RELATIVE PSEUDOMONADS, KLEISLI BICATEGORIES, AND SUBSTITUTION MONOIDAL STRUCTURES

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Abstract. We introduce the notion of a relative pseudomonad, which generalizes the notion of a pseudomonad, and define the Kleisli bicategory associated to a relative pseudomonad. We then present an efficient method to define pseudomonads on the Kleisli bicategory of a relative pseudomonad. The results are applied to define several pseudomonads on the bicategory of profunctors in an homogeneous way and provide a uniform approach to the definition of bicategories that are of interest in operad theory, mathematical logic, and theoretical computer science.

1. Introduction

Just as classical monad theory provides a general approach to study algebraic structures on objects of a category (see [5] for example), 2-dimensional monad theory offers an elegant way to investigate algebraic structures on objects of a 2-category [10, 30, 34, 37, 54, 56]. Even if the strict notion of a 2-monad is sufficient to develop large parts of the theory, the strictness requirements that are part of its definition are too restrictive for some applications and it is necessary to work with the notion of a pseudomonad [12], in which the diagrams expressing the associativity and unit axioms for a 2-monad commute up to specified invertible modifications, rather than strictly. In recent years, pseudomonads have been studied extensively [14, 39, 49, 50, 52, 53, 62].

Our general aim here is to develop further the theory of pseudomonads. In particular, we introduce relative pseudomonads, which generalize pseudomonads, define the associated Kleisli bicategory of a relative pseudomonad, and describe a method to extend a 2-monad on a 2-category to a pseudomonad on the Kleisli bicategory of a relative pseudomonad. We use this method to show how several 2-monads on the 2-category **Cat** of small categories and functors can be extended to pseudomonads on the bicategory **Prof** of small categories and profunctors (also known as bimodules or distributors) [8, 45, 60]. This result has applications in the theory of variable binding [22, 24, 55, 61], concurrency [13], species of structures [23], models of the differential λ-calculus [21], and operads and multicategories [15, 16, 17, 25, 27].

For these applications, one would like to regard the bicategory of profunctors as a Kleisli bicategory and then use the theory of pseudo-distributive laws [34, 49, 50], i.e. the 2-dimensional counterpart of Beck's fundamental work on distributive laws [6] (see [58] for an abstract treatment). In order to carry out this idea, one is naturally led to try to consider the presheaf construction, which sends a small category $\mathbb{X}$ to its category of presheaves $P(\mathbb{X}) =_{\text{def}} [\mathbb{X}^{\text{op}}, \mathbf{Set}]$, as a pseudomonad. Indeed, a profunctor $F\colon \mathbb{X} \to \mathbb{Y}$, i.e. a functor $F\colon \mathbb{Y}^{\text{op}} \times \mathbb{X} \to \mathbf{Set}$, can be identified with a functor $F\colon \mathbb{X} \to P(\mathbb{Y})$. However, the presheaf construction fails to be a pseudomonad for size reasons, since it sends small categories to locally small ones, making it impossible to define a multiplication. Although some aspects of the theory can be developed restricting the attention to small presheaves [19], which support the structure of a pseudomonad,

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some of our applications naturally involve general presheaves and thus require us to deal not only with coherence but also size issues.

In order to do so, we introduce the notion of a relative pseudomonad (Definition 3.1), which is based on the notions of a relative monad [1, Definition 2.1] and of a no-iteration pseudomonad [53, Definition 2.1]. These notions are, in turn, inspired by Manes' notion of a Kleisli triple [47], which is equivalent to that of a monad, but better suited to define Kleisli categories (see also [51, 64]). For a pseudofunctor between bicategories $J\colon \mathcal{C} \to \mathcal{D}$ (which in our main example is the inclusion $J\colon \mathbf{Cat} \to \mathbf{CAT}$ of the 2-category of small categories into the 2-category of locally small categories), the core of the data for a relative pseudomonad T over J consists of an object $TX \in \mathcal{D}$ for every $X \in \mathcal{C}$, a morphism $i_X\colon JX \to TX$ for every $X \in \mathcal{C}$, and a morphism $f^*\colon TX \to TY$ for every $f\colon JX \to TY$ in $\mathcal{D}$. This is as in a relative monad, but the equations for a relative monad are replaced in a relative pseudomonad by families of invertible 2-cells satisfying appropriate coherence conditions, as in a no-iteration pseudomonad. As we will see in Theorem 4.1, these conditions imply that every relative pseudomonad T over $J\colon \mathcal{C} \to \mathcal{D}$ has an associated Kleisli bicategory $\mathrm{Kl}(T)$, defined analogously to the one-dimensional case. In our main example, the presheaf construction gives rise to a relative pseudomonad over the inclusion $J\colon \mathbf{Cat} \to \mathbf{CAT}$ in a natural way and it is then immediate to identify its Kleisli bicategory with the bicategory of profunctors. It should be noted here that the presheaf construction is neither a no-iteration pseudomonad (because of size issues) nor a relative monad (because of strictness issues).

As part of our development of the theory of relative pseudomonads, we show how relative pseudomonads generalize no-iteration pseudomonads (Proposition 3.3) and hence (by the results in [53]) also pseudomonads. We then introduce relative pseudoadjunctions, which are related to relative pseudomonads just as adjunctions are connected to monads. In particular, we show that every relative pseudoadjunction gives rise to a relative pseudomonad (Theorem 3.8) and that the Kleisli bicategory associated to a relative pseudomonad fits in a relative pseudoadjunction (Theorem 4.4).

Furthermore, we introduce and study the notion of a lax idempotent relative pseudomonad, which appears to be the appropriate counterpart in our setting of the notion of a lax idempotent 2-monad (often called Kock-Zöberlein doctrines) [37, 38, 65] and pseudomonad [48, 52, 60]. In Theorem 5.3 we will give several equivalent characterizations of lax idempotent relative pseudomonad and combine this result with the analogous one in [52] to show that a pseudomonad is lax idempotent as a pseudomonad in the usual sense only if it is lax idempotent as a relative pseudomonad in our sense. This notion is of interest since it allows us to exhibit examples of relative pseudomonads by reducing the verification of the coherence axioms for a relative pseudomonad to the verification of certain universal properties. In particular, the relative pseudomonad of presheaves can be constructed in this way.

We then consider the question of when a 2-monad on the 2-category $\mathbf{Cat}$ of small categories can be extended to a pseudomonad on the bicategory $\mathbf{Prof}$ of profunctors. Rather than adapting the theory of distributive laws to relative pseudomonads along the lines of what has been done for no-iteration monads [51], which would involve complex calculations with coherence conditions, we establish directly that, for a pseudofunctor $J\colon \mathcal{C} \to \mathcal{D}$ of 2-categories, a 2-monad $S\colon \mathcal{D} \to \mathcal{D}$ restricting to $\mathcal{C}$ along J in a suitable way, and a relative pseudomonad T over J, if T admits a lifting to 2-categories of strict algebras or pseudoalgebras for S, then S admits an extension to the Kleisli bicategory of T (Theorem 6.3). We do so bypassing the notion of a pseudodistributive law in a counterpart of Beck's result.

This result is well-suited to our applications, where the structure that manifests itself most naturally is that of a lift of the relative pseudomonad of presheaves to various 2-categories of categories equipped with algebraic structure, often via forms of Day's convolution monoidal structure [18, 31]. In particular, our results will imply that the 2-monads for several important notions (categories with terminal object, categories with finite products, categories with finite limits, monoidal categories, symmetric monoidal categories, unbiased monoidal categories, unbiased symmetric monoidal categories, strict monoidal categories, and symmetric strict monoidal categories) can be extended to pseudomonads on the bicategory of profunctors. A reason for interest in this result is that the compositions in the Kleisli bicategories of these pseudomonads can be understood as variants of the substitution monoidal structure that can be used to characterize notions of operad [36, 57].

As an illustration of the applications of our theory, we discuss our results in the special case of the 2-monad S for symmetric strict monoidal categories, showing how it can be extended to a pseudomonad on the bicategory of profunctors. This result is the cornerstone of the understanding of the bicategory of generalized species of structures defined in [23] as a 'categorified' version of the relational model of linear logic [29, 28] and leads to showing that the substitution monoidal structure that gives rise to the notion of a coloured operad [4] is a special case of the composition in the Kleisli bicategory. The results presented here are intended to make these ideas precise by dealing with both size and coherence issues in a conceptually clear way.

Organization of the paper. Section 2 reviews some background material on 2-monads, pseudomonads and their algebras. Our development starts in Section 3, where we introduce relative pseudomonads, relate them to no-iteration pseudomonads and ordinary pseudomonads, introduce relative pseudoadjunctions and establish a connection between relative pseudoadjunctions and relative pseudomonads. Section 4 defines the Kleisli bicategory associated to a relative pseudomonad and discusses some of its basic properties. In Section 5 we introduce and study lax idempotent relative pseudomonads. Section 6 shows that an extension of a relative pseudomonad T to 2-categories of strict algebras or pseudoalgebras for a 2-monad S induces an extension of S to the Kleisli bicategory of T, as well as a composite relative pseudomonad TS. We conclude the paper in Section 7 by discussing applications of our theory and showing how several 2-monads on **Cat** can be extended to pseudomonads on **Prof**.

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2. **Background**

2-categories and 2-monads. We assume that readers are familiar with the fundamental aspects of the theory of 2-categories and of bicategories (as presented, for example, in [7, 11, 40]) and confine ourselves to review some facts that will be used in the following and to fix notation and conventions.

For a 2-category $\mathcal{C}$ and a pair of objects $X, Y \in \mathcal{C}$, we write $\mathcal{C}[X, Y]$ for the hom-category of morphisms $f\colon X \to Y$ and 2-cells between them, which we denote with lower-case Greek letters, $\phi\colon f \to f'$. Two parallel morphisms $f, f'\colon X \to Y$ are said to be *isomorphic* if they are isomorphic as objects of $\mathcal{C}[X, Y]$, and we write $f \cong f'$ in this case. We write **CAT** for the 2-category of locally small categories, functors, and natural transformations. Its full sub-2-category spanned by small categories will be written **Cat**. We then have an inclusion $J\colon \mathbf{Cat} \to \mathbf{CAT}$. We use the terms *pseudofunctor*, *pseudonatural transformation*, and *pseudoadjunction* rather than homomorphism, strong natural transformation, and biadjunction, respectively.

Let us now review some aspects of 2-dimensional monad theory [10]. By a 2-*monad* on a 2-category $\mathcal{C}$ we mean a 2-functor $S\colon \mathcal{C} \to \mathcal{C}$ equipped with 2-natural transformations $m\colon S^2 \to S$ and $e\colon 1_{\mathcal{C}} \to S$, called the *multiplication* and *unit* of the 2-monad, respectively, satisfying the usual axioms for a monad in a strict sense. As usual, we often refer to a 2-monad by mentioning only its underlying 2-functor, leaving implicit the rest of its data. Similar conventions will be used for other kinds of structures considered in the rest of the paper.

For a 2-category $\mathcal{C}$ and 2-monad $S\colon \mathcal{C} \to \mathcal{C}$, we write Ps-$S$-Alg$_{\mathcal{C}}$ (or Ps-S-Alg if no confusion arises) for the 2-category of pseudoalgebras, pseudomorphisms and algebra 2-cells, and S-Alg$_{\mathcal{C}}$ (or S-Alg) for the locally full sub-2-category of Ps-S-Alg$_{\mathcal{C}}$ spanned by strict algebras. Here, by a *pseudoalgebra* we mean an object $A \in \mathcal{C}$, called the *underlying object* of the algebra, equipped with a morphism $a\colon SA \to A$, called the *structure map* of the algebra, and invertible 2-cells

$$\begin{array}{ccc} S^2A & \xrightarrow{Sa} & SA \\ {\scriptstyle m_A}\downarrow & \Downarrow \bar{a} & \downarrow{\scriptstyle a} \\ SA & \xrightarrow[a]{} & A\,, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e_A} & SA \\ & {\scriptstyle 1_A}\searrow \;\overset{\tilde{a}}{\Rightarrow} & \downarrow{\scriptstyle a} \\ & & A\,, \end{array}$$

called the *associativity* and *unit* 2-cells of the algebra, subject to two coherence axioms [59]. We have a *strict algebra* when the associativity and unit 2-cells are identities, in which case (as in analogous cases below) the coherence conditions are satisfied trivially. For pseudoalgebras A and B (and in particular for strict algebras), a pseudomorphism from A to B consists of a morphism $f\colon A \to B$ and an invertible 2-cell

$$\begin{array}{ccc} SA & \xrightarrow{Sf} & SB \\ {\scriptstyle a}\downarrow & \Downarrow \bar{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B\,, \end{array} \tag{2.1}$$

required to satisfy two coherence axioms [10, 59]. For pseudomorphisms $f, g\colon A \to B$, an *algebra 2-cell* between them is a 2-cell $\alpha : f \to g$ that satisfies one coherence axiom [10]. We have a forgetful 2-functor $U\colon$ Ps-S-Alg $\to \mathcal{C}$ with a left pseudoadjoint $F\colon \mathcal{C} \to$ Ps-S-Alg, defined by mapping an object $X \in \mathcal{C}$ to the *free algebra* on X, which is the strict algebra having SX as its underlying object and $m_X\colon S^2X \to SX$ as its structure map. The components of the unit of the pseudoadjunction are the components of the unit of the 2-monad.

In our applications, we will consider several 2-monads on **CAT** (restricting to **Cat** in an evident way), for which we invite the readers to consult [10, 44, 46]. Among them, the 2-monads for (strict) monoidal categories, symmetric (strict) monoidal categories, categories with finite limits, categories with finite products, and categories with a terminal object.

Bicategories and pseudomonads. For a bicategory $\mathcal{C}$, we write the associativity and unit isomorphisms as natural families of invertible 2-cells

$$(h\,g)\,f \xrightarrow{\cong} h\,(g\,f)\ , \quad 1_Y f \xrightarrow{\cong} f\ , \quad f \xrightarrow{\cong} f\,1_X\,. \tag{2.2}$$

which we leave unnamed. By the coherence theorem for bicategories [43] (which also follows from the bicategorical Yoneda lemma [60], see [26]), every bicategory is biequivalent to a 2-category. In virtue of this, we shall often treat bicategories as if they were 2-categories.

Example 2.1. Fundamental to our applications is the bicategory **Prof** of profunctors [8, 45, 60]. Its objects are small categories; and for small categories $\mathbb{X}$ and $\mathbb{Y}$, the hom-category $\mathbf{Prof}[\mathbb{X}, \mathbb{Y}]$ is defined to be $\mathbf{CAT}[\mathbb{Y}^{\mathrm{op}} \times \mathbb{X}, \mathbf{Set}]$. The composite of profunctors $F\colon \mathbb{X} \to \mathbb{Y}$ and $G\colon \mathbb{Y} \to \mathbb{Z}$ is given by the profunctor $G \circ F \colon \mathbb{X} \to \mathbb{Z}$ defined by the coend formula

$$(G \circ F)(z,x) =_{\mathrm{def}} \int^{y \in \mathbb{Y}} G(z,y) \times F(y,x)\,. \tag{2.3}$$

For a small category $\mathbb{X}$, the identity profunctor $\mathrm{Id}_{\mathbb{X}} \colon \mathbb{X} \to \mathbb{X}$ is defined by letting

$$\mathrm{Id}_{\mathbb{X}}(x,y) =_{\mathrm{def}} \mathbb{X}[x,y]\,. \tag{2.4}$$

It remains to prove that these definitions give rise to a bicategory. In the literature, it is often suggested that this can be proved by direct calculations, left to the readers. In Section 3, instead, we will give a more conceptual proof by describing **Prof** as the Kleisli bicategory associated to the relative pseudomonad of presheaves.

A *pseudomonad* on a bicategory $\mathcal{C}$ is given by a pseudofunctor $T \colon \mathcal{C} \to \mathcal{C}$, pseudonatural transformations $n \colon T^2 \to T$ and $i \colon 1_{\mathcal{C}} \to T$, called the *multiplication* and *unit* of the pseudomonad, respectively, and invertible modifications α, ρ, and λ , called the *associativity*, *right unit*, and *left unit*, respectively, of T, fitting in the diagrams

$$\begin{array}{ccc} T^3 & \xrightarrow{Tn} & T^2 \\ {\scriptstyle nT}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle n} \\ T^2 & \xrightarrow[n]{} & T\,, \end{array} \qquad \begin{array}{ccccc} T & \xrightarrow{iT} & T^2 & \xleftarrow{Ti} & T \\ & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle n} & \overset{\rho}{\Rightarrow} & \\ & {\scriptstyle 1}\searrow & T & \swarrow{\scriptstyle 1} & \end{array} \tag{2.5}$$

and subject to two coherence conditions [39]. The notions of a strict algebra and pseudoalgebra, of strict morphism and pseudomorphism, and of algebra 2-cell make sense also for pseudomonads, giving rise to bicategories Ps-S-Alg and S-Alg. When $\mathcal{C}$ is a 2-category, these are again 2-categories.

Every pseudomonad has also an associated Kleisli bicategory [14], which can be defined in complete analogy with the one-dimensional case; but we do not spell this out since we will give an alternative account of the Kleisli construction in Section 3. Importantly, in constrast with the situation for algebras discussed above, the Kleisli construction for a pseudomonad $T \colon \mathcal{C} \to \mathcal{C}$ produces only a bicategory even when $\mathcal{C}$ is a 2-category, with the associativity and unit isomorphisms of T used to give the associativity and unit isomorphisms of the Kleisli bicategory (see also Theorem 4.1 below).

3. Relative pseudomonads

In ordinary category theory, the notion of a monad has an equivalent alternative presentation, via the notion of a Kleisli triple [47], which is particularly convenient to define Kleisli categories. The notion of a Kleisli triple admits a natural generalization, given by the notion of a relative monad [1], which is obtained by allowing the underlying mapping on objects of the Kleisli triple to be defined relative to a fixed functor (see [1] for details). Similarly, in 2-dimensional category theory, the notion of a pseudomonad can be rephrased equivalently as the notion of a no-iteration pseudomonad [53], which is the 2-dimensional analogue of the notion of a Kleisli triple. Here, we introduce relative pseudomonads, which generalize no-iteration pseudomonads in the same way as relative monads generalize Kleisli triples, i.e. by allowing the mapping on objects that is part of a no-iteration pseudomonad to be defined relatively to a fixed pseudofunctor between bicategories. From now until the end of this section, we consider a fixed pseudofunctor between bicategories $J \colon \mathcal{C} \to \mathcal{D}$.

Definition 3.1. A *relative pseudomonad* T over $J \colon \mathcal{C} \to \mathcal{D}$ consists of

- an object $TX \in \mathcal{D}$, for every $X \in \mathcal{C}$,

- a family of functors $(-)^*_{X,Y} : \mathcal{D}[JX, TY] \to \mathcal{D}[TX, TY]$ for $X, Y \in \mathcal{C}$,
- a family of morphisms $i_X : JX \to TX$ in $\mathcal{D}$ for $X \in \mathcal{C}$,
- a natural family of invertible 2-cells $\mu_{g,f} : (g^*\, f)^* \to g^*\, f^*$, for $f : JX \to TY$, $g : JY \to TZ$,
- a natural family of invertible 2-cells $\eta_f : f \to f^*\, i_X$, for $f : JX \to TY$ in $\mathcal{D}$,
- a family of invertible 2-cells $\theta_X : {i_X}^* \to 1_{TX}$, for $X \in \mathcal{C}$,

such that the following conditions hold:

- for every $f : JX \to TY$, $g : JY \to TZ$, $h : JZ \to TV$, the diagram

$$\begin{array}{ccc} & ((h^*\, g)^* f)^* & \\ {\scriptstyle (\mu_{h,g}\, f)^*}\swarrow & & \searrow{\scriptstyle \mu_{h^*g,f}} \\ ((h^* g^*) f)^* & & (h^* g)^* f^* \\ \cong\big\downarrow & & \big\downarrow{\scriptstyle \mu_{h,g}\, f^*} \\ (h^*(g^* f))^* & & (h^* g^*) f^* \\ {\scriptstyle \mu_{h,g^*f}}\big\downarrow & & \big\downarrow\cong \\ h^*(g^* f)^* & \xrightarrow[h^*\, \mu_{g,f}]{} & h^*(g^* f^*) \end{array} \tag{3.1}$$

  commutes, and
- for every $f : JX \to TY$, the diagram

$$\begin{array}{ccccc} f^* & \xrightarrow{(\eta_f)^*} & (f^*\, i_X)^* & \xrightarrow{\mu_{f,i_X}} & f^*\, {i_X}^* \\ & \searrow^{\cong} & & & \big\downarrow{\scriptstyle f^*\, \theta_X} \\ & & & & f^*\, 1_{TX} \end{array} \tag{3.2}$$

  commutes.

We introduce some terminology. For a relative pseudomonad T over $J : \mathcal{C} \to \mathcal{D}$ as in Definition 3.1, we refer to the family of morphisms $i_X : JX \to TX$, for $X \in \mathcal{C}$, as the *unit* of T, and to the family of 2-cells μ, η, and θ as the *associativity*, *right unit*, and *left unit* of T, respectively. Finally, we refer to the axioms in (3.1) and (3.2) as the *associativity* and *unit axioms* for T, respectively. Note that in order to simplify the notation we have omitted the subscripts on the functors $(-)^*_{X,Y}$ and we will henceforth continue to do so. We furthermore adopt the convention of writing X rather than i_X in a subscript of μ and θ. So, for example, we have

$$\mu_{f,X} : (f^* i_X)^* \to f^*\, {i_X}^*\,,$$
$$\eta_X : i_X \to {i_X}^*\, i_X\,.$$

We also omit the detailed definition of some 2-cells in diagrams, labelling arrows only with the main 2-cell involved in its definition, and omitting subscripts. In all such cases, the precise definition of the 2-cell can be easily deduced from its domain and codomain.

We wish to make precise in what sense relative pseudomonads are a generalization of no-iteration pseudomonads [53, Definition 2.1]. This will be useful in order to relate relative pseudomonads and ordinary pseudomonads.

Lemma 3.2. *Let T be a relative pseudomonad over $J : \mathcal{C} \to \mathcal{D}$.*

(i) For every $f : JX \to TY$ and $g : JY \to TZ$, the diagram

$$\begin{array}{ccc} g^* f & \xrightarrow{\eta_{g^* f}} & (g^* f)^* i_X \\ & \searrow^{g^* \eta_f} & \downarrow{\scriptstyle \mu_{g,f}} \\ & & g^* f^* i_X \end{array}$$

commutes.

(ii) For every $f : JX \to TY$, the diagram

$$\begin{array}{ccc} (i_Y{}^* f)^* & \xrightarrow{\mu_{Y,f}} & i_Y{}^* f^* \\ & \searrow^{(\theta_Y f)^*} & \downarrow{\scriptstyle \theta_Y f^*} \\ & & f^* \end{array}$$

commutes.

(iii) For every $X \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} i_X & \xrightarrow{\eta_X} & i_X{}^* i_X \\ & \searrow^{1} & \downarrow{\scriptstyle \theta_X i_X} \\ & & i_X \end{array}$$

commutes.

Proof. The proof is a modified version of the proof of the redundancy of three axioms in the original definition of a monoidal category [33] (see also [32]), which has a version also for pseudomonads [48, Proposition 8.1]. □

Proposition 3.3. *A no-iteration pseudomonad is the same thing as a relative pseudomonad over the identity.*

Proof. The two notions involve exactly the same data, except for the direction of the invertible 2-cells. Then, using the numbering of axioms for a no-iteration pseudomonad in [53, Definition 2.1], the equivalence between axioms for a relative pseudomonad and those for a no-iteration pseudomonad are given as follows:

Relative pseudomonads		*No-iteration pseudomonads*
Naturality of μ	$\Leftrightarrow$	Axioms 6 and 7
Naturality of η	$\Leftrightarrow$	Axiom 4
Associativity axiom	$\Leftrightarrow$	Axiom 8
Unit axiom	$\Leftrightarrow$	Axiom 2
Lemma 3.2, part (i)	$\Leftrightarrow$	Axiom 5
Lemma 3.2, part (ii)	$\Leftrightarrow$	Axiom 3
Lemma 3.2, part (iii)	$\Leftrightarrow$	Axiom 1.

Note that it follows that Axioms 1, 3 and 5 for a no-iteration pseudomonad in [53, Definition 2.1] are redundant, in that they can be derived from the others. □

The next remarks use Proposition 3.3 and the analysis of the relationship between ordinary pseudomonads and no-iteration pseudomonads in [53] to show how a pseudomonad can be regarded as a relative pseudomonad over the identity $1_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ and, conversely, how every relative pseudomonad over the identity determines a pseudomonad.

Remark 3.4 (From pseudomonads to relative pseudomonads)*.* The combination of [53, Theorem 6.1] and Proposition 3.3 shows that every pseudomonad $T \colon \mathcal{C} \to \mathcal{C}$ with data as in Section 2 induces a relative pseudomonad over the identity $1_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$. Explicitly, for $X \in \mathcal{C}$, we already have $TX \in \mathcal{C}$ and a morphism $i_X \colon X \to TX$ as part of the pseudomonad structure. For a morphism $f \colon X \to TY$, we define $f^* \colon TX \to TY$ by letting $f^* =_{\mathrm{def}} n_Y\, T(f)$. The three families of invertible 2-cells μ, η, θ for a pseudomonad are then obtained in an evident way. For example, for $f \colon X \to TY$, we let $\eta_f \colon f \to f^* i_X$ be the composite 2-cell

$$f \xrightarrow{\lambda} n_Y\, i_{TY}\, f \xrightarrow{\cong} n_Y\, T(f)\, i_X\,.$$

where the unnnamed isomorphism 2-cell is a pseudonaturality of i.

Remark 3.5 (From relative pseudomonad over the identity to pseudomonads)*.* The combination of Proposition 3.3 and [53, Theorem 3.6] shows that every relative pseudomonad over an identity pseudofunctor induces a pseudomonad. The explicit definitions are a bit involved, and therefore checking the coherence diagrams directly is not straightforward, but we shall outline a more conceptual account of the construction of a pseudomonad from a relative pseudomonad in Remark 4.5.

We introduce a generalization of the notion of pseudoadjunction between bicategories [12, 60], extending to the 2-categorical setting the notion of a relative adjunction considered in [63] and [1, Section 2.2].

Definition 3.6. Let $G \colon \mathcal{E} \to \mathcal{D}$ be a pseudofunctor. A *relative left pseudoadjoint* F to G over $J \colon \mathcal{C} \to \mathcal{D}$, denoted

$$\begin{array}{ccc} & & \mathcal{E} \\ & \overset{F}{\nearrow} & \big\downarrow G \\ \mathcal{C} & \xrightarrow[J]{} & \mathcal{D}\,, \end{array}$$

consists of

- an object $FX \in \mathcal{E}$, for every object $X \in \mathcal{C}$;
- a family of morphisms $i_X \colon JX \to GFX$, for $X \in \mathcal{C}$;
- a family of adjoint equivalences

$$\mathcal{D}[JX, GA] \mathrel{\substack{\xrightarrow{(-)^\sharp} \\ \bot \\ \xleftarrow[G(-)\, i_X]{}}} \mathcal{E}[FX, A]\,, \tag{3.3}$$

  for $X \in \mathcal{C}$, $A \in \mathcal{E}$.

For a relative pseudoadjunction as in Definition 3.6, the components of the unit and counit of the adjoint equivalences in (3.3) will be written

$$\eta_f \colon f \to G(f^\sharp)\, i_X\,, \quad \varepsilon_u \colon (G(u)\, i_X)^\sharp \to u\,,$$

respectively, where $f \colon JX \to GA$ and $u \colon FX \to A$. Note that a relative pseudoadjunction over the identity $1_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ is equivalent to a pseudoadjunction in the usual sense [35, 60]. We now establish a relative variant of a standard fact that a pseudoadjunction of bicategories gives rise to a pseudomonad [35, 60], namely that a relative pseudoadjunction determines a relative pseudomonad. The next lemma will be useful.

Lemma 3.7. *Let*

$$\begin{array}{ccc} & & \mathcal{E} \\ & \overset{F}{\nearrow} & \downarrow G \\ \mathcal{C} & \xrightarrow[J]{} & \mathcal{D} \end{array}$$

be a relative pseudoadjunction. Then there is an essentially unique way of extending the function mapping $X \in \mathcal{C}$ to $FX \in \mathcal{E}$ to a pseudofunctor $F \colon \mathcal{C} \to \mathcal{E}$ so that the maps $i_X \colon JX \to GFX$, for $X \in \mathcal{C}$, become the 1-cell components of a pseudonatural transformation $i \colon J \Rightarrow GF$.

Proof. For a morphism $f \colon X \to Y$ in $\mathcal{C}$, we define $F(f) =_{\mathrm{def}} (i_Y J(f))^\sharp$. The unit of the adjoint equivalence in (3.3) gives us invertible 2-cells

$$\psi_f \colon i_Y\, J(f) \to GF(f)\, i_X \tag{3.4}$$

for $f \colon X \to Y$. For $f \colon X \to Y$, $g \colon Y \to Z$, we need an invertible 2-cell $\phi_{g,f} \colon F(g\, f) \to F(g)\, F(f)$. By the definition, we have

$$F(g\, f) = (i_Z\, J(g\, f))^\sharp \cong (i_Z\, J(g)\, J(f))^\sharp\,, \quad F(g)\, F(f) = \big(G\big(F(g)\, F(f)\big)\, i_X\big)^\sharp\,.$$

So we can define $\phi_{g,f}$ using the composite

$$(i_Z\, J(g)\, J(f))^\sharp \xrightarrow{\psi^{-1}} \big(GF(g)\, i_Y\, J(f)\big)^\sharp \xrightarrow{\psi} \big(GF(g)\, GF(f)\, i_X\big)^\sharp \xrightarrow{\cong} \big(G\big(F(g)\, F(f)\big)\, i_X\big)^\sharp\,,$$

where the unnamed isomorphism is given by the pseudofunctoriality of G. For $X \in \mathcal{C}$, we need an invertible 2-cell $\phi_X \colon F(1_X) \to 1_{FX}$. By definition, $F(1_X) = (i_X)^\sharp = (G(1_{FX})\, i_X)^\sharp$, and so we define ϕ_X to be $\varepsilon_{1_{FX}} \colon (G(1_{FX})\, i_X)^\sharp \to 1_{FX}$. With these definitions, the pseudonaturality 2-cells for $i \colon J \Rightarrow GF$ are the 2-cells in (3.4). The proof of the coherence conditions is routine (cf. [35, 60]). □

Theorem 3.8. *Let*

$$\begin{array}{ccc} & & \mathcal{E} \\ & \overset{F}{\nearrow} & \downarrow G \\ \mathcal{C} & \xrightarrow[J]{} & \mathcal{D} \end{array}$$

be a relative pseudoadjunction. Then the function sending $X \in \mathcal{C}$ to $GF(X) \in \mathcal{D}$ admits the structure of a relative pseudomonad over J.

Proof. For $X \in \mathcal{C}$, define $TX =_{\mathrm{def}} GFX$. The relative pseudoadjunction gives morphisms $i_X \colon X \to TX$, for $X \in \mathcal{C}$, and adjoint equivalences

$$\mathcal{D}[JX, GFY] \underset{G(-)\, i_X}{\overset{(-)^\sharp}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle\perp}}}} \mathcal{E}[FX, FY] \tag{3.5}$$

for $X, Y \in \mathcal{C}$. We then define $(-)^* \colon \mathcal{D}[JX, TY] \to \mathcal{D}[TX, TY]$ by letting $f^* =_{\mathrm{def}} G(f^\sharp)$. It now remains to define the families of invertible 2-cells μ, η and θ. For $\mu_{g,f} \colon (g^*\, f)^* \to g^*\, f^*$, observe that

$$(g^*\, f)^* = G(G(g^\sharp)\, f)^\sharp\,, \quad g^*\, f^* = G(g^\sharp)\, G(f^\sharp)\,,$$

and so we define $\mu_{g,f}$ to be the composite

$$G\big(G(g^\sharp)\,f\big)^\sharp \xrightarrow{\ \eta\ } G\big(G(g^\sharp)\,G(f^\sharp)\,i_X\big)^\sharp \xrightarrow{\ \cong\ } G(G(g^\sharp\,f^\sharp)\,i_X)^\sharp \xrightarrow{\ \varepsilon\ } G(g^\sharp\,f^\sharp) \xrightarrow{\ \cong\ } G(g^\sharp)\,G(f^\sharp)\,.$$

The 2-cells $\eta_f\colon f \to f^*\,i_X$ are given by the units of the adjunction (3.5), which satisfy the required naturality condition. For $\theta_X : {i_X}^* \to 1_{TX}$, we recall that ${i_X}^* = G({i_X}^\sharp)$, and so we define θ_X to be the composite 2-cell

$$G({i_X}^\sharp) \xrightarrow{\ \eta\ } G\big((G(1_{FX})\,i_X)^\sharp\big) \xrightarrow{\ \varepsilon\ } G(1_{FX}) \xrightarrow{\ \cong\ } 1_{GFX}\,.$$

It remains to establish the coherence conditions. While it is possible to show this directly, it is more illuminating to argue in terms of universal properties. Simply restating the adjunction in (3.5), we observe that, given $f\colon JX \to GA$ in $\mathcal{D}$ and $u\colon FX \to A$ in $\mathcal{E}$, for every 2-cell $\phi\colon f \to G(u)\,i_X$, there is a unique 2-cell $\psi\colon f^\sharp \to u$, the adjoint transpose, such that the diagram

$$\begin{array}{ccc} f & \xrightarrow{\ \eta_f\ } & G(f^\sharp)\,i_X \\ & \searrow^{\phi} & \downarrow{\scriptstyle G(\psi)\,i_X} \\ & & G(u)\,i_X \end{array}$$

commutes. Accordingly, we can characterize $\mu_{g,f}$ and θ_X as follows. There are 2-cells

$$\tilde{\kappa}_{g,f} : G(G(g^\sharp)\,f)^\sharp \to G(g^\sharp\,f^\sharp)\,, \qquad \tilde{\kappa}_X : G({i_X}^\sharp) \to G(1_{FX})$$

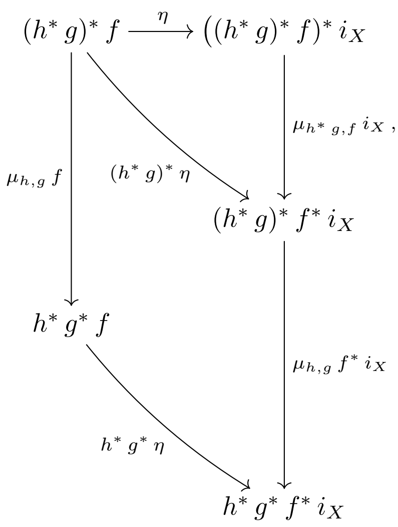
being the image under G of the unique 2-cells $(G(g^\sharp)\,f)^\sharp \to g^\sharp\,f^\sharp$ and ${i_X}^\sharp \to 1_{FX}$ such that the diagrams

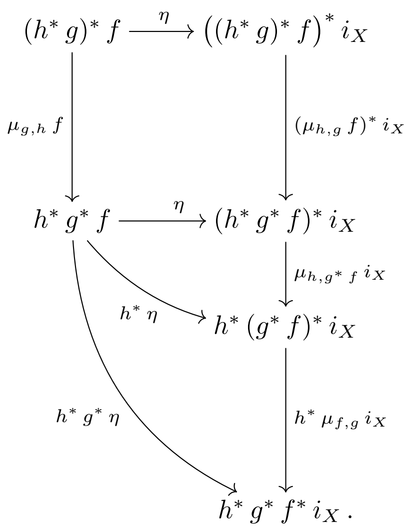
$$\begin{array}{ccc} G(g^\sharp)\,f & \xrightarrow{\ \eta\ } & G(G(g^\sharp)\,f)^\sharp\,i_X \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\tilde{\kappa}_{g,f}\,i_X} \\ G(g^\sharp)\,G(f^\sharp)\,i_X & \xrightarrow[\cong]{} & G(g^\sharp\,f^\sharp)\,i_X \end{array} \qquad \begin{array}{ccc} i_X & \xrightarrow{\ \eta\ } & G({i_X}^\sharp)\,i_X \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\tilde{\kappa}_X\,i_X} \\ 1_{GFX}\,i_X & \xrightarrow[\cong]{} & G(1_{FX})i_X \end{array}$$

commute. The 2-cells $\mu_{g,f}$ and θ_X then arise by composing these 2-cells with pseudofunctoriality 2-cells of G. The coherence diagrams follow readily, and we give details in the 2-categorical case, where the characterizing diagrams for $\mu_{g,f}$ and θ_X reduce to the diagrams

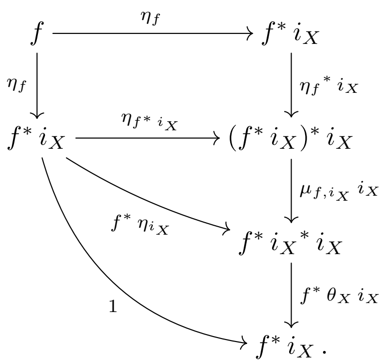
$$\begin{array}{ccc} i_X & \xrightarrow{\ \eta\ } & {i_X}^*\,i_X \\ & \searrow^{1} & \downarrow{\scriptstyle\theta_X\,i_X} \\ & & i_X\,, \end{array} \qquad \begin{array}{ccc} g^*\,f & \xrightarrow{\ \eta\ } & (g^*\,f)^*\,i_X \\ & \searrow^{g^*\,\eta_f} & \downarrow{\scriptstyle\mu_{g,f}\,i_X} \\ & & g^*\,f^*\,i_X\,. \end{array}$$

For the associativity condition in (3.1), we have commuting diagrams





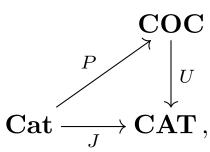
Note that the triangles in these diagrams commute by part (i) of Lemma 3.2. Since both of the composites on the right-hand side of the diagrams lie in the image of G, we deduce by universality that they are equal, as required. For the unit condition in (3.2) we have a commuting diagram



Here, the triangles commute by part (i) and (iii) of Lemma 3.2. Again, since the composite of $(f^* \, \theta_X)\,(\mu_{f,i_X})\,(\eta_f{}^*)$ lies in the image of G we deduce by universality that it equals the identity, as required. ☐

Using Theorem 3.8, we can introduce our fundamental example of a relative pseudomonad, given by the presheaf construction.[1]

Example 3.9. There is a relative pseudoadjunction of the form



[1]The possibility of introducing a notion of relative pseudomonad to include the presheaf construction as an example was mentioned in [1, Example 2.7].

where **COC** is the 2-category of locally small cocomplete categories, cocontinuous functors, and natural transformations, and $U\colon \mathbf{COC} \to \mathbf{CAT}$ is the evident forgetful functor. The category $P(\mathbb{X}) =_{\mathrm{def}} [\mathbb{X}^{\mathrm{op}}, \mathbf{Set}]$ of presheaves over a small category $\mathbb{X}$ is the colimit completion of $\mathbb{X}$ in the sense that, for every locally small cocomplete $\mathbb{A}$, composition with the Yoneda embedding $\mathrm{y}_{\mathbb{X}}\colon \mathbb{X} \to P(\mathbb{X})$ induces an equivalence of categories

$$\mathbf{CAT}[\mathbb{X}, \mathbb{A}] \xleftarrow{U(-)\,\mathrm{y}_{\mathbb{X}}} \mathbf{COC}[P(\mathbb{X}), \mathbb{A}]\,.$$

Thus P provides a relative left pseudoadjoint to U. By Theorem 3.8 we obtain a relative pseudomonad over the inclusion $J\colon \mathbf{Cat} \to \mathbf{CAT}$. For a functor $F\colon \mathbb{X} \to P(\mathbb{Y})$, $F^*\colon P(\mathbb{X}) \to P(\mathbb{Y})$ is the left Kan extension of F along the Yoneda embedding, defined by the coend formula

$$F^*(p)(y) =_{\mathrm{def}} \int^{x \in \mathbb{X}} F(x)(y) \times p(x)\,. \tag{3.6}$$

The invertible 2-cells η_F fit into the diagram

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\mathrm{y}_{\mathbb{X}}} & P(\mathbb{X}) \\ & \overset{\eta_F}{\Rightarrow} & \downarrow F^* \\ & \searrow_{F} & P(\mathbb{Y}). \end{array}$$

The 2-cells $\mu_{F,G}$ and $\theta_{\mathbb{X}}$ are uniquely determined by the universal property of left Kan extensions.

There is an analogous relative pseudomonad arising from the relative pseudoadjunction

$$\begin{array}{ccc} & & \mathbf{FIL} \\ & \nearrow^{D} & \downarrow U \\ \mathbf{Cat} & \xrightarrow[J]{} & \mathbf{CAT}\,, \end{array}$$

where **FIL** is the 2-category of locally small categories with filtered colimits, functors preserving such colimits, and all natural transformations [2]. Here, for $\mathbb{X} \in \mathbf{Cat}$, we define $D(\mathbb{X}) \in \mathbf{CAT}$ to be the full subcategory of $P(\mathbb{X})$ spanned by small filtered colimits of representables.

4. Kleisli bicategories

We introduce the Kleisli bicategory of a relative pseudomonad, extending to the 2-dimensional setting the definition of the Kleisli category of a relative monad [1, Section 2.3].

Theorem 4.1. *Let T be a relative pseudomonad over $J\colon \mathcal{C} \to \mathcal{D}$. Then there is a bicategory* $\mathrm{Kl}(T)$*, called the Kleisli bicategory of T, having the objects of $\mathcal{C}$ as objects, and hom-categories given by* $\mathrm{Kl}(T)[X, Y] =_{\mathrm{def}} \mathcal{D}[JX, TY]$*, for $X, Y \in \mathcal{C}$.*

Proof. We begin by defining composition in $\mathrm{Kl}(T)$. Let $f\colon JX \to TY$ and $g\colon JY \to TZ$. We define $g \circ f\colon JX \to TZ$ as the composite in $\mathcal{D}$

$$JX \xrightarrow{f} TY \xrightarrow{g^*} TZ\,.$$

This obviously extends to 2-cells, so as to obtain the required composition functors. For $X \in \mathcal{C}$, the identity morphism on X in $\mathrm{Kl}(T)$ is $i_X\colon JX \to TX$. For the associativity isomorphisms, let $f\colon JX \to TY$, $g\colon JY \to TZ$ and $h\colon JZ \to TV$. Since

$$(h \circ g) \circ f = (h^*\, g)^* f\,, \quad h \circ (g \circ f) = h^*\,(g^*\, f)\,,$$

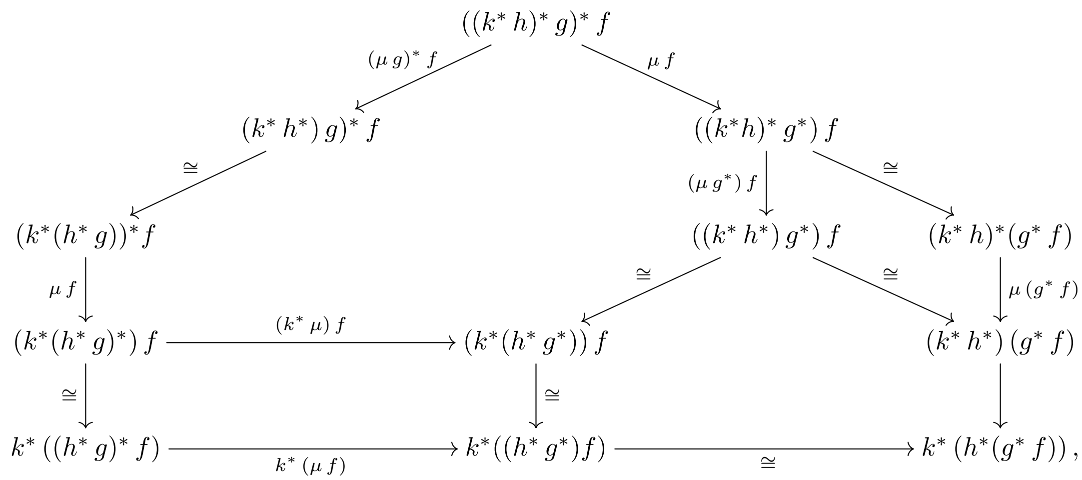
we define the associativity isomorphism $\alpha_{h,g,f}\colon (h\circ g)\circ f \to h\circ(g\circ f)$ to be the composite 2-cell

$$(h^*\, g)^*\, f \xrightarrow{\mu_{h,g}\, f} (h^*\, g^*)\, f \xrightarrow{\cong} h^*\,(g^*\, f)\,.$$

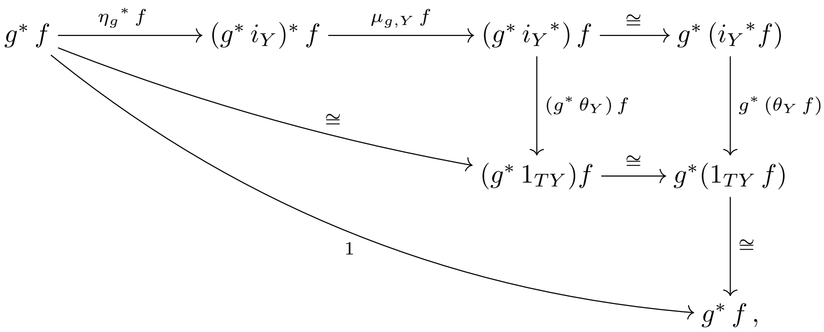
For the right and left unit, let $f\colon JX \to TY$. Since $f\circ i_X = f^*\, i_X$, we define $\rho_f\colon f\circ i_X \to f$ to be the 2-cell $\eta_f\colon f \to f^*\, i_X$. Since $i_Y\circ f = {i_Y}^*\, f$, we define $\lambda_f\colon i_Y\circ f\to f$ to be the composite 2-cell

$${i_Y}^*\, f \xrightarrow{\theta_Y\, f} 1_{TY}\, f \xrightarrow{\cong} f\,.$$

We now need to show that these natural isomorphisms satisfy the required coherence conditions. We give the proof making explicit the bicategorical structure of $\mathcal{C}$ and $\mathcal{D}$. We only need to show that the associativity, left unit, and right unit isomorphisms satisfy the coherence conditions for a bicategory. The coherence axiom for associativity is obtained via the following diagram:



where, starting from the rhombus on the right-hand side and proceeding clockwise, we use naturality of the associativity in $\mathcal{D}$, coherence of associativity in $\mathcal{D}$, naturality of the associativity in $\mathcal{D}$ again, and finally the associativity coherence axiom for a relative pseudomonad in (3.1). The coherence axiom for the units is obtained via the following diagram:



where, starting from the triangle on the top left-hand side, we use the coherence axiom for units of the relative pseudomonad in (3.2), naturality of the associativity in $\mathcal{D}$, and the coherence axiom for units of $\mathcal{D}$. □

Note that, as mentioned in Section 2 for ordinary pseudomonads, $\mathrm{Kl}(T)$ is only a bicategory even if $\mathcal{C}$ and $\mathcal{D}$ are 2-categories.

Example 4.2. It is straightforward to identify the bicategory of profunctors of Example 2.1 with the Kleisli bicategory associated to the relative pseudomonad of presheaves P of Example 3.9. First of all, both bicategories have small categories as objects. Secondly, for small categories $\mathbb{X}$ and $\mathbb{Y}$ we have

$$\mathbf{Prof}[\mathbb{X}, \mathbb{Y}] = [\mathbb{Y}^{\mathrm{op}} \times \mathbb{X}, \mathbf{Set}], \qquad \mathrm{Kl}(P)[\mathbb{X}, \mathbb{Y}] = \mathbf{CAT}[\mathbb{X}, P(\mathbb{Y})].$$

Thus, we have a canonical isomorphism of hom-categories

$$\tau \colon \mathbf{Prof}[\mathbb{X}, \mathbb{Y}] \to \mathrm{Kl}(P)[\mathbb{X}, \mathbb{Y}],$$

given by exponential adjoint transposition. Furthermore, these isomorphisms are compatible with composition and identities. For composition, it suffices to observe that, for profunctors $F \colon \mathbb{X} \to \mathbb{Y}$ and $G \colon \mathbb{Y} \to \mathbb{Z}$, there is a canonical natural isomorphism

$$\tau(G \circ F) \cong (\tau G) \circ (\tau F),$$

where the composition on the left-hand side is that of **Prof**, as defined in (2.3), while the composition on the right is the one of $\mathrm{Kl}(P)$, which is given by the functorial composite of $\tau(F) \colon \mathbb{X} \to P(\mathbb{Y})$ and $(\tau G)^* \colon P(\mathbb{Y}) \to P(\mathbb{Z})$, the latter being defined by the formula for left Kan extensions in (3.6). For identities, simply note that, for a small category $\mathbb{X}$, the adjoint transpose of the identity profunctor on $\mathbb{X}$, as defined in (2.4), is exactly the Yoneda embedding, which is the identity on $\mathbb{X}$ in $\mathrm{Kl}(P)$.

In one-dimensional category theory, every monad determines two universal adjunctions relating the base category with the category of Eilenberg-Moore algebras and the Kleisli category for the monad. For pseudomonads, the construction of the bicategory of pseudoalgebras is well-known and it has been considered for no-iteration pseudomonads in [53, Section 4], but we do not need its counterpart for relative pseudomonads here. We focus instead on the counterpart of the Kleisli adjunction, which has not been considered for no-iteration pseudomonads. The first step is the following lemma.

Lemma 4.3. *Let T be a relative pseudomonad over $J \colon \mathcal{C} \to \mathcal{D}$. Then the function sending $X \in \mathrm{Kl}(T)$ to $TX \in \mathcal{D}$ admits the structure of a pseudofunctor $G^T \colon \mathrm{Kl}(T) \to \mathcal{D}$.*

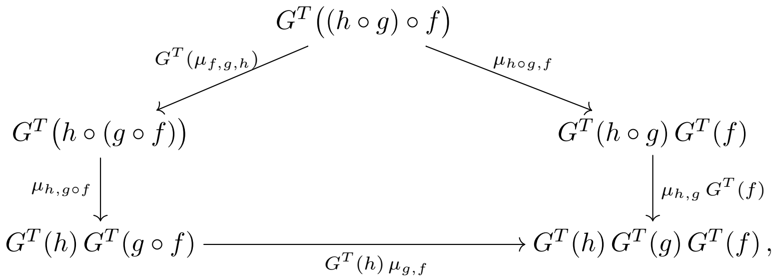
Proof. For $X, Y \in \mathrm{Kl}(T)$, we define the functor $G^T_{X,Y} \colon \mathrm{Kl}(T)[X, Y] \to \mathcal{D}[G^T X, G^T Y]$ to be

$$(-)^* \colon \mathcal{D}[JX, TY] \longrightarrow \mathcal{D}[TX, TY].$$

By inspection of the definitions, we can define the pseudofunctoriality 2-cells of G^T to be exactly some of the 2-cells that are part of the data of a relative pseudomonad, namely

$$\mu_{g,f} \colon G^T(g \circ f) \to G^T(g)\, G^T(f), \quad \theta_X \colon G^T(i_X) \to 1_{G^T X}.$$

In order to have a pseudofunctor, we need to verify that the following three coherence diagrams commute:



$$\begin{array}{ccccc} G^T(f) & \xrightarrow{G^T(\rho_f)} & G^T(f \circ i_X) & \xrightarrow{\mu_{f,X}} & G^T(f)\, G^T(1_X) \\ & & & & \big\downarrow {\scriptstyle G^T(f)\,\theta_X} \\ & & \xrightarrow[1_{G^T(f)}]{\qquad\qquad} & & G^T(f)\,, \end{array}$$

$$\begin{array}{ccc} G^T(i_Y \circ f) & \xrightarrow{\mu_{Y,f}} & G^T(i_Y)\, G^T(f) \\ & & \big\downarrow {\scriptstyle \theta_Y\, G^T(f)} \\ & \xrightarrow[G^T(\lambda_f)]{\qquad\qquad} & G^T(f)\,. \end{array}$$

The first and second diagrams follow at once from the coherence conditions in (3.1) and (3.2) that are part of the definition of a relative pseudomonad. The third is part (ii) of Lemma 3.2. □

By analogy with the one-dimensional case, we expect that the pseudofunctor $G^T \colon \mathrm{Kl}(T) \to \mathcal{D}$ has some form of left pseudoadjoint. The next result makes this precise.

Theorem 4.4. *Let $T \colon \mathcal{C} \to \mathcal{D}$ be a relative pseudomonad over $J \colon \mathcal{C} \to \mathcal{D}$. Then $G^T \colon \mathrm{Kl}(T) \to \mathcal{D}$ has a relative left pseudoadjoint over $J \colon \mathcal{C} \to \mathcal{D}$,*

$$\begin{array}{ccc} & & \mathrm{Kl}(T) \\ & \overset{F^T}{\nearrow} & \big\downarrow {\scriptstyle G^T} \\ \mathcal{C} & \xrightarrow[J]{} & \mathcal{D}\,. \end{array}$$

Proof. For $X \in \mathcal{C}$, we define $F^T X =_{\text{def}} X$. Then we have $G^T F^T X = TX$, so the relative pseudomonad provides a morphism $i_X \colon JX \to G^T F^T X$. For these to act like the components of the unit of a relative pseudoadjunction one needs to show that the functor

$$\mathrm{Kl}(T)[F^T X, Y] \xrightarrow{G^T(-)\, i_X} \mathcal{D}[JX, G^T Y]$$

is an adjoint equivalence. Indeed, $\mathrm{Kl}(T)[F^T X, Y] = \mathcal{D}[JX, TY] = \mathcal{D}[JX, G^T Y]$ and the functor $G^T(-)\, i_X$ is naturally isomorphic to the identity. This is because, for $f \colon JX \to TY$, we have $G^T(f) \circ i_X = f^*\, i_X$ and there is an invertible 2-cell $\eta_f \colon f \to f^*\, i_X$, suitably natural. □

Remark 4.5. Theorem 4.4 allows us to give a more conceptual account of the construction of the pseudomonad associated to a relative pseudomonad over the identity $1_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$ in Remark 3.5. Given a relative pseudomonad T over $1_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$, we can construct a pseudoadjunction between $\mathcal{C}$ and $\mathrm{Kl}(T)$ as in Theorem 4.4. Then, the pseudomonad associated to this pseudoadjunction is exactly the pseudomonad described in Remark 3.5. So we have established again the coherence conditions for a pseudomonad, in a clean (albeit indirect) way.

Let us also note that if we start with a relative pseudomonad T over $J \colon \mathcal{C} \to \mathcal{D}$, form the associated Kleisli relative pseudoadjunction (as in Theorem 4.4), and take the induced relative pseudomonad (as in Theorem 3.8), we then retrieve the original relative pseudomonad. The only issues arise at the 2-cell level and we leave the verifications to the interested readers. In the other direction, suppose that we start with a relative pseudoadjunction

$$\begin{array}{ccc} & & \mathcal{E} \\ & {}^{F}\nearrow & \downarrow{\scriptstyle G} \\ \mathcal{C} & \xrightarrow[J]{} & \mathcal{D}\,, \end{array}$$

form the induced relative pseudomonad $T = GF$ over $J\colon \mathcal{C} \to \mathcal{D}$ (as in Theorem 3.8), and then take the induced relative pseudoadjunction

$$\begin{array}{ccc} & & \mathrm{Kl}(T) \\ & {}^{F^T}\nearrow & \downarrow{\scriptstyle G^T} \\ \mathcal{C} & \xrightarrow[J]{} & \mathcal{D}\,, \end{array}$$

as in Theorem 4.4. We expect a comparison and indeed we have a pseudofunctor $C\colon \mathrm{Kl}(T) \to \mathcal{E}$ defined on objects by letting $C(X) =_{\mathrm{def}} FX$, for $X \in \mathcal{C}$. On hom-categories, for $X, Y \in \mathcal{C}$, we define

$$C_{X,Y}\colon \mathcal{D}[JX, TY] \to \mathcal{E}[FX, FY]$$

by letting $C(f) =_{\mathrm{def}} f^\sharp$, where we used that $TY = GF(Y)$.

One can compose adjunctions between categories and, similarly, pseudoadjunctions between bicategories. It does not make sense to compose relative pseudoadjunctions, but one can form the composite of a relative pseudoadjunction and a pseudoadjunction.

Proposition 4.6. *Let*

$$\begin{array}{ccc} & & \mathcal{E} \\ & {}^{F}\nearrow & \downarrow{\scriptstyle G} \\ \mathcal{C} & \xrightarrow[J]{} & \mathcal{D}\,, \end{array} \qquad\qquad \mathcal{E} \underset{G'}{\overset{F'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{E}'$$

be a relative pseudoadjunction and a pseudoadjunction, respectively. Then there is a relative pseudoadjunction of the form

$$\begin{array}{ccc} & & \mathcal{E}' \\ & {}^{F'F}\nearrow & \downarrow{\scriptstyle GG'} \\ \mathcal{C} & \xrightarrow[J]{} & \mathcal{D}\,. \end{array}$$

Proof. The construction is evident and thus omitted. □

We conclude this section by extending some results on no-iteration pseudomonads to relative pseudomonads. In the next proposition, part (i) generalizes [53, Proposition 3.1], part (ii) generalizes [53, Proposition 3.2], while part (iii) does not seem to have been made explicit yet for no-iteration pseudomonads.

Proposition 4.7. *Let T be a relative pseudomonad over $J\colon \mathcal{C} \to \mathcal{D}$.*

(i) The function $T\colon \mathcal{C} \to \mathcal{D}$ admits the structure of a pseudofunctor.
(ii) The family of morphisms $i_X\colon JX \to TX$, for $X \in \mathcal{C}$, admits the structure of a pseudonatural transformation $i\colon J \to T$.
(iii) The family of functions $(-)^*\colon \mathcal{D}[JX, TY] \to \mathcal{D}[TX, TY]$ for $X, Y \in \mathcal{C}$, admits the structure of a pseudonatural transformation.

Proof. Parts (i) and (ii) follow from Lemma 3.7 and Theorem 4.4 via Remark 4.5, but we also give explicit proofs. We begin from part (i). For $f\colon X \to Y$, we define $T(f)\colon TX \to TY$ by letting $Tf =_{\text{def}} (i_Y J(f))^*$. We then define the pseudofunctoriality 2-cells. First, we need invertible 2-cells $\tau_{g,f}\colon T(g\,f) \to T(g)\,T(f)$, for $f\colon X \to Y$ and $g\colon Y \to Z$. By definition, we have

$$T(g\,f) = (i_Z\,J(g)\,J(f))^*\,, \quad T(g)\,T(f) = (i_Z\,J(g))^*\,(i_Y\,J(f))^*\,.$$

We then define $\tau_{g,f}$ as the composite 2-cell

$$(i_Z\,J(g)\,J(f))^* \xrightarrow{\ \eta\ } ((i_Z\,J(g))^*\,i_Y\,J(f))^* \xrightarrow{\ \mu\ } (i_Z\,J(g))^*\,(i_Y\,J(f))^*$$

Secondly, we need invertible 2-cells $\tau_X\colon T(1_X) \to 1_{TX}$ for $X \in \mathcal{C}$. But since $T(1_X) = {i_X}^*$ by definition, we let $\tau_X =_{\text{def}} \theta_X$, the component of the left unit of the relative pseudomonad.

One should check the three coherence laws for a pseudofunctor. The coherence law for $\tau_{g,f}$ involves a pasting of the associativity condition in (3.1), part (i) of Lemma 3.2 and all the naturality conditions for the families 2-cells of a relative pseudomonad. One of the coherence laws for τ_X comes from the unit condition in (3.2), while the other is from part (ii) of Lemma 3.2.

For part (ii), the required pseudonaturality 2-cell for $f\colon X \to Y$ fits into the diagram

$$\begin{array}{ccc} JX & \xrightarrow{\ J(f)\ } & JY \\ {\scriptstyle i_X}\downarrow & \Downarrow\,\bar{i}_f & \downarrow{\scriptstyle i_Y} \\ TX & \xrightarrow[\ T(f)\ ]{} & TY \end{array}$$

Since $T(f) = (i_Y\,J(f))^*$, we can simply let

$$\bar{i}_f =_{\text{def}} \eta_{i_Y\,J(f)}\,. \tag{4.1}$$

We should check two coherence conditions for pseudonatural transformations. The composition condition involves a pasting of a naturality of η to a diagram coming from part (i) of Lemma 3.2. The identity condition is just part (iii) of Lemma 3.2 .

Finally, for part (iii), to see the pseudonaturality in X, take $u\colon X' \to X$, observe that $f^*\,T(u) = f^*\,(i_X\,u)^*$ and note the 2-cell

$$(f\,u)^* \xrightarrow{\ \eta\ } (f^*\,i_X\,u)^* \xrightarrow{\ \mu\ } f^*\,(i_X\,u)^*\,.$$

For the pseudonaturality in Y, take $v\colon Y \to Y'$, observe that $(T(v)\,f)^* = ((i_{Y'}\,v)^*\,f)^*$ and $T(v)\,f^* = (i_{Y'}\,v)^*\,f^*$, and note the 2-cell

$$\big((i_{Y'}\,v)^*\,f\big)^* \xrightarrow{\ \mu\ } (i_{Y'}\,v)^*\,f^*\,.$$

There are coherence conditions to check, but they are straightforward. □

5. Lax idempotent relative pseudomonads

We isolate a special class of relative pseudomonads, which appears to be the appropriate generalization to our setting of the notion of a lax idempotent 2-monad, or Kock-Zöberlein 2-monad, or KZ-doctrine [38, 65]. An extensive analysis of these 2-monads, with useful equivalent formulations, was given in the course of a study of general property-like 2-monads in [37]. For pseudomonads, the more general notion of lax idempotent pseudomonad on a bicategory was introduced in [60], with yet another characterisation of the notion, and studied further in [48, 52].

In order to state the definition of a lax idempotent relative pseudomonad, it is convenient to use the notion of a left extension in a bicategory [40, §2.2.], which we now recall. We consider a fixed morphism $i\colon X \to X'$ in a bicategory $\mathcal{C}$. By definition, a *left extension* of a morphism $f\colon X \to Y$

along i consists of a morphism $f' \colon X' \to Y$ and a 2-cell $\eta \colon f \to f'i$ such that composition with η induces a bijection between 2-cells $f \to gi$ and 2-cells $f' \to g$ for every morphism $g \colon X' \to Y$. In this case, one says that η exhibits f' as the left extension of f along i.

Let us fix again a pseudofunctor between bicategories $J \colon \mathcal{C} \to \mathcal{D}$.

Definition 5.1. A *lax idempotent relative pseudomonad* over J is a relative pseudomonad T over $J \colon \mathcal{C} \to \mathcal{D}$ such that the following conditions hold:

- for all $f \colon JX \to TY$, the 2-cell $\eta_f \colon f \to f^* \, i_X$ exhibits $f^* \colon TX \to TY$ as a left extension of f along i_X,
- for all $f \colon JX \to TY$, $g \colon JY \to TZ$, the diagram

$$\begin{array}{ccc} g^* f & \xrightarrow{\eta_{g^* f}} & (g^* \, f)^* i_X \\ & \searrow^{g^* \eta_f} & \downarrow{\scriptstyle \mu_{f,g} i_X} \\ & & g^* \, f^* i_X \end{array}$$

  commutes,
- for all $X \in \mathcal{C}$, the diagram

$$\begin{array}{ccc} i_X & \xrightarrow{\eta_{i_X}} & {i_X}^* \, i_X \\ & \searrow^{1} & \downarrow{\scriptstyle \theta_X \, i_X} \\ & & i_X \end{array}$$

  commutes.

Our next goal is to give alternative characterizations of lax idempotent relative pseudomonads, which we will use to discuss further the relative pseudomonad of presheaves. For this, we formulate the relative version of the notion of a left Kan pseudomonad introduced in [52, Definition 3.1].

Definition 5.2. A *relative left Kan pseudomonad* over $J \colon \mathcal{C} \to \mathcal{D}$ consists of:

- an object $TX \in \mathcal{D}$, for every $X \in \mathcal{C}$,
- a morphism $i_X \colon JX \to TX$ in $\mathcal{D}$, for every $X \in \mathcal{C}$,
- a morphism $f^* \colon TX \to TY$, for every $f \colon JX \to TY$,
- an invertible 2-cell $\eta_f \colon f \to f^* \, i_X$ which exhibits f^* as the left extension of f along i_X, for every $f \colon JX \to TY$,

such that the following conditions hold:

- the 2-cell $g^* \, \eta_f \colon g^* f \to g^* f^* i_X$ exhibits $g^* f^*$ as the left extension of $g^* f$ along i_X, for all $f \colon JX \to TX$, $g \colon JY \to TZ$,
- the identity 2-cell $1 \colon i_X \to i_X$ exhibits 1_{TX} as a left extension of i_X along i_X, for all $X \in \mathcal{C}$.

Let us now assume we have an object $TX \in \mathcal{D}$ for every $X \in \mathcal{C}$, a morphism $i_X \colon JX \to TX$ in $\mathcal{D}$ for every $X \in \mathcal{C}$, a morphism $f^* \colon TX \to TY$ for every $f \colon JX \to TY$ and an invertible 2-cell $\eta_f \colon f \to f^* \, i_X$ exhibiting f^* as the left extension of f along i_X for every $f \colon JX \to TY$. Note that this gives us all the data for a relative left Kan pseudomonad, but does not require all its axioms. Below, we refer to this data simply as $T \colon \mathcal{C} \to \mathcal{D}$. It is evident that, for all $X, Y \in \mathcal{C}$ we have an adjunction

$$\mathcal{D}[JX, TY] \underset{(-)\, i_X}{\overset{(-)^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{D}[TX, TY] \, , \tag{5.1}$$

whose unit has components the invertible 2-cells $\eta_f \colon f \to f^* i_X$, for $f \colon JX \to TY$, and whose counit has components 2-cells that will be written $\varepsilon_u \colon (u\, i_X)^* \to u$, for $u \colon TX \to TY$. We can then state our characterizations of lax idempotent relative pseudomonads as follows.

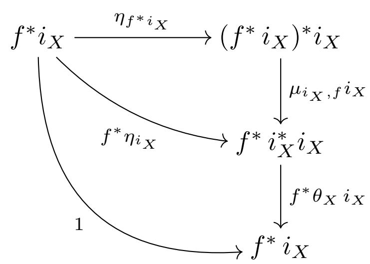
Theorem 5.3. *The following conditions are equivalent:*

(i) *$T \colon \mathcal{C} \to \mathcal{D}$ admits the structure of a lax idempotent relative pseudomonad over $J \colon \mathcal{C} \to \mathcal{D}$.*
(ii) *For every $f \colon JX \to TY$, the 2-cell $\varepsilon_{f^*} \colon (f^* i_X)^* \to f^*$ is invertible and there are isomorphisms*

$$\mu_{f,g} \colon (g^* f)^* \to g^* f^* \, , \quad \theta_X \colon (i_X)^* \to 1_{TX} \, .$$

(iii) *The bicategory $\mathcal{E}$ with objects of the form TX, for $X \in \mathcal{C}$, and hom-categories given by defining $\mathcal{E}[TX, TY]$ to be the full subcategory of $\mathcal{D}[TX, TY]$ spanned by the morphisms $u \colon TX \to TY$ such that $u \cong f^*$, for some $f \colon JX \to TY$ in $\mathcal{D}$, is a sub-bicategory of $\mathcal{D}$.*
(iv) *There exists a sub-bicategory $\mathcal{E}$ of $\mathcal{D}$ such that $f^* \colon TX \to TY$ is in $\mathcal{E}$ for all $f \colon JX \to TY$ in $\mathcal{D}$ and $\varepsilon_u \colon (u i_X)^* \to u$ is invertible for all $u \colon TX \to TY$ in $\mathcal{E}$.*
(v) *$T \colon \mathcal{C} \to \mathcal{D}$ is a relative left Kan pseudomonad over $J \colon \mathcal{C} \to \mathcal{D}$.*

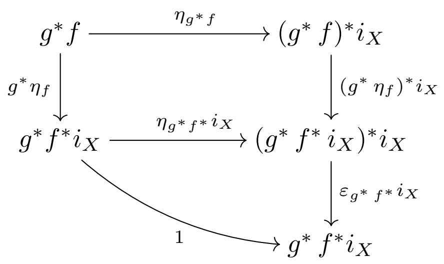
Proof. To prove that (i) implies (ii), observe that the axioms for a lax idempotent relative pseudomonad imply that the following diagram commutes:



By one of the triangular laws for the adjunction in (5.1), ε_{f^*} is the composite

$$(f^* i_X)^* \xrightarrow{\mu_{f, i_X}} f^* (i_X)^* \xrightarrow{f^* \theta_X} f^*$$

and it is therefore invertible. For (ii) implies (iii), the given isomorphisms show that $\mathcal{E}$ as defined is closed under composition and contains identities. The implication from (iii) to (iv) is immediate. For the implication from (iv) to (v), observe that the following diagram commutes:



This shows that $g^* \eta_f$ is the composite of $\eta_{g^* f}$ (which exhibits $(g^* f)^*$ as an extension of $g^* f$ along i_X) with an invertible 2-cell. It then follows that $g^* \eta_f$ exhibits $g^* f^*$ as the left extension of g^*f along i_X, as required. Similarly, we have

$$\begin{array}{ccc} i_X & \xrightarrow{\eta_{i_X}} & {i_X}^* \, i_X \\ & \searrow^{1} & \downarrow{\scriptstyle \varepsilon_{1_{TX}}} \\ & & i_X \end{array}$$

which implies that the identity $1 \colon i_X \to i_X$ exhibits 1_{TX} as a left extension of i_X along i_X.

Finally, we show that (v) implies (i). We begin by defining the remaining parts of the data for a relative pseudomonad, namely the families of invertible 2-cells $\mu_{f,g} \colon (g^* \, f)^* \to g^* \, f^*$ and $\theta_X : {i_X}^* \to 1_{TX}$. Using the universal property of $\eta_{g^* f}$, we define $\mu_{f,g}$ to be the unique 2-cell such that

$$\begin{array}{ccc} g^* \, f & \xrightarrow{\eta_{g^* \, f}} & (g^* \, f)^* \, i_X \\ & \searrow^{g^* \, \eta_f} & \downarrow{\scriptstyle \mu_{f,g} \, i_X} \\ & & (g^* \, f)^* \, i_X \end{array}$$

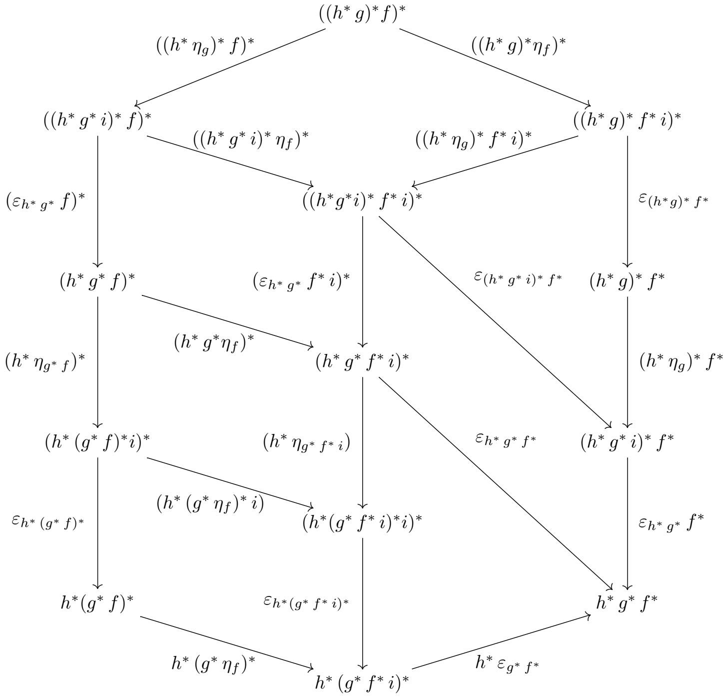
Similarly, using the universal property of η_{i_X}, we define θ_X to be the unique 2-cell such that

$$\begin{array}{ccc} i_X & \xrightarrow{\eta_X} & {i_X}^* \, i_X \\ & \searrow^{1} & \downarrow{\scriptstyle \theta_X \, i_X} \\ & & 1_{TX} \, i_X \, . \end{array}$$

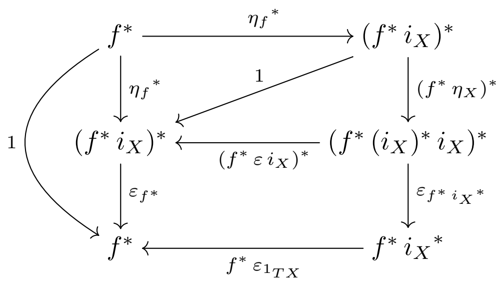
It then remains only to check the coherence conditions of Definition 3.1. There are two ways of doing this. The first is by a diagram-chasing arguments using the universal properties defining $\mu_{f,g}$ and θ_X. The second is to express $\mu_{f,g}$ and θ_X in terms of the unit and counit of the adjunction. Taking that approach, first observe that the diagram

$$\begin{array}{ccccc} (h^* \, u \, i_X)^* & \xrightarrow{(h^* \, \eta_{u \, i_X})^*} & (h^* \, (u \, i_X)^* \, i_X)^* & \xrightarrow{\varepsilon_{h^* \, (u \, i_X)^*}} & h^* \, (u \, i_X)^* \\ & \searrow & \downarrow{\scriptstyle (h^* \, \varepsilon_u \, i_X)^*} & & \downarrow{\scriptstyle h^* \, \varepsilon_u} \\ & & (h^* \, u \, i_X)^* & \xrightarrow[\varepsilon_{h^* \, u}]{} & h^* \, u \end{array} \qquad (5.2)$$

commutes by a triangle identity and the naturality of ε. The associativity coherence condition is then given by the diagram



where, starting from the top in a clockwise direction, we use interchange, two naturalities of ε, the diagram in (5.2), a naturality of ε, a naturality of η, and finally an interchange again. Finally, the unit condition is given by the following diagram:



where we have two uses of the triangle identities and a naturality ☐

Example 5.4. We can apply Theorem 5.3 to show that the relative pseudomonad of presheaves of Example 3.9 is lax idempotent. For this, observe that for $\mathbb{X}, \mathbb{Y} \in \mathbf{Cat}$, there is an adjunction of the form

$$\mathbf{CAT}[\mathbb{X}, P(\mathbb{Y})] \underset{(-)\,\mathrm{y}_X}{\overset{(-)^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{CAT}[P(\mathbb{X}), P(\mathbb{Y})] \tag{5.3}$$

in which the components of the unit are natural isomorphisms. The left adjoints in (5.3) factor through **COC**, the sub-2-category of cocomplete locally small categories and cocontinuous functors, and if $U \colon P(\mathbb{X}) \to P(\mathbb{Y})$ is cocontinuous, then ε_U is an isomorphism. Part (iii) of Theorem 5.3 then applies. A similar example arises by considering the Ind-completion, which is also lax idempotent. This is because the corresponding left adjoint factors through **FIL** the sub-2-category of Ind-complete categories and functors preserving filtered colimits; and if $U\colon D(\mathbb{X}) \to D(\mathbb{Y})$ preserves filtered colimits then ε_U is an isomorphism.

Remark 5.5. Theorem 5.3 and [52, Theorems 4.1 and 4.2] imply that a pseudomonad T on a bicategory $\mathcal{C}$ is lax idempotent in the usual sense if and only if it is lax idempotent as a relative pseudomonad over the identity $1_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ in the sense of Definition 5.1.

6. **Liftings, extensions, and compositions**

We now discuss a general method to extend a 2-monad to the Kleisli bicategory of a relative pseudomonad, which we will apply in Section 7 to extend several 2-monads from the 2-category **Cat** of small categories and functors to the bicategory **Prof** of small categories and profunctors.

Let us begin by introducing the setting in which we will be working. We fix a pseudofunctor between 2-categories $J\colon \mathcal{C} \to \mathcal{D}$, a relative pseudomonad T over J with data as in Definition 3.1 and a 2-monad $S\colon \mathcal{D} \to \mathcal{D}$ with data as in Section 2. We assume that the 2-monad S restricts along J. Explicitly, this means that we have a dotted functor

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;J\;} & \mathcal{D} \\ {\scriptstyle S}\downarrow & & \downarrow{\scriptstyle S} \\ \mathcal{C} & \xrightarrow[\;J\;]{} & \mathcal{D}\,, \end{array}$$

and that, for $X \in \mathcal{C}$, the components of the multiplication and the unit, written $m_X\colon S^2X \to SX$ and $e_X\colon X \to SX$, respectively, are in $\mathcal{C}$. This implies that the pseudofunctor $J\colon \mathcal{C} \to \mathcal{D}$ can be lifted to pseudofunctors $J\colon \text{Ps-}S\text{-Alg}_{\mathcal{C}} \to \text{Ps-}S\text{-Alg}_{\mathcal{D}}$ and $J\colon S\text{-Alg}_{\mathcal{C}} \to S\text{-Alg}_{\mathcal{D}}$ (the definition of these 2-categories is recalled in Section 2), making the following diagram commute:

$$\begin{array}{ccc} S\text{-Alg}_{\mathcal{C}} & \xrightarrow{\;J\;} & S\text{-Alg}_{\mathcal{D}} \\ \downarrow & & \downarrow \\ \text{Ps-}S\text{-Alg}_{\mathcal{C}} & \xrightarrow{\;J\;} & \text{Ps-}S\text{-Alg}_{\mathcal{D}} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow[\;J\;]{} & \mathcal{D}\,, \end{array} \tag{6.1}$$

where the vertical arrows in the top square are inclusions and those in the bottom square are forgetful 2-functors. We shall deal with two types of liftings, one involving only strict algebras (Definition 6.1) and another one involving both strict algebras and pseudoalgebras (Definition 6.2). We begin by defining the simpler type of lifting, involving only strict algebras.

Definition 6.1. A *lifting of T to strict algebras for S*, denoted

$$\begin{array}{ccc} S\text{-Alg}_{\mathcal{C}} & \xrightarrow{\;\bar{T}\;} & S\text{-Alg}_{\mathcal{D}} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{C} & \xrightarrow[\;T\;]{} & \mathcal{D}\,, \end{array}$$

consists of

- a strict algebra structure on TA, for every $A \in S\text{-Alg}_{\mathcal{C}}$;

- a pseudomorphism structure on $f^* \colon TA \to TB$, for every pseudomorphism $f \colon JA \to TB$;
- a pseudomorphism structure on $i_A \colon JA \to TA$, for every $A \in S\text{-Alg}_{\mathcal{C}}$;

such that

- $\mu_{f,g} : (g^* \, f)^* \to g^* \, f^*$ is an algebra 2-cell for every pair of pseudomorphisms $f \colon JA \to TB$ and $g \colon JB \to TC$;
- $\eta_f \colon f \to f^* \, i_A$ is an algebra 2-cell for every pseudomorphism $f \colon JA \to TB$;
- $\theta_A \colon {i_A}^* \to 1_{TA}$ is an algebra 2-cell for $A \in S\text{-Alg}_{\mathcal{C}}$.

Note that a lifting of T to strict algebras gives immediately a relative pseudomonad $\bar{T}$ over the pseudofunctor $J \colon S\text{-Alg}_{\mathcal{C}} \to S\text{-Alg}_{\mathcal{D}}$ such that applying the forgetful 2-functors to the data of $\bar{T}$ returns the corresponding data of T. We shall give several examples of liftings of the relative monad of presheaves to strict algebras for some 2-monads in Section 7. However, it is not useful to work with liftings to categories of strict algebras for other 2-monads, since typically for a strict algebra A there is no evident structure of strict algebra structure on TA. In order to address this situation, we introduce the following definition.

Definition 6.2. A *lifting of T to pseudoalgebras for S*, denoted

$$\begin{array}{ccc} S\text{-Alg}_{\mathcal{C}} & \xrightarrow{\ \bar{T}\ } & \text{Ps-}S\text{-Alg}_{\mathcal{D}} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow[\ T\ ]{} & \mathcal{D} \end{array}$$

consists of the following data:

- a pseudoalgebra structure on TA, for every $A \in S\text{-Alg}_{\mathcal{C}}$;
- a pseudomorphism structure on $f^* \colon TA \to TB$, for every pseudomorphism $f \colon JA \to TB$;
- a pseudomorphism structure on $i_A \colon JA \to TA$, for every $A \in S\text{-Alg}_{\mathcal{C}}$;

such that

- $\mu_{f,g} \colon (g^* \, f)^* \to g^* \, f^*$ is an algebra 2-cell for every pair of pseudomorphisms $f \colon JA \to TB$ and $g \colon JB \to TC$;
- $\eta_f \colon f \to f^* \, i_A$ is an algebra 2-cell for every pseudomorphism $f \colon JA \to TB$;
- $\theta_A \colon {i_A}^* \to 1_{TA}$ is an algebra 2-cell for every $A \in S\text{-Alg}_{\mathcal{C}}$.

Similarly to what happened for liftings to strict algebras, a lifting of T to pseudoalgebras gives a relative pseudomonad $\bar{T}$, but now over the pseudofunctor $J \colon S\text{-Alg}_{\mathcal{C}} \to \text{Ps-}S\text{-Alg}_{\mathcal{D}}$, again suitably related to $\bar{T}$ via the appropriate forgetful 2-functors. Note here that for the inclusion $J \colon \mathbf{Cat} \to \mathbf{CAT}$, the corresponding inclusion $J \colon S\text{-Alg}_{\mathbf{Cat}} \to \text{Ps-}S\text{-Alg}_{\mathbf{CAT}}$ is not merely about size distinction, but involves both strict algebras and pseudoalgebras. Indeed, the notion of a relative pseudomonad was designed to encompass these situations as well.

Our next goal is to show how a lifting of a relative pseudomonad T gives rise to a pseudomonad on the Kleisli bicategory of T. In the one-dimensional situation, such a step commonly involves passing via a distributive law [6]. In our setting, where we are dealing with both coherence and size issues, such an approach would be rather complicated, as one would have to adapt the theory of pseudo-distributive laws [34, 49, 50] to relative pseudomonads. However, it is possible to take a more direct approach.

Theorem 6.3. *Assume that T has a lifting to either strict algebras or pseudoalgebras for S. Then S has an extension to a pseudomonad $\tilde{S} \colon \mathrm{Kl}(T) \to \mathrm{Kl}(T)$ on the Kleisli bicategory of T.*

Proof. We only deal with the case of a lifting to pseudoalgebras, since the case of lifting to strict algebras is completely analogous. First, we consider the relative pseudomonad $\bar{T}$ over

$J \colon S\text{-Alg}_{\mathcal{C}} \to \text{Ps-}S\text{-Alg}_{\mathcal{D}}$ and its Kleisli bicategory $\text{Kl}(\bar{T})$. The objects of $\text{Kl}(\bar{T})$ are strict algebras with underlying object in $\mathcal{C}$, and its hom-categories are given by

$$\text{Kl}(\bar{T})[A, B] = \text{Ps-}S\text{-Alg}_{\mathcal{D}}[JA, TB] \, .$$

Secondly, we observe that there is a forgetful pseudofunctor $U \colon \text{Kl}(\bar{T}) \to \text{Kl}(T)$, defined on objects by sending a strict algebra to its underlying object. To define the action on hom-categories, let $A, B \in S\text{-Alg}_{\mathcal{C}}$. Then, the required functor is determined by the diagram

$$\begin{array}{ccc} \text{Kl}(\bar{T})[A, B] & \xrightarrow{U_{A,B}} & \text{Kl}(T)[A, B] \\ \| & & \| \\ \text{Ps-}S\text{-Alg}[JA, TB] & \xrightarrow[U_{A,B}]{} & \mathcal{D}[JA, TB] \, . \end{array}$$

We claim that U has a left pseudoadjoint. The action of the left pseudoadjoint on objects is defined by sending X to SX, the free pseudoalgebra on X (which is in fact a strict algebra since S is a 2-monad). Next, for $X \in \mathcal{C}$, we define morphisms $\tilde{e}_X \colon X \to SX$ in $\text{Kl}(T)$ as the composite

$$JX \xrightarrow{e_X} SJX = JSX \xrightarrow{i_{SX}} TSX$$

in $\mathcal{D}$. We wish to show that these are suitably universal. For this, let us observe that the diagram

$$\begin{array}{ccc} \text{Kl}(T)[X, A] & \xleftarrow{U(-) \circ \tilde{e}_X} & \text{Kl}(\bar{T})[SX, A] \\ \| & & \| \\ \| & & \text{Ps-}S\text{-Alg}_{\mathcal{D}}[JSX, TA] \\ \| & & \| \\ \mathcal{D}[JX, TA] & \xleftarrow[U(-)\, e_X]{} & \text{Ps-}S\text{-Alg}_{\mathcal{D}}[SJX, TA] \end{array} \tag{6.2}$$

commutes up to natural isomorphism, since if $f \colon SJX \to TA$ is a pseudomorphism, then

$$f \circ \tilde{e}_X = f^* \, \tilde{e}_X = f^* \, i_{SX} \, e_X \cong f \, e_X \, .$$

Since the horizontal arrow at the bottom of (6.2) is an equivalence, we have the desired universality of the morphism $\tilde{e}_X$. Now that we have a pseudoadjunction

$$\text{Kl}(T) \underset{U}{\overset{F}{\rightleftarrows}} \text{Kl}(\bar{T}) \, , \qquad (F \dashv U)$$

we obtain the desired extension $\tilde{S}$ as the pseudomonad associated to this pseudoadjunction. □

We conclude this section by showing how to compose a relative pseudomonad and a 2-monad.

Theorem 6.4. *Assume that T admits a lifting to pseudoalgebras of S. Then the function sending $X \in \mathcal{C}$ to $TS(X) \in \mathcal{D}$ admits the structure of a relative pseudomonad over $J \colon \mathcal{C} \to \mathcal{D}$.*

Proof. First, recall that, by Theorem 4.4, we have a relative pseudoadjunction

$$\begin{array}{ccc} & & \text{Kl}(T) \\ & \nearrow^{F^T} & \downarrow{\scriptstyle G^T} \\ \mathcal{C} & \xrightarrow[J]{} & \mathcal{D} \, . \end{array} \tag{6.3}$$

Secondly, let us consider the pseudomonad $\tilde{S}\colon \mathrm{Kl}(T) \to \mathrm{Kl}(T)$ constructed in the proof of Theorem 6.3 and its associated Kleisli bicategory $\mathrm{Kl}(\tilde{S})$. Applying again Theorem 4.4, this time in the case of an ordinary pseudomonad, we have a pseudoadjunction

$$\mathrm{Kl}(T) \underset{G^{\tilde{S}}}{\overset{F^{\tilde{S}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Kl}(\tilde{S})\,. \tag{6.4}$$

By Proposition 4.6, we can then compose the pseudoadjunctions in (6.3) and the relative pseudoadjunction in (6.4) so as to obtain a new relative pseudoadjunction

$$\begin{array}{ccc} & & \mathrm{Kl}(\tilde{S}) \\ & {\scriptstyle F^{\tilde{S}}F^{T}} \nearrow & \Big\downarrow {\scriptstyle G^{T}G^{\tilde{S}}} \\ \mathcal{C} & \xrightarrow[J]{} & \mathcal{D}\,. \end{array}$$

By Theorem 3.8 we then obtain a relative pseudomonad over $J\colon \mathcal{C} \to \mathcal{D}$. Unfolding the definitions, one readily checks that the underlying function of this relative pseudomonad sends $X \in \mathcal{C}$ to $TS(X) \in \mathcal{D}$, as required. □

7. Substitution monoidal structures

We apply our results to obtain a homogeneous method for extending several 2-monads from the 2-category **Cat** of small categories and functors to the bicategory **Prof** of small categories and profunctors, encompassing all the examples considered in the theory of variable binding [24, 55, 61], concurrency [13], species of structures [23], models of the differential λ-calculus [21], and operads [25].

The simplest examples of liftings for the relative pseudomonad for presheaves are with respect to 2-monads on **CAT** whose strict algebras are locally small categories equipped with suitable classes of limits. These 2-monads are co-lax, as discussed in [37]. The specific examples of 2-monads that we consider here are those for categories with terminal object, categories with chosen finite products (by which we mean categories with chosen terminal object and binary products) and categories with chosen finite limits (by which we mean categories with chosen terminal object and pullbacks). Each of these 2-monads is flexible in the sense of [9, 10] and restricts along the inclusion $J\colon \mathbf{Cat} \to \mathbf{CAT}$ to a 2-monad on the 2-category **Cat** of small categories, so as to determine a situation as in (6.1). We speak of small (or locally small) strict algebras to indicate small (or locally small) categories equipped with a strict algebra structure.

We make some preliminary observations about the pseudomorphisms (cf. (2.1)) in the cases under consideration. Since limits are determined up to a unique isomorphism, the pseudomorphisms are exactly the functors that preserve the specified limits in the usual up to isomorphism sense: the coherence conditions for a pseudomorphism are automatic [37]. Similarly, one sees directly that any 2-cell between functors preserving the relevant limits is an algebra 2-cell. In the terminology of [37], these 2-monads S are fully property-like. It follows in particular that $S\text{-}\mathrm{Alg}_{\mathbf{CAT}}[\mathbb{A}, \mathbb{B}]$ can be regarded as a full subcategory of $\mathbf{CAT}[\mathbb{A}, \mathbb{B}]$. All this is in fact an abstract consequence of the fact that the 2-monads in question are all co-lax. That fact is evident and the general theory appears in [37].

Theorem 7.1. *Let $S\colon \mathbf{CAT} \to \mathbf{CAT}$ be the 2-monad for categories with terminal object, or categories with finite products, or categories with finite limits. Then the relative pseudomonad of presheaves $P\colon \mathbf{Cat} \to \mathbf{CAT}$ has a lifting to strict S-algebras,*

$$\bar{P}\colon S\text{-}\mathrm{Alg}(\mathbf{Cat}) \to S\text{-}\mathrm{Alg}(\mathbf{CAT})\,.$$

Proof. Let us begin by observing that we have a choice of limits in **Set**, so for any $\mathbb{X} \in \mathbf{Cat}$, $P(\mathbb{X})$ has chosen limits defined pointwise. Thus, there is a strict S-algebra structure on $P(\mathbb{X})$. Furthermore, the Yoneda embedding $\mathrm{y}_{\mathbb{X}} \colon \mathbb{X} \to P(\mathbb{X})$ preserves those limits. Hence, if $\mathbb{A} \in S\text{-Alg}_{\mathbf{Cat}}$ is a small strict algebra, then $\mathrm{y}_{\mathbb{A}} \colon \mathbb{A} \to P(\mathbb{A})$ is a pseudomorphism of S-algebras in an evident fashion. Composition with $\mathrm{y}_{\mathbb{A}}$ thus gives us a functor

$$S\text{-Alg}_{\mathbf{CAT}}[\mathbb{A}, P(\mathbb{B})] \xleftarrow[(-)\,\mathrm{y}_{\mathbb{A}}]{} S\text{-Alg}_{\mathbf{CAT}}[P(\mathbb{A}), P(\mathbb{B})]\,.$$

Now, suppose that $F \colon \mathbb{A} \to P(\mathbb{B})$ is a pseudomorphism, that is to say, F preserves the relevant limits. Then the left Kan extension $F^* \colon P(\mathbb{A}) \to P(\mathbb{B})$ also preserves these limits. This is critical, and for the separate classes of limits needs to be proved on a case by case basis. The case when S is the 2-monad for a terminal object is simple. If F preserves the terminal, then so does $F^*\,\mathrm{y}_{\mathbb{A}}$ (being naturally isomorphic to F). But the Yoneda $\mathrm{y}_{\mathbb{A}} \colon \mathbb{A} \to P(\mathbb{A})$ preserves the terminal object, and hence so does F^*. The case when S is the 2-monad for finite products can be seen as a corollary of the results in [31] (see also Theorem 7.3), but we provide a direct argument. Suppose that F and hence $F^*\,\mathrm{y}_{\mathbb{A}}$ preserves finite products. As the Yoneda $\mathrm{y}_{\mathbb{A}} \colon \mathbb{A} \to P(\mathbb{A})$ preserves finite products, F^* preserves finite products of representables. But the objects of $P(\mathbb{A})$ are colimits of representables. Since F^* and products with objects (are left adjoints and so) preserve colimits, it follows that F^* preserves finite products. Finally, the case when S is the monad for finite limits is similar, though in this case the result is standard. If $\mathbb{A}$ has finite limits and $F \colon \mathbb{A} \to P(\mathbb{B})$ preserves finite limits, then F is flat [42, §VII.10, Corollary 3] and hence F^* preserves finite limits.

Thus, in each case, F^* is a pseudomorphism of strict S-algebras; and, as we observed above, any 2-cell between pseudomorphisms will be an algebra 2-cell. Hence, the left Kan extension gives us a functor

$$S\text{-Alg}_{\mathbf{CAT}}[\mathbb{A}, P(\mathbb{B})] \xrightarrow{(-)^*} S\text{-Alg}_{\mathbf{CAT}}[P(\mathbb{A}), P(\mathbb{B})]\,.$$

Now we exploit the fact that the relative pseudomonad for presheaves is lax idempotent (see Example 5.4). So we have an adjunction

$$\mathbf{CAT}[\mathbb{A}, P(\mathbb{B})] \underset{(-)\,\mathrm{y}_{\mathbb{A}}}{\overset{(-)^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{CAT}[P(\mathbb{A}), P(\mathbb{B})]\,. \tag{7.1}$$

We observed that $S\text{-Alg}[\mathbb{A}, P(\mathbb{B})]$ and $S\text{-Alg}[P(\mathbb{A}), P(\mathbb{B})]$ are full subcategories of $\mathbf{CAT}[\mathbb{A}, P(\mathbb{B})]$ and $\mathbf{CAT}[P(\mathbb{A}), P(\mathbb{B})]$, and it is clear from the above discussion that this adjunction restricts to an adjunction

$$S\text{-Alg}_{\mathbf{CAT}}[\mathbb{A}, P(\mathbb{B})] \underset{(-)\,\mathrm{y}_{\mathbb{A}}}{\overset{(-)^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} S\text{-Alg}_{\mathbf{CAT}}[P(\mathbb{A}), P(\mathbb{B})]\,.$$

In view of Theorem 5.3, the claim is proved. □

Remark 7.2. With the experience of these examples of liftings, it is easy to give examples of 2-monads which do not lift as above.

- Consider the 2-monad for a category with zero object (i.e. an object which is both terminal and initial). No category of presheaves of sets over a non-empty category has a zero object. So the 2-monad cannot lift. The same applies to the monad for direct sums or biproducts (in the terminology of [41]).

- Consider the 2-monad for a category with initial object. Given a category $\mathbb{A}$ with initial object, while the presheaf category $P(\mathbb{A})$ does indeed have an initial object, the Yoneda embedding does not preserve it. Hence the 2-monad cannot lift.
- Consider the 2-monad for a category with equalisers. Given a category $\mathbb{A}$ with equalisers, the presheaf category $P(\mathbb{A})$ also has equalisers, and the Yoneda embedding $\mathrm{y}_{\mathbb{A}} \colon \mathbb{A} \to P(\mathbb{A})$ preserves them. But now suppose that $\mathbb{A}$ has equalisers and that $F \colon \mathbb{A} \to \mathbf{Set}$ preserves them. It does not follow that $F^* \colon P(\mathbb{A}) \to \mathbf{Set}$ preserves equalisers. For a counterexample one can obviously just take $\mathbb{A}$ to be the fork (i.e. the generic equaliser). Then for example take $F \colon \mathbb{A} \to \mathbf{Set}$ mapping the parallel pair to the identity and twist on 2 with equaliser 0. Because of this failure it follows that the 2-monad cannot lift.

Next, we consider 2-monads associated with various notions of monoidal category. To start with, we consider 2-monads which are flexible in the sense of [9, 10] and we have again a situation as in (6.1).

Theorem 7.3. *Let $S \colon \mathbf{CAT} \to \mathbf{CAT}$ be the 2-monad for monoidal categories, or symmetric monoidal categories, or monoidal categories in which the unit is a terminal object, or symmetric monoidal categories in which the unit is a terminal object. The relative pseudomonad of presheaves $P \colon \mathbf{Cat} \to \mathbf{CAT}$ has a lifting to strict S-algebras,*

$$\bar{P} \colon S\text{-}\mathrm{Alg}_{\mathbf{Cat}} \to S\text{-}\mathrm{Alg}_{\mathbf{CAT}} \,.$$

Proof. The base case is that of a monoidal category. We discuss that case and derive the others. We use the analysis in [31] of the univeral property of Day's convolution tensor product [18]. We write **Mon** (respectively, **MON**) for the 2-category of small (respectively, locally small) monoidal categories, strong monoidal functors and monoidal natural transformations. For cocomplete categories $\mathbb{A}$, $\mathbb{B}$, a functor $F \colon \mathbb{A} \times \mathbb{B} \to \mathbb{C}$ is *separately cocontinuous* if for every $a \in \mathbb{A}$, $b \in \mathbb{B}$ both $F(a, -) \colon \mathbb{B} \to \mathbb{C}$ and $F(-, b) \colon \mathbb{A} \to \mathbb{C}$ are cocontinous. We write $\mathbf{COC}[\mathbb{A}, \mathbb{B}; \mathbb{C}]$ for the category of such functors and natural transformations between them. A cocomplete category $\mathbb{A}$ equipped with a monoidal structure is *monoidally cocomplete* if the tensor product is separately cocontinuous. We then have a straightforward 2-category **MONCOC** of monoidally cocomplete locally small categories, strong monoidal cocontinuous functors, and monoidal transformations.

In [18] Day showed how for any small monoidal category $\mathbb{A}$, the category $P(\mathbb{A})$ of presheaves on $\mathbb{A}$ can be equipped with a monoidal structure, called the convolution tensor product, which makes $P(\mathbb{A})$ into a biclosed monoidally cocomplete category, defined by letting

$$(F_1 \hat{\otimes} F_2)(a) =_{\mathrm{def}} \int^{a_1, a_2 \in \mathbb{A}} F_1(a_1) \times F_2(a_2) \times \mathbb{A}[a, a_1 \otimes a_2]$$

for $F_1, F_2 \in P(\mathbb{A})$ and $a \in \mathbb{A}$. Furthermore, the Yoneda embedding $\mathrm{y}_{\mathbb{A}} \colon \mathbb{A} \to P(\mathbb{A})$ has then the structure of a strong monoidal functor. For $\mathbb{A} \in \mathbf{Mon}$ and $\mathbb{B} \in \mathbf{MONCOC}$, we have the adjoint equivalence obtained in [31]

$$\mathbf{MON}[\mathbb{A}, \mathbb{B}] \underset{(-)\,\mathrm{y}_{\mathbb{A}}}{\overset{(-)^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{MONCOC}[P(\mathbb{A}), \mathbb{B}] \,, \tag{7.2}$$

as required. In particular, for any $\mathbb{A}, \mathbb{B} \in \mathbf{Mon}$, we have an adjoint equivalence

$$\mathbf{MON}[\mathbb{A}, P(\mathbb{B})] \underset{(-)\,\mathrm{y}_{\mathbb{A}}}{\overset{(-)^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbf{MONCOC}[P(\mathbb{A}), P(\mathbb{B})] \,.$$

In our terminology, the adjoint equivalences in (7.2) amount to saying that we have a relative pseudoadjunction

$$\begin{array}{ccc} & & \mathbf{MONCOC} \\ & \overset{P}{\nearrow} & \downarrow \\ \mathbf{Mon} & \longrightarrow & \mathbf{MON}\,. \end{array}$$

This provides exactly a lifting of the relative pseudomonad $P\colon \mathbf{Cat} \to \mathbf{CAT}$ to a relative pseudomonad $\bar{P}\colon S\text{-Alg}_{\mathbf{Cat}} \to S\text{-Alg}_{\mathbf{CAT}}$. All these considerations extend to symmetric monoidal categories, again by the results in [18, 31]. For the 2-monads for monoidal categories with the condition that the unit is terminal, the lift follows from the above, observing that the unit of the convolution monoidal structure is the Yoneda embedding of the unit on the base category and so it remains a terminal object. □

Our final group of examples of a lifting involve 2-monads on **CAT** which are not flexible. In this case, we have a lifting to pseudoalgebras in the sense of Definition 6.2.

Theorem 7.4. *Let $S\colon \mathbf{Cat} \to \mathbf{Cat}$ be the 2-monad for either strict monoidal categories, or symmetric strict monoidal categories, or strict monoidal category in which the unit is terminal, or symmetric strict monoidal categories in which the unit is terminal. Then the relative pseudomonad $P\colon \mathbf{Cat} \to \mathbf{CAT}$ has a lifting to pseudo-S-algebras,*

$$\bar{P}\colon S\text{-Alg}_{\mathbf{Cat}} \to \text{Ps-}S\text{-Alg}_{\mathbf{CAT}}\,.$$

Proof. There is a direct and an indirect approach to this. Directly, one follows through the arguments of the previous section making the necessary adjustments. Indirectly, observe that in each case S', the flexible 2-monad associated to S, is the 2-monad whose strict algebras are categories with unbiased structure (in the sense of [46]) as in the list in Theorem 7.3. Now S-Alg is a full sub-2-category of S'-Alg $\cong$ Ps-S-Alg. So the lifting of the relative pseudomonad of presheaves $P\colon \mathbf{Cat} \to \mathbf{CAT}$ to $\bar{P}\colon S'\text{-Alg}(\mathbf{Cat}) \to S'\text{-Alg}(\mathbf{CAT})$ restricts to $S\text{-Alg}(\mathbf{Cat}) \to S'\text{-Alg}(\mathbf{Cat})$. □

Corollary 7.5. *All the 2-monads on* **Cat** *listed in Theorems 7.1,7.3, and 7.4 admit an extension to pseudomonads on* **Prof***.*

Proof. Immediate consequence of Theorem 6.3 and Theorems 7.1, 7.3, and 7.4. □

For each of the monads $S\colon \mathbf{Cat} \to \mathbf{Cat}$ above, one can consider the Kleisli bicategory associated to the pseudomonad $\tilde{S}\colon \mathbf{Prof} \to \mathbf{Prof}$ determined by Corollary 7.5. The composition functors of these Kleisli bicategories can be understood as generalizations of various kinds of substitution monoidal structures [22, 23, 24, 25, 27, 36, 57], among those giving rise to the notions of a many-sorted Lawvere theory and of a coloured operad. We conclude the paper by illustrating this idea in the case of coloured operads.

Example 7.6. As an illustration of the theory developed here, we revisit the construction of the bicategory of generalized species of structures of [23] and relate more precisely its composition with the substitution monoidal structure for coloured operads [4] (see also [20]). For this, let us begin by recalling the definition of the 2-monad $S\colon \mathbf{Cat} \to \mathbf{Cat}$ for symmetric strict monoidal categories. Let $\mathbb{X} \in \mathbf{Cat}$. For $n \in \mathbb{N}$, define $S_n(\mathbb{X})$ to be the category having as objects n-tuples $\bar{x} = (x_1, \ldots, x_n)$ of objects $x_i \in \mathbb{X}$ and as morphisms $(\sigma, \bar{f})\colon \bar{x} \to \bar{x}'$ given by pairs consisting of a permutation $\sigma \in \Sigma_n$ and an n-tuple of morphisms $f_i\colon x_i \to x'_{\sigma(i)}$. We then let

$$S(\mathbb{X}) =_{\text{def}} \bigsqcup_{n \in \mathbb{N}} S_n(\mathbb{X})\,.$$

The category $S(\mathbb{X})$ is equipped with a strict symmetric monoidal structure: the tensor product, written $\bar{x} \oplus \bar{x}'$, is given by concatenation of sequences, and the unit is given by the empty sequence; the symmetry is given by a permutation of identity maps. This definition can be extended easily to obtain a 2-functor $S\colon \mathbf{Cat} \to \mathbf{Cat}$. The multiplication of the monad is given by taking a sequence of sequences and forgetting the bracketing, while the unit has components $e_{\mathbb{X}}\colon \mathbb{X} \to S(\mathbb{X})$ mapping $x \in \mathbb{X}$ to the singleton sequence $(x) \in S(\mathbb{X})$. By the theory developed above, and Corollary 7.5 in particular, we obtain a pseudomonad

$$\tilde{S}\colon \mathbf{Prof} \to \mathbf{Prof}\,. \tag{7.3}$$

For our purposes, it is convenient to describe explicitly the relative pseudomonad associated to $\tilde{S}$. Its action on objects is the function mapping $\mathbb{X} \in \mathbf{Cat}$ to $S(\mathbb{X}) \in \mathbf{Cat}$. The component of the unit for $\mathbb{X} \in \mathbf{Cat}$ is the profunctor $\tilde{e}_{\mathbb{X}}\colon \mathbb{X} \to S(\mathbb{X})$ corresponding to the functor

$$\mathbb{X} \xrightarrow{\;e_{\mathbb{X}}\;} S(\mathbb{X}) \xrightarrow{\;\mathrm{y}_{S(\mathbb{X})}\;} PS(\mathbb{X})\,.$$

The extension functors of the relative pseudomonad have the form

$$(-)^\sharp\colon \mathbf{Prof}[\mathbb{X}, S(\mathbb{Y})] \to \mathbf{Prof}[S(\mathbb{X}), S(\mathbb{Y})]\,,$$

where $\mathbb{X}, \mathbb{Y}$ are small categories. For a functor $F\colon \mathbb{X} \to PS(\mathbb{Y})$, we can define the functor $F^\sharp\colon S(\mathbb{X}) \to PS(\mathbb{Y})$ recalling that, since $S(\mathbb{Y})$ has a symmetric strict monoidal structure, $PS(\mathbb{Y})$ has an unbiased (in the sense of [46]) symmetric monoidal structure. Hence, by the universal property of $S(\mathbb{X})$, we have an essentially unique $F^\sharp$ fitting into a diagram of the following form:

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{\;e_{\mathbb{X}}\;} & S(\mathbb{X}) \\ & \cong & \big\downarrow F^\sharp \\ & \underset{F}{\searrow} & PS(\mathbb{Y})\,. \end{array}$$

For brevity, we omit the description of the invertible natural transformations

$$\eta_F\colon F \Rightarrow F^\sharp \circ \tilde{e}_{\mathbb{X}}\,, \qquad \mu_{F,G}\colon (G^\sharp \circ F)^\sharp \Rightarrow G^\sharp \circ F^\sharp\,, \qquad \kappa_{\mathbb{X}}\colon (\tilde{e}_{\mathbb{X}})^\sharp \Rightarrow \mathrm{Id}_{\mathbb{X}}\,.$$

The Kleisli bicategory of $\tilde{S}$ is the bicategory S-$\mathbf{Prof}$ of S-profunctors defined in [25], which has small categories as objects and hom-categories defined by

$$S\text{-}\mathbf{Prof}[\mathbb{X}, \mathbb{Y}] = \mathbf{Prof}[\mathbb{X}, S(\mathbb{Y})] = \mathbf{CAT}[S(\mathbb{Y})^{\mathrm{op}} \times \mathbb{X}, \mathbf{Set}]\,.$$

Indeed, one can readily check that composition and identity morphisms of S-$\mathbf{Prof}$, as defined in [25], coincide with those given by instantiating the general definition of a Kleisli bicategory. Following [25], we write $\mathbf{CatSym}$ for the bicategory of categorical symmetric sequences, which is defined as the opposite of S-$\mathbf{Prof}$. Thus, the objects of $\mathbf{CatSym}$ are small categories and its hom-categories are given by

$$\mathbf{CatSym}[\mathbb{X}, \mathbb{Y}] =_{\mathrm{def}} S\text{-}\mathbf{Prof}[\mathbb{Y}, \mathbb{X}] = \mathbf{CAT}[S(\mathbb{X})^{\mathrm{op}} \times \mathbb{Y}, \mathbf{Set}]\,.$$

We write $F[\bar{x}; y]$ for the result of applying $F\colon S(\mathbb{X})^{\mathrm{op}} \times \mathbb{Y} \to \mathbf{Set}$ to $(\bar{x}, y) \in S(\mathbb{X})^{\mathrm{op}} \times \mathbb{Y}$. Given categorical symmetric sequences $F\colon \mathbb{X} \to \mathbb{Y}$ and $G\colon \mathbb{Y} \to \mathbb{Z}$, i.e. functors $F\colon S(\mathbb{X})^{\mathrm{op}} \times \mathbb{Y} \to \mathbf{Set}$ and $G\colon S(\mathbb{Y})^{\mathrm{op}} \times \mathbb{Z} \to \mathbf{Set}$, their composite $G \circ F\colon \mathbb{X} \to \mathbb{Z}$ in $\mathbf{CatSym}$ is given by considering F and G as S-profunctors in the opposite direction, taking their composition in S-$\mathbf{Prof}$ using the definition of composition in a Kleisli bicategory, and then regarding the result as a categorical symmetric sequence from $\mathbb{X}$ to $\mathbb{Z}$. Explicitly, one obtains that

$$(G \circ F)(\bar{x}; z) =_{\text{def}} \bigsqcup_{m \in \mathbb{N}} \int^{(y_1, \ldots, y_m) \in S_m(\mathbb{Y})} G[y_1, \ldots, y_m; z] \times$$
$$\int^{\bar{x}_1 \in S(\mathbb{X})} \cdots \int^{\bar{x}_m \in S(\mathbb{X})} F[\bar{x}_1; y_1] \times \ldots \times F[\bar{x}_m; y_m] \times S(\mathbb{X})[\bar{x}, \bar{x}_1 \oplus \ldots \oplus \bar{x}_m] \,. \quad (7.4)$$

Happily, this formula yields the definition of the substitution monoidal structure for coloured operads given in [3] by considering the special case where $\mathbb{X}$ and $\mathbb{Y}$ are discrete and coincide with a fixed set of colours of the coloured operads under consideration.

The bicategory **CatSym** can be seen to be equivalent to the bicategory of generalized species of structures **Esp** previously introduced in [23]. To see this, let us recall the definition of **Esp**. For this, observe that the duality pseudofunctor $(-)^{\perp} \colon \mathbf{Prof} \to \mathbf{Prof}$ defined by $\mathbb{X}^{\perp} =_{\text{def}} \mathbb{X}^{\text{op}}$ allows us to turn this pseudomonad in (7.3) into a pseudocomonad. The bicategory **Esp** is then defined as the co-Kleisli bicategory of this pseudocomonad. More explicitly, its objects are small categories and its hom-categories are given by

$$\mathbf{Esp}[\mathbb{X}, \mathbb{Y}] = \mathbf{Prof}[S(\mathbb{X}), \mathbb{Y}] = \mathbf{CAT}[\mathbb{Y}^{\text{op}} \times S(\mathbb{X}), \mathbf{Set}] \,.$$

The bicategory **Esp** is then equivalent to **CatSym**, via the pseudofunctor that sends $\mathbb{X}$ to $\mathbb{X}^{\text{op}}$. Indeed,

$$\mathbf{CatSym}[\mathbb{X}, \mathbb{Y}] = \mathbf{CAT}[S(\mathbb{X}^{\text{op}}) \times \mathbb{Y}, \mathbf{Set}] \cong [\mathbb{Y} \times S(\mathbb{X})^{\text{op}}, \mathbf{Set}] \cong \mathbf{Esp}[\mathbb{X}^{\text{op}}, \mathbb{Y}^{\text{op}}] \,.$$

Furthermore, the composition operation of categorical symmetric sequences defined in (7.4) corresponds exactly to composition of generalized species of structures defined via co-Kleisli composition given in [23].

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