MULTICATEGORIES

IN AND AROUND

ALGEBRA

AND

LOGIC

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CONNECTIONS

(Not apparent, not in this order.)

- Algebraic logic

  Categorical proof theory

  Usual proof theory

- Duality

  Theories ⟷ Spaces of Models

  Completions

- Domains

  (Categorical generalizations)

  Completions
  Duality

- Coalgebras

  (Winskel's processes as profunctors)

  Linear logic

WHAT IS AN ALGEBRAIC THEORY

- Signatures, terms, equations
- Clones in universal algebra
- Lawvere theories
- (Finitary) monads
- Cartesian multicategories

ALGEBRAIC THEORIES

GROUPS

$$a.(b.c) = (a.b).c$$

$$a.e = a = e.a$$

$$a.\bar{a} = e = \bar{a}.a$$

BOOLEAN ALGEBRAS

$$\vdots$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$\vdots$$

$$a \wedge \bar{a} = 0 \qquad a \vee \bar{a} = 1$$

$$\vdots$$

RINGS

$$\vdots$$

OR

Monoids in abelian groups

OPERADIC THEORIES

(Linear setting)

Lie algebras

$$[x,y]+[y,x]=0$$

$$[x,[y,z]]=[[x,y],z]+[y,[x,z]]$$

Pre-Lie algebras

$$x.(y.z)-(x.y).z$$

$$=x.(z.y)-(x.z).y$$

$$\boxed{\text{COMBINATORICS}}$$

OPERADIC THEORIES

Associative algebras

$$a.(b.c) = (a.b).c$$

(Monoids + $a.e = a = e.a$)

Commutative associative algebras

$$a.(b.c) = (a.b).c$$

$$a.b = b.a \qquad \text{symmetric}$$

'Projection algebras'

$$a.(b.c) = (a.b).c$$

$$a.(b.c) = a.(c.b) \qquad \text{symmetric}$$

Dialgebras

Two associative operations $\dashv, \vdash$

\+

$$a \dashv (b \dashv c) = a \dashv (b \vdash c)$$

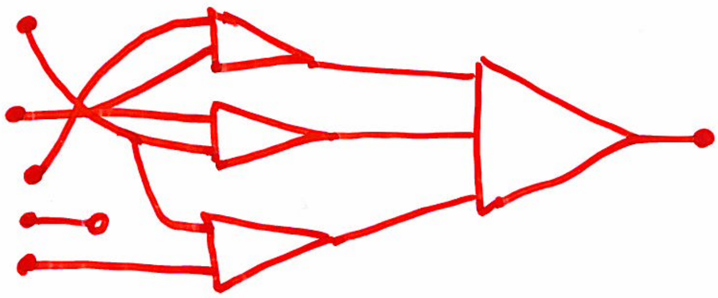
$$(a \vdash b) \vdash c = (a \dashv b) \vdash c$$

$$(a \vdash b) \dashv c = a \vdash (b \dashv c)$$

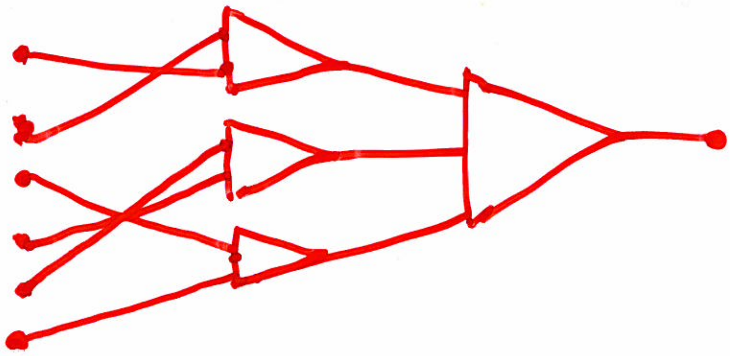
$\boxed{\text{COMBINATORICS}}$

TERMS : WIRING DIAGRAMS

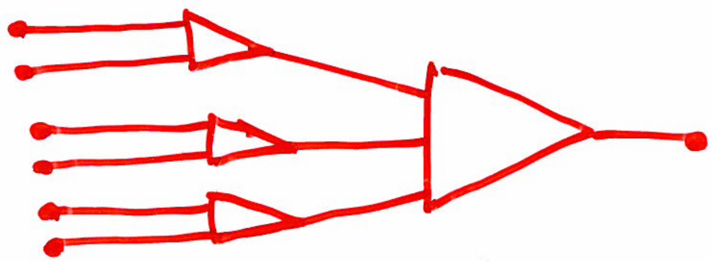
Algebraic theories



Operads



Non-symmetric operads



COMPUTATION

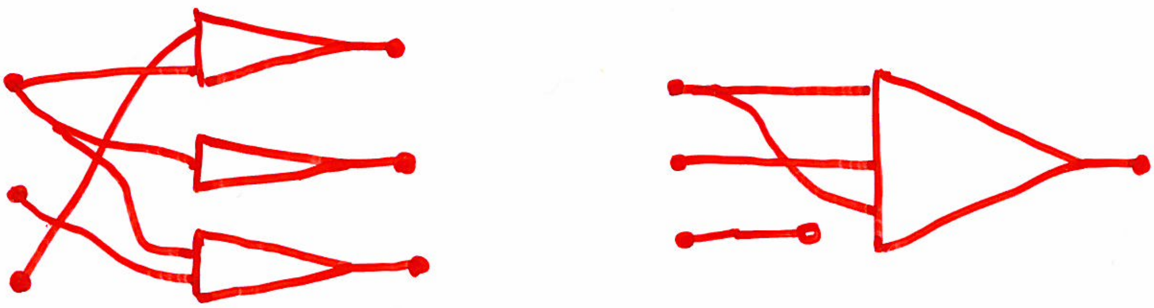
TO NORMAL FORM

Substitute

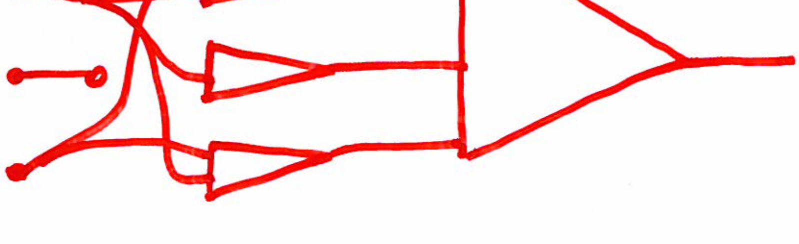
$g(x_3, x_1)$ for y_1

$h(x_1)$ for y_2 in $f(y_1, y_2, y_1)$

$k(x_1, x_2)$ for y_3



$$\Downarrow$$



ALGEBRAIC THEORIES

An outline

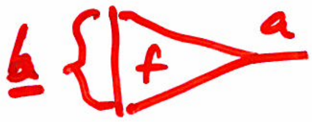
A : the $a \in A$ are sorts

SA : the arities of kind S based on A

$A \xrightarrow[F]{\;|\;} SB$: $f \in F$ has

inputs $\underline{b} \in SB$

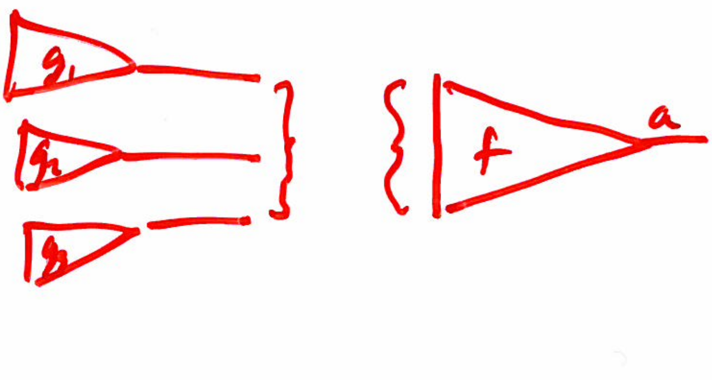
output $a \in A$



$$\frac{A \xrightarrow[F]{\;|\;} SB \qquad B \xrightarrow[G]{\;|\;} SC}{A \xrightarrow[GF]{\;\;|\;\;} SC}$$

composition controlled by S of

$f \in F$ and $g \in G$ viz



ALGEBRAIC THEORIES

Outline continued

An S-Algebraic theory on A

is a (pseudo-) monoid in $\mathrm{End}(A)$

viz

$$\begin{tikzcd} A \arrow[r, "M", "\shortmid" marking] & SA \end{tikzcd} \qquad \boxed{S\text{-Multicategory}}$$

viz with

$$\begin{tikzcd} A \arrow[r, bend left, "I", "\shortmid" marking] \arrow[r, bend right, "M"', "\shortmid" marking] \arrow[r, phantom, "\Downarrow"] & SA \end{tikzcd}$$

variables are terms

$$\begin{tikzcd} A \arrow[r, "M", "\shortmid" marking] \arrow[rrr, bend right, "M"', "\shortmid" marking] & SA \quad A \arrow[rr, "M", "\shortmid" marking] \arrow[d, phantom, "\Downarrow"] & & SA \\ & & & \end{tikzcd}$$

terms closed under substitution

and satisfying evident axioms

$$\boxed{\text{What can } S \text{ be?}}$$

THE RELATIONAL MODEL

FOR LINEAR LOGIC

Rel category of sets and relations

is compact closed ($\otimes = ⅋$)

has biproducts ($\oplus = \&$ of Hom sets)

can be equipped with a linear exponential comonad

(! is multisets : this is a modality $\square$)

Extension of Monads

Rel is the Kleisli category $Kl(P)$ of the power set monad on **Set**

$$\frac{A \nrightarrow B}{A \longrightarrow PB}$$

Given a monad S on **Set** we seek an extension $\tilde{S}$ on **Rel**

FACT We get an extension

$$\begin{tikzcd} \mathbf{Rel} \arrow[loop right, "\tilde{S}"] \\ \mathbf{Set} \arrow[u] \arrow[loop right, "S"] \end{tikzcd}$$

so long as S is (Eilenberg?) an operadic monad

So take S = commutative monoids extend to Rel and dualize to get !

DISTRIBUTIVE LAWS

$$\begin{tikzcd} \mathcal{C} \arrow[loop left, "S"] \arrow[loop right, "P"] \end{tikzcd}$$

monads

$$\begin{tikzcd} S\text{-Alg} \arrow[loop right, "\hat{P}"] \arrow[d, "\text{lifting}"] & \Longleftrightarrow & Kl(P) \arrow[loop right, "\tilde{S}"] \\ \mathcal{C} \arrow[loop right, "P"] & & \mathcal{C} \arrow[loop right, "S"] \arrow[u, "\text{extension}"'] \end{tikzcd}$$

$\Longleftrightarrow$ Distributive law $\Longleftrightarrow$

$$SP \longrightarrow PS$$

Then

PS is a monad

$$\hat{P}\text{-Alg} \simeq PS\text{-Alg} \qquad Kl(\tilde{S}) \simeq Kl(PS)$$

ANALOGY

• **Set**	**Cat**
category of sets	2-category of categories
• $\mathbf{Set} \circlearrowleft P$	$\mathbf{Cat} \circlearrowleft P$
power set monad	presheaf not quite monad
• $\mathbf{Rel} = Kl(P)$	$\mathbf{Prof} = Kl(P)$
$\uparrow$	$\uparrow$
Set	**Cat**
sets and functions extend to sets and relations	categories and functors extend to categories and profunctors

EXTENDING 2-MONADS

To extend S to $\mathbf{Prof}$ is to lift P to S-$\mathbf{Alg}$:-

$$
\begin{tikzcd}
A \arrow[rr, "y"] \arrow[dr, "f"'] & & P(A) \arrow[dl, dashed, "f^{+}"] & \text{Yoneda} \\
& P(B) & & \text{Left Kan extension}
\end{tikzcd}
$$

- If A an S-algebra then so is $P(A)$ and y 'preserves the structure'

- If f an S-algebra map then so is f^{+}

(Theory developed with Fiore, Gambino, Winskel)

MONADS WHICH EXTEND

S = categories with

- finite limits
- finite products *
- terminal object

Conjecture: not many property monads

S = categories with

- monoidal structure *
- symmetric monoidal structure *

⋮

Non-examples

colimits

compact closed

(cf conjecture) equalizers

S-MULTICATEGORIES

For any S which extends, an S-Multicategory is a monad in the Kleisli bicategory $Kl(\tilde{S})$.

That is,

a profunctor

$$A \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} SA$$

with

identity (variables)

composition (substitution)

2-cells, as in the outline earlier.

(General theory developed with John Power.)

THEORY OF OPERADS

Change of point of view.

E.g.

the natural place to find algebras for an operad $\mathcal{O}$ is in $Kl(\tilde{S})$: these are the
'algèbres tordues'
'twisted algebras'.

Other notions of algebra involve a change of base

LAMBDA CALCULUS

Definition An interpretation of the typed λ-calculus is a semi-closed cartesian multicategory. It is closed when we have η. An interpretation of the pure λ-calculus is a one-object (semi) closed cartesian multicategory

(Similar definition for linear case.)

A pre-closed category is a closed multicategory.

(Cf Eilenberg - Kelly 1966)

DIFFERENTAL CALCULUS I

- Additive setting
- New arities SA

the symmetric monoidal category generated by two versions (a) and a of $a \in A$

linear non-linear

with

$$a \to 1$$

$$a \to a \times a$$

$$a \to (a)$$

Differentiation on an S-multicategory

suitably natural plus axioms

DIFFERENTIAL CALCULUS II

Axioms

Structural — Weakening — 0

Contraction — $+$

Linear — constant + linear

Chain rule

(Project with Christine Tasson)