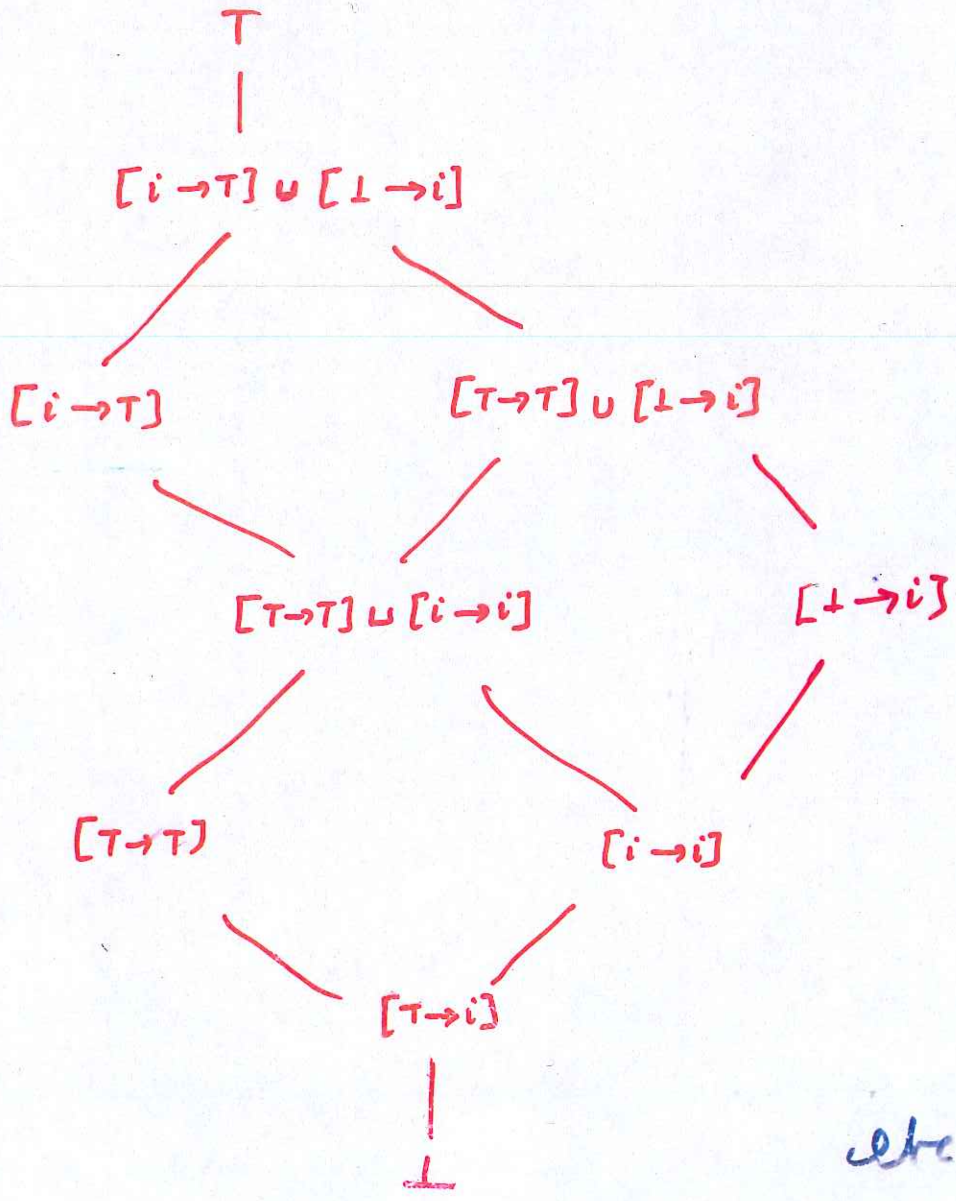
D_∞ Model

Start with $D_0 = \begin{pmatrix} \top \\ \bot \end{pmatrix}$

Set $D_1 = [D_0 \to D_0] = \begin{pmatrix} \top \\ i \\ \bot \end{pmatrix} = [\top \to \top]$

Then $D_2 = [D_1 \to D_1] =$



etc.

D_∞ CTD

Use retraction

$$\begin{tikzcd}
\top \arrow[r, shift left] & \top \arrow[l, shift left] \\
 & i \arrow[dl] \\
\bot \arrow[r, shift left] & \bot \arrow[l, shift left]
\end{tikzcd}
\qquad\qquad D_0 \triangleleft D_1$$

to generate

$$D_0 \triangleleft D_1 \triangleleft D_2 \triangleleft \cdots .$$

D_∞ is limit = colimit

i.e. take colimit in $\vee$-semilatts

to get $(D_\infty)_f$ and complete:

the algebraic lattice

$$D_\infty = \mathrm{Idl}\,((D_\infty)_f).$$

ALTERNATIVE VIEW

T = all terms

U = unknown

Use $\rightarrow$ with $X \rightarrow T = T$ all X

$T \rightarrow U = U$

$\cap$ theory $\cap \emptyset = T$

$\cap (A \rightarrow B_i) = A \rightarrow \cap B_i$

order theory $A' \subseteq A \Rightarrow (A \rightarrow B) \subseteq (A' \rightarrow B)$

Start with

$$\begin{array}{c} T \\ \cup\!\mid \\ U \end{array}$$

get

$$\begin{array}{c} T \\ \cup\!\mid \\ (U \rightarrow U) \\ \cup\!\mid \\ U \end{array}$$

and then

SECOND ITERATE

$$
\begin{tikzcd}
 & & T \arrow[d, phantom, "\supseteq"] & \\
 & & {[u \to [u \to u]]} \arrow[dl, phantom, "\supseteq"] \arrow[dr, phantom, "\supseteq"] & \\
 & {[[u \to u] \to [u \to u]]} \arrow[dl, phantom, "\supseteq"] \arrow[dr, phantom, "\supseteq"] & & {[u \to u]} \arrow[dl, phantom, "\supseteq"] \\
{[T \to [u \to u]]} \arrow[dr, phantom, "\supseteq"] & & {[[u \to u] \to [u \to u]] \cap [u \to u]} \arrow[dl, phantom, "\supseteq"] \arrow[dr, phantom, "\supseteq"] & \\
 & {[T \to [u \to u]] \cap [u \to u]} \arrow[dr, phantom, "\supseteq"] & & {[[u \to u] \to u]} \arrow[dl, phantom, "\supseteq"] \\
 & & {[T \to [u \to u]] \cap [[u \to u] \to u]} \arrow[d, phantom, "\supseteq"] & \\
 & & u &
\end{tikzcd}
$$

$\cap$ - Type Reading

M λ - term (A a type d_A corr'ing element)

$$[\![M]\!]_{D_\infty} = \bigvee \{ d_A \mid M : A \}$$

natural typing deductions

TYPE (IN)EQUALITY

$$A \leq A$$

$$A \leq B \;,\; B \leq C \implies A \leq C$$

$$A \leq C \quad B \leq D \implies A \cap B \leq C \cap D$$

$$\Omega \cap A = A \;,\; A \cap (B \cap C) = (A \cap B) \cap C \;,\; A \cap B = B \cap A$$

$$A \leq \Omega$$

$$A \cap B \leq A \qquad\qquad A \cap B \leq B$$

$$A \leq A \cap A$$

$$C \leq A \quad B \leq D \implies (A \to B) \leq (C \to D)$$

$$A \to \Omega = \Omega$$

$$A \to B \cap C = (A \to B) \cap (A \to C).$$

NATURAL TYPING

$$(x : A \text{ in } \Gamma) \qquad \Gamma \vdash x : A$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x . M : A \to B}$$

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B}$$

$$(A \leq B) \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash M : B}$$

NATURAL INTENSIONAL MODELS

[Think : ~~remove order~~, treat $\cap$ like a tensor product,

so types simply inductively generated in form

$$\bigcap_i (A_i \longrightarrow B_i)$$].

Take Rel with $! = M$ finite multisets as model for linear logic.

Solve

$$D \cong !(D \multimap D)$$

[$D = M(D \times D)$ as a set]

Have

$$E = (D \multimap D) \quad \text{and} \quad D \cong !E$$

$$D \underset{\beta}{\overset{\alpha}{\rightleftarrows}} !E$$

ABSTRACT FACT

CLAIM

E is a model for the λ-calculus (i.e. E is reflexive in the ccc).

Pf :-

The ccc map $E \longrightarrow [E \Rightarrow E] = \,!E \multimap E$

is obtained from the linear

$$E = [D \multimap D] \xrightarrow[\beta \multimap \alpha]{} [!E \multimap !E] \xrightarrow[!E \multimap \varepsilon]{} [!E \multimap E] = [E \Rightarrow E]$$

The ccc map $[E \Rightarrow E] \longrightarrow E$ is non-linear given by

$$!(!E \multimap E) \xrightarrow{m} (!!E \multimap !E) \xrightarrow{\delta \multimap !E} [!E \multimap !E] \xrightarrow[\alpha \multimap \beta]{} (D \multimap D) = E$$

The composite

$$(E \Rightarrow E) \longrightarrow (E \Rightarrow E)$$

is given by

$$\begin{tikzcd}
!(!E \multimap E) \arrow[r, "m"] \arrow[d, "\varepsilon"'] & (!!E \multimap !E) \arrow[r, "\delta \multimap !E"] \arrow[d, "!!E \multimap \varepsilon"] & (!E \multimap !E) \arrow[d, "!E \multimap \varepsilon"] \\
(!E \multimap E) \arrow[r, "\varepsilon \multimap E"'] \arrow[rr, bend right, "\text{identity}"'] & (!!E \multimap E) \arrow[r, "\delta \multimap E"'] & (!E \multimap E)
\end{tikzcd}$$

Done.

MODEL FOR $\beta\eta$

[Think: add the equations of form

$$(A \to \bigcap_i B_i) = \bigcap_i (A \to B_i)$$

Then normal form for types is

$$\bigcap_i [A_i \to B_i]$$

"purely functional type"]

Write F for the purely functional
D for all types,

and solve in Rel

$$F \cong D \multimap F$$
$$D \cong\ !F$$

and so

$$F \cong\ !F \multimap F \qquad \left[\begin{array}{c} F = M(F) \times F \\ (F \neq \emptyset) \end{array}\right]$$

Then $F \cong [F \Rightarrow F]$ in the ccc.

ALGEBRAIC LATTICE MODELS

An algebraic lattice $A = \mathrm{Idl}(A_f)$

$\uparrow$ $\vee$-semi-lattice

To give $A \Rightarrow B$ take the free $\vee$-semilatt on $[a \Rightarrow b]$ subject to

$$(a \Rightarrow \vee b_i) = \vee (a \Rightarrow b_i)$$

$$a_1 \leq a_2 \text{ implies } [a_2 \Rightarrow b] \leq [a_1 \Rightarrow b]$$

Notn $(A \Rightarrow B)_f = (A_f \Rightarrow B_f)$

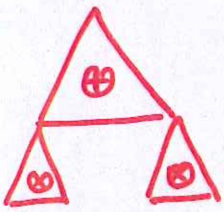
Suppose D an algebraic lattice with $(D_f \Rightarrow D_f)$ a sublattice of D_f.

Then $(D \Rightarrow D) \triangleleft D$ and D is a model of the λ-calculus.

[D_∞ has $(D_\infty)_f = (D_\infty)_f \Rightarrow (D_\infty)_f$]

LUDICS

Game semantics in a positive (Player starts) version



$$\bigoplus_i \left(\bigotimes_j P_{ij} \right)$$

1st move : P plays an i

$$P_{ij} = \downarrow \left(\&_k \parr_\ell N_{k\ell} \right)$$

2nd move : O chooses to respond to a j and plays a corresponding k.

Ludics a universal setting where

$$\bigoplus_{J \in \mathcal{P}} \qquad \mathcal{P} = \text{finite power set of } \mathbb{N}$$

and for each J we take the J-fold tensor of the dual.

ABSTRACT LUDICS

Positive view

$$D = \bigoplus_{J \in P} \left(\bigotimes_{J} D^{\perp} \right)$$

$$= \bigoplus_{J \in P} \left(\left(\bigparr_{J} D \right)^{\perp} \right)$$

Negative view

$$E = \mathop{\&}_{J \in P} \left(\bigparr_{J} E^{\perp} \right)$$

$$= \mathop{\&}_{J \in P} \left(\left(\bigotimes_{J} E \right)^{\perp} \right)$$

$$= \prod_{J \in P} \left(\bigotimes_{J} E \multimap R \right)$$

$$\left(E^{\otimes J} \multimap R \right)$$

SOME MODELS

Linear case

Solve
$$D = \prod_{J \in \mathcal{P}} \left(\bigotimes_J D \multimap R \right)$$
in Rel (with $R = 1$ say) that is
$$D = \sum_{J \in \mathcal{P}} D^J$$
(Compare $D = 1 + D + D^2 + \cdots$)

Non linear case

Solve
$$D = \prod_{J \in \mathcal{P}} \left(D^J \Rightarrow R \right)$$
in Algebraic lattices (with $R = \left(\begin{smallmatrix} \top \\ \Omega \end{smallmatrix}\right)$ say).

NEGATIVE VIEW

(Think types)

Positive type $\quad R$

Negative types $\quad \Pi(\otimes N \multimap R)$

Curien's terms (for type free)

Positive $\quad P = \Omega \mid \dagger \mid N_I \quad (N_i)_{i \in I}$

Negative $\quad N = (\lambda x_J . P)_J$