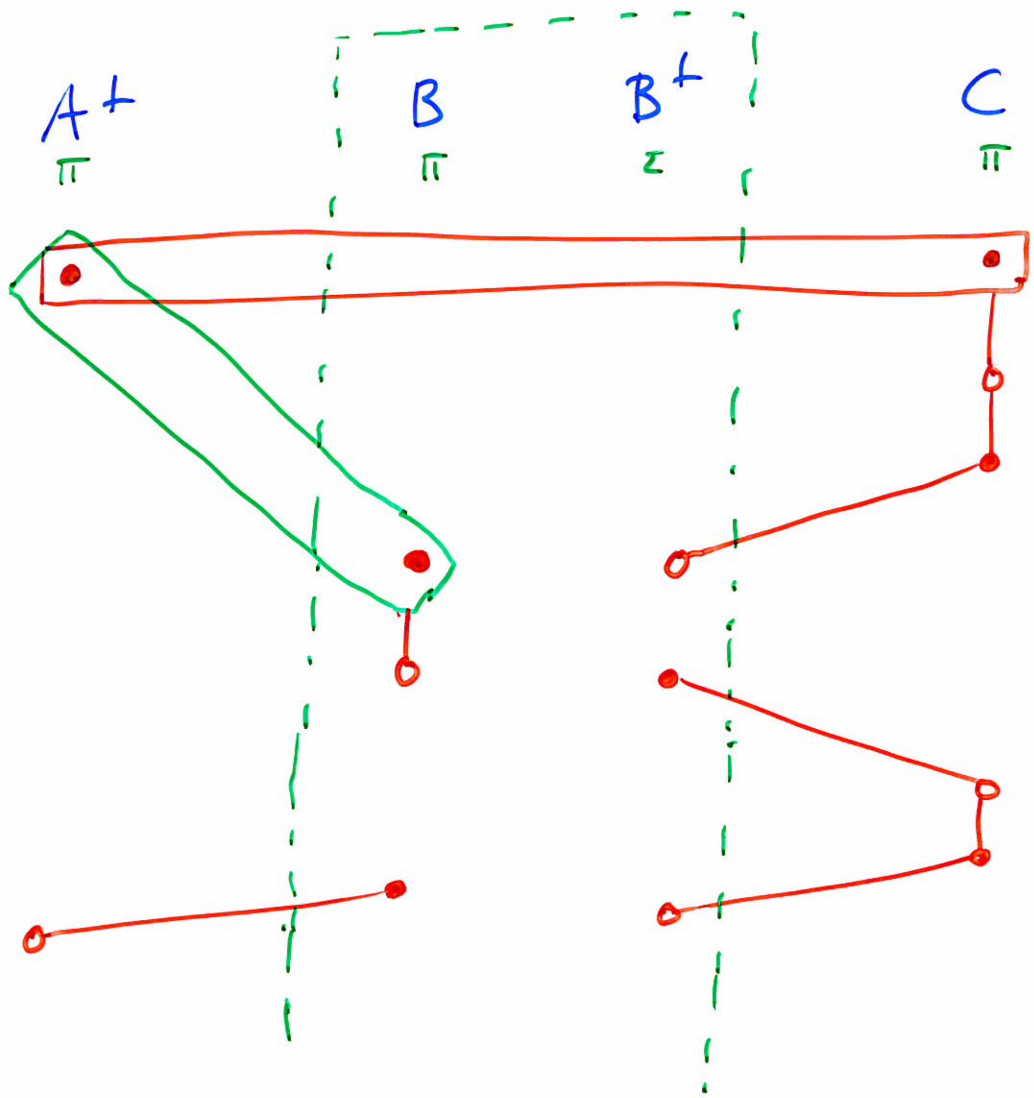


Another play:-



COMPOSITION

Case $\Sigma \to \Pi \to \Pi$



A COMPOSITIONAL NOTION OF PROOF?

AIM A proof σ of B from A is a strategy (in standard sense) for Us in

$$A \multimap B = A^{\perp} \parr B$$

We need to compose such

$$\frac{A \xrightarrow{\sigma} B \qquad B \xrightarrow{\tau} C}{A \xrightarrow{\sigma ; \tau} C}$$

Cases

$\Pi \to \Pi \to \Pi$ $(\Sigma \to \Sigma \to \Sigma)$

$\Pi \to \Pi \to \Sigma$ $(\Pi \to \Sigma \to \Sigma)$

$\Pi \to \Sigma \to \Pi$ $(\Sigma \to \Pi \to \Sigma)$

$\Sigma \to \Pi \to \Pi$ $(\Sigma \to \Sigma \to \Pi)$

THE CATEGORY OF Π-GAMES

Restricting above to Π-games we get a category.

It is
- symmetric monoidal closed
- has a good 'exponential comonad'
- has products

and so gives a model of ILL (Intuitionistic Linear Logic).

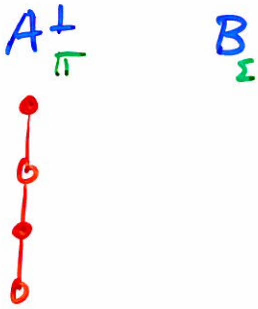
Via duality this gives rise to model of classical linear logic without good additives.

(Analogous models much studied.)

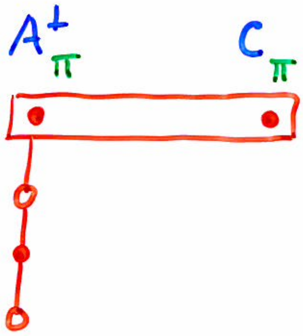
FAILURE OF ASSOCIATIVITY

Consider $\underset{\Sigma}{A} \overset{\sigma}{\multimap} \underset{\Sigma}{B}$ $\quad$ $\underset{\Sigma}{B} \overset{\tau}{\multimap} \underset{\Pi}{C}$ $\quad$ $\underset{\Pi}{C} \overset{\rho}{\multimap} \underset{\Pi}{D}$

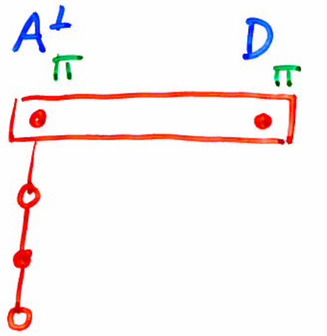
Suppose σ starts



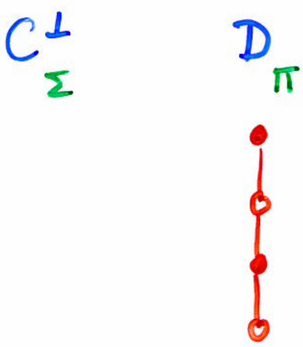
then $\sigma;\tau$ starts



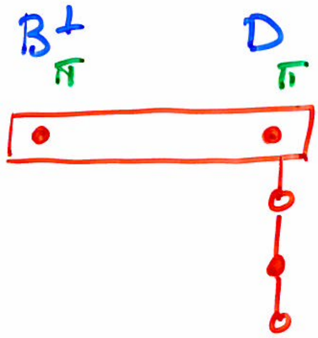
and so $(\sigma;\tau);\rho$ starts



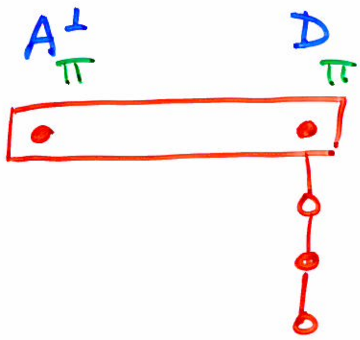
Suppose ρ starts



then $\tau;\rho$ starts



and so $\sigma;(\tau;\rho)$ starts



Hence whatever τ may be

$$\boxed{(\sigma;\tau);\rho \neq \sigma;(\tau;\rho)}$$

FREE BICOMPLETIONS OF (LOCALLY) SMALL CATEGORIES

(Joyal 1995) Suppose $\mathbb{C}$ is a (locally) small category. Then there exists a locally small bicomplete (= complete and cocomplete) category $\mathcal{B}(\mathbb{C})$ and an embedding $\mathbb{C} \xrightarrow{E} \mathcal{B}(\mathbb{C})$ such that for any map $\mathbb{C} \xrightarrow{F} \mathbb{D}$ of categories with $\mathbb{D}$ bicomplete there is a unique (up to natural isomorphism) $\bar{F} : \mathcal{B}(\mathbb{C}) \longrightarrow \mathbb{D}$ such that

$$\begin{tikzcd} \mathbb{C} \arrow[r, "E"] \arrow[dr, "F"'] & \mathcal{B}(\mathbb{C}) \arrow[d, "\bar{F}"] \\ & \mathbb{D} \end{tikzcd}$$

commutes (up to natural isomorphism).

(Cf Old theory of free lattices).

FREE BICOMPLETIONS AND $*$-AUTONOMOUS CATEGORIES

(Joyal 1995) If $\mathbb{C}$ is a $*$-autonomous category, then so is the free bicompletion $\mathcal{B}(\mathbb{C})$, and the embedding $\mathbb{C} \longrightarrow \mathcal{B}(\mathbb{C})$ preserves $*$-autonomous structure.

AIM As $\mathbb{2} = \{0 \to 1\}$, the two element Boolean Algebra is $*$-autonomous, $\mathcal{B}(\mathbb{2})$ is $*$-autonomous. Wish to describe a suitable subcategory as a basis for a 'coherent' compositional account of constructive content of classical logic.

CONSTRUCTION OF FREE BICOMPLETION

(first steps of construction over the ordinals)

- Start with $\mathbb{C}_0 = \mathbb{C}$
- First add colimits say.

Let $\mathbb{C}_1$ consist of formal colimits

$$\Sigma_i X_i \qquad (X_i \in \mathbb{C}_0)$$

non-standard symbol for arbitrary colimit

Let $\mathbb{C}_1(\Sigma_i X_i, \Sigma_j Y_j) = \Pi_i \Sigma_j \mathbb{C}_0(X_i, Y_j)$

So $\mathbb{C}_1$ is the free cocompletion of $\mathbb{C}$.

CONSTRUTION (Defn of $\mathbb{C}_2$)

- Next add limits (taking care)

Let $\mathbb{C}_2$ consist of formal limits of of objects of $\mathbb{C}_1$, so

$$\Pi_i \Sigma_j X_{ij} \qquad X_{ij} \in \mathbb{C}$$

Now $\mathbb{C}_2(\Pi_i \Sigma_j X_{ij}, \Pi_k \Sigma_\ell Y_{k\ell})$

$$= \Pi_k \underline{\mathbb{C}_2(\Pi_i \Sigma_j X_{ij}, \Sigma_\ell Y_{k\ell})}$$

Idea: only maps which must exist $\Pi_i A_i \to \Sigma_\ell B_\ell$
are $\Pi_i A_i \longrightarrow A_{i_0} \longrightarrow \Sigma_\ell B_\ell$
or $\Pi_i A_i \longrightarrow B_{\ell_0} \longrightarrow \Sigma_\ell B_\ell$
+ some maps have both forms.

So for ______ :

$$\begin{tikzcd}
\Sigma_{i\ell}\, \mathbb{C}_2(\Sigma_j X_{ij}, Y_{k\ell}) = \Sigma_{i\ell}\, \Pi_j\, \mathbb{C}_0(X_{ij}, Y_{k\ell}) \arrow[r] \arrow[d] & \Sigma_\ell\, \mathbb{C}_2(\Pi_i \Sigma_j X_{ij}, Y_{k\ell}) = \Sigma_{i\ell}\, \Pi_j\, \mathbb{C}_0(X_{ij}, Y_{k\ell}) \ (Y_{k\ell} \text{ atom}) \arrow[d] \\
\Sigma_i\, \mathbb{C}_2(\Sigma_j X_{ij}, \Sigma_\ell Y_{k\ell}) = \Sigma_i\, \Pi_j\, \Sigma_\ell\, \mathbb{C}_0(X_{ij}, Y_{k\ell}) \arrow[r] & \ulcorner \ \text{p.o.}
\end{tikzcd}$$

$$\boxed{\text{So } \mathbb{C}_2(\Pi_i \Sigma_j X_{ij}, \Pi_k \Sigma_\ell Y_{k\ell}) = \Pi_k \Sigma_i \Pi_j \Sigma_\ell\, \mathbb{C}_0(X_{ij}, Y_{k\ell})}$$

Compare Gödel's Dialectica Interpretation.

CONSTRUCTION (Defn of $\mathbb{C}_3$)

• Next add colimits (& we really need to take care)

Let $\mathbb{C}_3$ consist of $\boxed{\text{formal}}$ colimits of objects of $\mathbb{C}_2$, so

$$\Sigma_i \Pi_j \Sigma_k X_{ijk} \qquad X_{ijk} \in \mathbb{C}$$

Now $\mathbb{C}_3(\Sigma_i \Pi_j \Sigma_k X_{ijk}, \Sigma_\ell \Pi_m \Sigma_n Y_{\ell mn})$

$$= \Pi_i \underline{\mathbb{C}_3(\Pi_j \Sigma_k X_{ijk}, \Sigma_\ell \Pi_m \Sigma_n Y_{\ell mn})}$$

Again apply idea about maps $\Pi_j B_j \to \Sigma_\ell C_\ell$

So for $\underline{\qquad\qquad}$:

$$\begin{tikzcd}
\Sigma_{j\ell}\, \mathbb{C}_3(\Sigma_k X_{ijk}, \Pi_m \Sigma_n Y_{\ell mn}) \arrow[r] \arrow[d, equal] & \Sigma_j\, \mathbb{C}_3(\Sigma_k X_{ijk}, \Sigma_\ell \Pi_m \Sigma_n Y_{\ell mn}) \arrow[d, equal] \\
\Sigma_{j\ell}\, \Pi_{km}\, \Sigma_n\, \mathbb{C}(X, Y) \arrow[d] & \Sigma_j \Pi_k \Sigma_\ell \Pi_m \Sigma_n\, \mathbb{C}(X, Y) \arrow[d] \\
\Sigma_\ell\, \mathbb{C}_3(\Pi_j \Sigma_k X_{ijk}, \Pi_m \Sigma_n Y_{\ell mn}) \arrow[r] \arrow[d, equal] \arrow[ur, phantom, "\text{p.o.}"] & \underline{\qquad} \\
\Sigma_\ell \Pi_m \Sigma_j \Pi_k \Sigma_n\, \mathbb{C}(X, Y) &
\end{tikzcd}$$

So $\mathbb{C}_3(\Sigma_i \Pi_j \Sigma_k X_{ijk}, \Sigma_\ell \Pi_m \Sigma_n Y_{\ell mn})$

$$= \left(\Pi_i \begin{array}{c} \diagup\ \Sigma_j - \Pi_k\ \diagdown \\ \diagdown\ \Sigma_\ell - \Pi_m\ \diagup \end{array} \Sigma_n \right) \mathbb{C}(X_{ijk}, Y_{\ell mn})$$

* AUTONOMOUS STRUCTURE

Assume $\mathbb{C}$ is $*$-autonomous.

- The duality on $\mathcal{B}(\mathbb{C})$ is clear

  limits $\longleftrightarrow$ colimits

- Recursive definition of $\otimes$:-

  - If $A, B \in \mathbb{C}$ then $A \otimes B = A \otimes B$
  - $X \otimes \Sigma_j Y_j = \Sigma_j (X \otimes Y_j)$
  - If $A \in \mathbb{C}$ then $A \otimes \Pi_i X_i = \Pi_i (A \otimes X_i)$
  - $\Pi_i X_i \otimes \Pi_j Y_j = ?$

where

$$\begin{tikzcd}
? \arrow[r] \arrow[d] & \Pi_i (X_i \otimes \Pi_j Y_j) \arrow[d] \\
\Pi_j (\Pi_i X_i \otimes Y_j) \arrow[r] & \Pi_i \Pi_j (X_i \otimes Y_j)
\end{tikzcd}$$

[One has to check this works.]

A SIMPLE $\otimes$ IN $\mathcal{B}(2)$

Consider

$$\Pi_i \Sigma_j A_{ij} \otimes \Pi_k \Sigma_l B_{kl}$$

where $A_{ij}, B_{kl} \in 2 = \left\{ \begin{smallmatrix} T \\ F \end{smallmatrix} \right\}$.

$$\begin{array}{ccc} ? & \longrightarrow & \Pi_i \Sigma_j \Pi_k \Sigma_l (A_{ij} \wedge B_{kl}) \\ \downarrow & & \downarrow \\ \Pi_k \Sigma_l \Pi_i \Sigma_j (A_{ij} \wedge B_{kl}) & \longrightarrow & \Pi_{ik} \Sigma_{jl} (A_{ij} \wedge B_{kl}) \end{array}$$

This identifies

$$\Pi_i \Sigma_j A_{ij} \otimes \Pi_k \Sigma_l B_{kl}$$

with the standard Henkin quantifier applied to the matrix, i.e.

$$\begin{pmatrix} \Pi_i & \Sigma_j \\ \Pi_k & \Sigma_l \end{pmatrix} (A_{ij} \wedge B_{kl})$$

HENKIN QUANTIFIERS

The traditional form

$$\begin{pmatrix} \forall u & \exists x \\ \forall v & \exists y \end{pmatrix} A(u,x;v,y)$$

interpreted as

$$\exists f : X^u, g : Y^v \ \forall u,v.\ A(u,f(u);v,g(v)).$$

The 'dual' is

$$\begin{pmatrix} \exists u & \forall x \\ \exists v & \forall y \end{pmatrix} \neg A$$

whose interpretation <u>should</u> be <u>roughly</u>

$$\exists u \forall x \exists v \forall y \neg A \vee \exists v \forall y \exists u \forall x \neg A.$$

Clearly duality is not 'classical not'.

An imaginable quantifier form which we do not have but which makes sense in another context:

$$\begin{pmatrix} \exists u \\ \forall x \end{pmatrix} A$$

(VERY) SIMPLE GENERALIZED HENKIN QUANTIFIERS

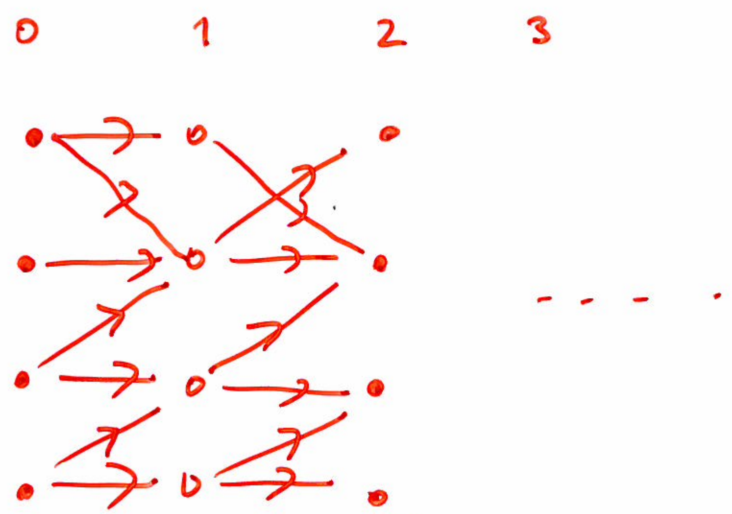
Natural context: Many sorted logic i.e. collection of basic sets X, Y, Z which we close under finite products

A generalized quantifier $\alpha(x_1, \dots x_n)$ binds sorted variables $x_1, \dots x_n$; and we apply it to matrices $A(x_1, \dots x_n) : \prod X_i \to 2$.
So we have simple propositional forms

$\alpha(\underline{x}).\ A(\underline{x})$. (GH-game)

Q'fiers are either of Π or of Σ type.

A Π Q-fier is given by a finite graded-bipartite directed graph with nodes to be explained



such that all nodes are accessible from nodes of degree 0.

NODES OF A Π Q-FIER

Nodes of $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$ degree are $\begin{matrix}\Pi\\ \Sigma\end{matrix}$ nodes.

Variables of the q-fier α are partitioned into sets $V_0, V_1, \ldots$ and the nodes of degree i are indexed by (pairwise incompatible) collections of ~~nos~~ variables from V_i.

So an edge of the graph looks like

$$\Pi\vec{x} \longrightarrow \Sigma\vec{y}$$

and is read 'values for the variables $\vec{x}$ enable a choice of values for the variables $\vec{y}$'.

Π q'fier say

PLAYS IN $\alpha(\underline{x})\ A(\underline{x})$

Players now Π and Σ.

Standing fairness assumption: (FA).

Π, Σ always play so that the other has some move until game ends with all variables $\underline{x}$ given values.

- Π plays some (≥1) nodes of degree 0 by giving values to all the variables (a node is played when all its variables have values)

- By (FA) some Σ nodes (of degree 1) are now enabled and Σ plays some of these.

- By (FA) some Π nodes (of degree 0, 2) are now enabled and Π plays some of these.

ETC

Eventually Π, Σ have set $\underline{x} = \underline{a}$ and Σ wins if $A(\underline{a})$ is 1 = True.

N.B. Not all plays can occur in accord with the strategies we consider

"SKOLEM" STRATEGIES

Sketch Σ-Strategies are presented as collections of Skolem functions i.e. for each Σ variable $y : Y$ we have a function $\prod X_i \xrightarrow{\eta} Y$ from a product of sorts for Π variables (not necessarily of smaller degree) which $\boxed{\text{work}}$ in the following sense.

- Let Π play according to the rules and whenever They have managed to play $\underline{x} = \underline{a} \in \prod X_i$ we are prepared to play $\eta(\underline{a}) \in Y$ and we do so as soon thereafter as we can.
- However Π plays our Skolem functions always provide us with a next move according to the rules.

Finally we want strategies which win

WS - strategies

(A couple of trivial nuances suppressed.)

* AUTONOMOUS STRUCTURE

- Definition of $(\ )^\perp$

$$(\alpha(\vec{x}).A(\vec{x}))^\perp = \bar{\alpha}(\vec{x}).\neg A(\vec{x})$$

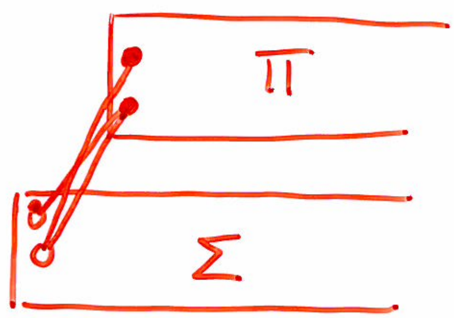
dual quantifier

- Definition of $\otimes$

By cases (assume all variables distinct)

$\Pi \otimes \Pi$ graphs laid out side by side

$\Pi \otimes \Sigma$



$\Sigma \otimes \Sigma$ nodes of degree 0 are pairs of nodes and these enable whatever either member does then side by side.

(Very like Blass.)

LITTLE THEOREM

Take GH-games $\alpha.A$, $\beta.B$ etc, and, as maps $\alpha.A \to \beta.B$, WS-strategies in $(\alpha.A)^\perp ⅋ \beta.B$. Then we have a $*$-autonomous category.

A simplified version of a category in which we might analyze Coquand's Intuition.

ASSOCIATIVITY REVISITED

Why do we avoid the problem for Blass games?

The example $\quad A, B\ \Sigma \quad C, D\ \Pi$

$$A \xrightarrow{\sigma}\!\!\circ\, B \qquad B \xrightarrow{\tau}\!\!\circ\, C \qquad C \xrightarrow{\rho}\!\!\circ\, D$$

$$\underset{\Pi}{A^{\perp}} \parr \underset{\Sigma}{B} \qquad \underset{\Pi}{B^{\perp}} \parr \underset{\Pi}{C} \qquad \underset{\Sigma}{C^{\perp}} \parr \underset{\Pi}{D}$$

Skolem function $\to 0$

Skolem function $\to 0$

Now in the composites

$$(\sigma ; \tau) ; \rho = \sigma ; (\tau ; \rho),$$

Both Skolem functions act (at once) independently.

PROOF AS PROCESS

Think of a WS strategy as a process in some kind of chemical abstract machine. That is the Skolem functions float around waiting to be triggered by appropriate inputs & (then waiting to) give up their output.

There is some degree of parallelism.