

Algebra and Logic

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Modern algebra and logic emerged at about the same time and were met with equal suspicion by many mathematicians. Hilbert, himself responsible for new forms of mathematical argument, later proposed to justify abstract mathematics in logical terms. This is known as the Hilbert programme - to establish the consistency of higher order mathematics in finitary terms. Gödel's celebrated Incompleteness Theorem shows that this idea cannot succeed generally; but there is a profound intuition behind it. Results in logic show that in many instances abstract ideas can be eliminated in favour of concrete ones. This has an interesting manifestation in the case of abstract algebra.

From a modern perspective, the concrete aspects of abstract algebra can be explained via the notion of classifying topos which arose in categorical logic. Typically the concrete algebraic manipulations reflect elementary properties of a classifying topos which can be presented in a completely explicit fashion. Then the seemingly problematic modern abstract formulation reflects a use of some form of the axiom of choice to establish the existence of enough points of the classifying topos, that is, enough models of the theory classified.

The basic ideas are best understood in terms of properties of very simple logical theories, which are easily appreciated. This talk will use these to explain arguments in abstract algebra with hidden computational content. It will be illustrated by leading examples derived from familiar Linear Algebra and elementary Commutative Algebra. Just at the end there will be a brief outline of the perspective of categorical logic.