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Hopf measuring comonoids and enrichment

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ABSTRACT

We study the existence of universal measuring comonoids $P(A, B)$ for a pair of monoids A, B in a braided monoidal closed category, and the associated enrichment of the category of monoids over the monoidal category of comonoids. In symmetric categories, we show that if A is a bimonoid and B is a commutative monoid, then $P(A, B)$ is a bimonoid; in addition, if A is a cocommutative Hopf monoid then $P(A, B)$ always is Hopf. If A is a Hopf monoid, not necessarily cocommutative, then $P(A, B)$ is Hopf if the fundamental theorem of comodules holds; to prove this we give an alternative description of the dualizable $P(A, B)$-comodules and use the theory of Hopf (co)monads. We explore the examples of universal measuring comonoids in vector spaces and graded spaces.

1. *Introduction*

The *finite* or *Sweedler dual* of a k-algebra [**29**] plays a central role in the duality theory of Hopf algebras. If A is an algebra over a field k, its finite dual A° is a coalgebra with the property that coalgebra morphisms $C \to A^\circ$ are in natural bijection with algebra morphisms $A \to C^*$, for any coalgebra C. When A has finite dimension, A° is isomorphic to the linear dual A^*, but in arbitrary dimension A^* may not have a natural coalgebra structure.

The classical construction of the finite dual A° depends on the fact that k is a field, a hypothesis that was somewhat weakened in [**2**]. The existence of a coalgebra A° satisfying the universal property described in the previous paragraph can be proven in great generality (see [**24**], but also Section 4 where a braiding is not required); in particular, A° exists over any commutative ring, but its classical description may no longer hold true.

That fact that the category of k-algebras admits an enrichment in the category of k-coalgebras has long been part of mathematical folklore. It seems that Gavin Wraith was the first to appreciate this fact and that he lectured on it and related matters at the Oberwolfach meeting *Universelle und Kategorische Algebra*, 3–10 July 1968. When Sweedler's book [**29**] on Hopf algebras came out, Wraith immediately recognised that the enrichment is given by what Sweedler called the *universal measuring coalgebra* of a pair of algebras.

In this paper we explore the existence of a generalisation of the finite dual construction, called *universal measuring comonoids*$P(A, B)$, for a pair of monoids A, B in a monoidal closed category, and the properties of this construction. The comonoid $P(A, B)$ is defined by the the property that monoid morphisms $A \to [C, B]$ are in natural bijection with comonoid morphisms $C \to P(A, B)$, for all comonoids C; note that $A^\circ = P(A, I)$, where I is the monoidal unit. We show that, when the monoidal category has a braiding, the functor with mapping on objects $(A, B) \mapsto P(A, B)$ is monoidal, so there are coherent comonoid morphisms

$$P(A, B) \otimes P(A', B') \longrightarrow P(A \otimes A', B \otimes B') \quad \text{and} \quad I \longrightarrow P(I, I) \tag{1}$$

(the latter is invertible), and when the braiding is a symmetry, P is a braided monoidal functor.

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The enrichment of the opposite of the category $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ of monoids in $\mathcal{V}$ over the category $\mathbf{Comon}(\mathcal{V})$ of comonoids in $\mathcal{V}$ arises from an action of the latter, viewed as a monoidal category, on the former. We are lead to consider actions of monoidal categories and answer the following question: what extra structure on an action of the monoidal category $\mathcal{C}$ on $\mathcal{A}$ ensures that the associated $\mathcal{C}$-enriched category is monoidal? This extra structure is what we call an *opmonoidal action*, and we use it to deduce that for a symmetric $\mathcal{V}$, the category of monoids is a symmetric monoidal $\mathbf{Comon}(\mathcal{V})$-category.

The classical construction of the Sweedler dual A° of a k-algebra [**29**] satisfies two important properties: A° is a bialgebra if A is so, and A° is Hopf algebra if A is so. We show in complete generality that $P(A,B)$ is a bimonoid if A is a bimonoid and B is a commutative monoid. We then prove that $P(A,B)$ is a Hopf monoid in two situations. First, if A is a Hopf cocommutative bimonoid and B is commutative; secondly, if A is a Hopf monoid (not necessarily cocommutative) and the base symmetric monoidal category satisfies the fundamental theorem of comodules. To prove this last result, we provide an alternative description of the dualizable $P(A,B)$-comodules, as dualizable objects X equipped with a morphism $A\otimes X\to X\otimes B$ that satisfies two axioms.

We now briefly outline the organisation of the article. Section 2 collects some known facts about monoidal closed categories, monoids and comonoids, and locally presentable categories. After recalling the connection between actions of monoidal categories and enrichment, Section 3 introduces *opmonoidal actions* and *braided opmonoidal actions*, and proves that they give rise to monoidal and braided monoidal enriched categories. Section 4 studies the existence of universal measuring comonoids in a more general setting than the category of R-modules of [**24**], namely, in locally presentable monoidal categories. The enrichment of monoids in comonoids is recovered in Section 5, while Section 6 gives some tools to compute universal measuring comonoids, especially via their comodules. Section 7 explores induced monoidal structures of dualizable comodules, and 8 certain (co)commutativity relations for $P(A,B)$. In Section 9, we move to the Hopf setting by proving that the universal measuring comonoids of cocommutative Hopf monoids are Hopf monoids, in the general context of a locally presentable symmetric monoidal closed category. The case when the Hopf monoid is not necessarily cocommutative is dealt with in Section 10, which also contains some aspects of the theory of Hopf monads and comonads. The example of graded coalgebras is given its own in Section 11.

The presentation of Sections 2, 4, 5 and part of Section 3 is similar to that found in [**31**, **32**]. Soon after the first version of this manuscript was made public, the preprint [**26**] appeared, containing some overlapping material.

2. *Background*

Let us start the section with a few words on terminology and notation around monoidal categories, for which [**14**] is a standard reference. Throughout the paper, the tensor product and unit object of monoidal categories will be denoted by $\otimes$ and I, and the associativity and unit constraints will be omitted in many occasions (something that is allowed by the coherence theorem for monoidal categories). A *left closed* monoidal category $\mathcal{V}$ will be one for which the functor $(-\otimes X)$ has a right adjoint $[X,-]$, for all objects X; the resulting functor $[-,-]$ is called the *left internal hom*. Symmetrically, a *right closed* monoidal category is one for which each $(X\otimes -)$ has a right adjoint $[X,-]'$, called the *right internal* hom. Braidings will be denoted by the letter c, and they induce a *biclosed* monoidal structure on $\mathcal{V}$, should it be right or left closed.

A dual pair in a monoidal category is a pair of objects X, $X^\vee$ with two morphisms $\mathrm{ev}\colon X^\vee\otimes X\to I$ and $\mathrm{coev}\colon I\to X\otimes X^\vee$, satisfying two 'triangular equalities'; $X^\vee$ is said to be a *left dual* of X, and, reciprocally, X a *right dual* of $X^\vee$. A dual pair induces an adjunction $(-\otimes X)\dashv$

$(-\otimes X^\vee)$, and $Y \otimes X^\vee$ is the left internal hom from X to Y. When the monoidal category is braided, right duals can be obtained from left duals, via the braiding, and the adjectives 'left' and 'right' may be dropped. For example, a k-module X has a dual if and only if it is projective and finitely presentable, in which case the dual is the usual linear dual X^*.

An object of a braided monoidal category $\mathcal{V}$ is *dualizable* if it has a dual (left dual, or equivalently, right dual). Given a functor $U\colon \mathcal{C} \to \mathcal{V}$, an object $X \in \mathcal{C}$ is *U-dualizable*, or simply dualizable when U is implicit, if $U(X)$ is dualizable in $\mathcal{V}$.

A *monoidal functor* between monoidal categories $\mathcal{V}$ and $\mathcal{W}$ will be a functor $F\colon \mathcal{V} \to \mathcal{W}$ equipped with a transformation $F_{2,X,Y}\colon F(X) \otimes F(Y) \to F(X \otimes Y)$ and a morphism $F_0 : I \to F(I)$ satisfying coherence axioms; see, for example, [**14**, § 1]. Other names in use for this notion are *tensor functor* or *lax monoidal functor*. The dual notion will be called an *opmonoidal functor*, that is, F is equipped with a transformation $F_{2,X,Y}\colon F(X \otimes Y) \to F(X) \otimes F(Y)$ and a morphism $F_0\colon F(I) \to I$, satisfying coherence axioms. Other names in use for this notion are *colax monoidal functor* and *oplax monoidal functor*. If $F_{2,X,Y}$ and F_0 are isomorphisms (respectively, identities), F is a *strong* (respectively, *strict*) monoidal functor, and it is moreover *braided* monoidal when it preserves the braiding in the appropriate sense.

Throughout the paper, we employ the well-known fact that the right adjoint of a strong monoidal functor between monoidal categories has a unique monoidal structure such that the unit and counit of the adjunction become monoidal natural transformations. This generalises to the case of parametrised adjoints; a higher dimension version of the following proposition appeared in [**8**, Proposition 2].

PROPOSITION 2.1. *Suppose $F : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{B}$ are parametrised adjoints, that is, $F(-,C) \dashv G(C,-)$ for all C, and all the categories are monoidal. Then there is a bijection between opmonoidal structures on F and monoidal structures on G.*

Let $\mathcal{V}$ be a braided monoidal closed category. Recall from [**14**, § 5] that the braiding endows $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ with a strong monoidal structure, given by

$$1 \otimes c_{Y \otimes Z \otimes W}\colon X \otimes Y \otimes Z \otimes W \xrightarrow{\sim} X \otimes Z \otimes Y \otimes W \quad \text{and} \quad I \cong I \otimes I. \tag{2}$$

By definition, the internal hom is a parametrised right adjoint to $(-\otimes A) \cong (A \otimes -)$. As a result, the bifunctor $[-,-] : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ has a monoidal structure, by Proposition 2.1. In the case that the braiding is a symmetry, both the tensor product and internal hom become braided monoidal functors.

2.1. *Monoids, comonoids and bimonoids*

A *monoid* in a monoidal category $\mathcal{V}$ is an object A equipped with a multiplication and unit morphisms $\mu\colon H \otimes H \to H \leftarrow I : \iota$ that satisfy the usual associativity and unit axioms, that we depict.

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes 1"] \arrow[d, "1\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\iota\otimes 1"] \arrow[dr, "1"'] & A \otimes A \arrow[d, "\mu"] & A \arrow[l, "1\otimes\iota"'] \arrow[dl, "1"] \\ & A & \end{tikzcd} \tag{3}$$

A morphism of monoids $(A, \iota, \mu) \to (A', \iota', \mu')$ is a morphism $f\colon A \to A'$ in $\mathcal{V}$ compatible with multiplication and unit; that is, such that $\mu' \cdot (f \otimes f) = f \cdot \mu$ and $f \cdot \iota = \iota'$.

A *comonoid* in $\mathcal{V}$ is a monoid in the opposite monoidal category $\mathcal{V}^{\mathrm{op}}$; it consists of an object C with a comultiplication $\Delta\colon C \to C \otimes C$ and a counit $\varepsilon\colon C \to I$ satisfying axioms dual to those of a monoid. Morphisms of comonoids are defined in a dual fashion to morphisms of monoids.

The categories of monoids and comonoids in a monoidal category $\mathcal{V}$ will be denoted, respectively, by $\mathbf{Mon}(\mathcal{V})$ and $\mathbf{Comon}(\mathcal{V})$. For $\mathcal{V}$ braided, these categories are monoidal: if A and A' are monoids, then $A \otimes A'$ has a monoid structure with multiplication $(\mu \otimes \mu') \cdot (A \otimes c_{A',A} \otimes A')$. The respective forgetful functors into $\mathcal{V}$ are strict monoidal. These categories need not support a braiding unless $\mathcal{V}$ is symmetric, as explained below.

REMARK 2.2. The monoidal category $\mathbf{Comon}(\mathcal{V})$ on a braided monoidal category $\mathcal{V}$ has a braiding given by $c_{AB}\colon A \otimes B \to B \otimes A$, that is, c_{AB} is a morphism of comonoids, if c is a symmetry. The analogous result holds for $\mathbf{Mon}(\mathcal{V})$.

Monoidal functors preserve monoids, in the sense that, given a monoid A and a monoidal functor $F\colon \mathcal{V} \to \mathcal{W}$, then $F(A)$ is a monoid with multiplication $F(\mu) \cdot F_{2,A,A}$ and unit $F(\iota) \cdot F_0$. We denote the induced functor between the categories of monoids by $\mathbf{Mon}(F) : \mathbf{Mon}(\mathcal{V}) \to \mathbf{Mon}(\mathcal{W})$. Dually, opmonoidal functors preserve comonoids.

As an example, the monoidal structure of $\mathbf{Mon}(\mathcal{V})$ and $\mathbf{Comon}(\mathcal{V})$ can be deduced from $\otimes$ being strong monoidal. Also, the internal hom functor induces

$$\mathbf{Mon}[-,-]\colon \mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \longrightarrow \mathbf{Mon}(\mathcal{V}) \quad (C, A) \mapsto [C, A]. \tag{4}$$

In particular, whenever C is a comonoid and A a monoid, the object $[C, A]$ is endowed with the structure of a monoid, sometimes called the *convolution* monoid structure. We record for later reference:

LEMMA 2.3. *The internal hom functor $[-,-]$ of a braided monoidal closed category $\mathcal{V}$ induces a functor $\mathbf{Comon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \to \mathbf{Mon}(\mathcal{V})$. When the braiding is a symmetry, the domain and codomain are symmetric monoidal categories and this functor is braided.*

In any braided monoidal category $\mathcal{V}$, the braiding allows us to define *opposite* monoids and comonoids. In contrast to the case of a symmetric monoidal category (that is, when $c^{-1}_{X,Y} = c_{Y,X}$), there are two choices of opposite, one that employs c and the other that employs c^{-1}. If (A, ι, μ) is a monoid, we denote by $A^{\mathrm{op},c}$ and $A^{\mathrm{op},c^{-1}}$ the monoids with multiplication $\mu \cdot c_{A,A}$ and $\mu \cdot c^{-1}_{A,A}$ respectively. Similarly, if (C, ε, Δ) is a comonoid, we denote by $C^{\mathrm{cop},c}$ and $C^{\mathrm{cop},c^{-1}}$ the comonoids with comultiplication $c_{C,C} \cdot \Delta$ and $c^{-1}_{C,C} \cdot \Delta$. If c is a symmetry, we clearly only have A^{op} and C^{cop}. A monoid A is then called *commutative* if $A = A^{\mathrm{op}}$, and dually for a *cocommutative* comonoid $C = C^{\mathrm{cop}}$.

DEFINITION 2.4. (1) A *bimonoid* in a braided $\mathcal{V}$ is an object B with a monoid structure (ι, μ) and a comonoid structure (ε, Δ) such that $\varepsilon\colon B \to I$ and $\Delta\colon B \to B \otimes B$ are monoid morphisms, where $B \otimes B$ is a monoid with the structure described earlier.

(2) An *antipode* for a bimonoid $(H, \iota, \mu, \varepsilon, \Delta)$ is a morphism $s\colon H \to H$ for which $\mu \cdot (s \otimes 1_H) \cdot \Delta = \iota \cdot \varepsilon = \mu \cdot (1_H \otimes s) \cdot \Delta$. In other words, it is an inverse for 1_H in the convolution monoid induced by the bimonoid H. A bimonoid is called a *Hopf monoid* if it has an antipode.

(3) An *opantipode* is an inverse for 1_H in the convolution monoid induced by the comonoid $H^{\mathrm{cop},c^{-1}}$ and the monoid H; see [7, § 5.2]. The bimonoid H is called an *op-Hopf monoid* if it has an opantipode.

(4) Finally, for any monoid A in a monoidal category $\mathcal{V}$, we shall denote its category of left modules by $\mathbf{Mod}_{\mathcal{V}}(A)$; its objects are pairs (M, ν) where M is an object of $\mathcal{V}$ and $\nu\colon A \otimes M \to M$ is an action of A, by which we mean that it satisfies the usual module axioms. Dually, we denote the category of left comodules over a comonoid C by $\mathbf{Comod}_{\mathcal{V}}(C)$.

2.2. *Accessible and locally presentable monoidal categories*

This section compiles some facts about accessible and locally presentable monoidal categories, present in [**25**], that will be useful later. We refer the reader to [**21**] or [**3**] for the theory of accessible and locally presentable categories. We limit ourselves here to mentioning only a few facts. An object X of a category $\mathcal{C}$ is *κ-presentable* if the representable $\mathcal{C}(X,-)$ preserves κ-filtered colimits; here κ is a regular cardinal. The category $\mathcal{C}$ is *κ-accessible* if it has κ-filtered colimits, there is, up to isomorphism, a small set of κ-presentable objects, and each object of $\mathcal{C}$ is the colimit of a κ-filtered diagram of κ-presentable objects. In particular, κ-presentable objects form a small dense subcategory of $\mathcal{C}$. If $\mathcal{C}$ is cocomplete, one says that it is *locally κ-presentable*. A category is *accessible* or *locally presentable* if it is κ-accessible or locally κ-presentable for some regular cardinal κ. Locally presentable categories are automatically complete. Also, if $\mathcal{C}$ is accessible or locally presentable, then so is the functor category $[\mathcal{A},\mathcal{C}]$ for any small $\mathcal{A}$.

A functor between κ-accessible categories is *κ-accessible* if it preserves κ-filtered colimits. A functor between two accessible categories is said to be *accessible* if it is κ-accessible for some κ. This makes sense because given two (or a small set L of) accessible categories, there exists a regular cardinal κ such that both (all the elements of L) are κ-accessible (see [**21**, 2.4.9]). Clearly, if F is κ-accessible and $\kappa \leqslant \mu$, then F is μ-accessible.

REMARK 2.5. The following two facts about functors between locally presentable categories will be needed in later sections.

(1) Any cocontinuous functor between locally presentable categories is a left adjoint. In fact, by [**18**, 5.33], any cocontinuous functor between cocomplete categories whose domain has a small dense category is a left adjoint.

(2) Any continuous accessible functor between locally presentable categories is a right adjoint, by [**9**, Satz 14.6].

DEFINITION 2.6. A *κ-accessible monoidal category* is a monoidal category $\mathcal{V}$ that satisfies the following: it is a κ-accessible category; the tensor product is a κ-accessible functor; the unit object is κ-presentable; and, κ-presentable objects are closed under the tensor product. A *locally presentable monoidal category* is an accessible monoidal category that is cocomplete.

Locally presentable monoidal categories that are symmetric are called *admissible* monoidal categories in [**25**]. Examples include the monoidal category of modules over a commutative ring; the monoidal category of chain complexes over a commutative ring; any locally presentable cartesian closed category, such as for example Grothendieck toposes. In fact, the results obtained in the current paper require more relaxed conditions on the base monoidal category $\mathcal{V}$, that is, local presentability of its underlying category, as we explain below.

Any monoidal biclosed structure $(\otimes, I, [-,-])$ on an accessible category $\mathcal{V}$ is automatically an accessible monoidal category. Indeed, I is κ-presentable for some κ, by [**21**, 2.3.12]. If X, Y are κ-presentable objects, then so is $X \otimes Y$, since $\mathcal{V}(X \otimes Y, -) \cong \mathcal{V}(X, [Y,-])$ is the composition of two κ-accessible functors: $\mathcal{V}(X,-)$ by hypothesis, and $[Y,-]$ as a right adjoint [**21**, 2.4.8]. Clearly, the tensor product preserves all colimits, as it has a right adjoint.

Later we will use the following easy lemma. Recall that a left adjoint $U \dashv R$ is of *codescent type* if the components $\eta_X \colon X \to RU(X)$ of the unit are equalisers. If this is the case, η_X is necessarily the equaliser of the pair of morphisms $\eta_{RU(X)}, RU\eta_X \colon RU(X) \rightrightarrows RURU(X)$.

LEMMA 2.7. *Let $U \dashv R \colon \mathcal{B} \to \mathcal{A}$ be an adjunction of codescent type between accessible categories. Then $X \in \mathcal{A}$ is presentable if and only if $U(X) \in \mathcal{B}$ is presentable. Furthermore, the presentability degree of X is at least the maximum of presentability degree of $U(X)$ and the accessibility degree of RU.*

Proof. The functor $\mathcal{B}(U(X),-) \cong \mathcal{A}(X,R-)$ is accessible if X is presentable, being the composition of the accessible functors $\mathcal{A}(X,-)$ and R, so the direct implication is obvious. For the converse, suppose that $U(X)$ is presentable, so $\mathcal{A}(X,R-)$ is accessible. There is an equaliser, natural in Y, by the hypothesis that the adjunction is of codescent type:

$$\mathcal{A}(X,Y) \rightarrowtail \mathcal{A}(X,RU(Y)) \rightrightarrows \mathcal{A}(X,RURU(Y)). \tag{5}$$

Thus, $\mathcal{A}(X,-)$ is an equaliser of accessible functors $\mathcal{A} \to \mathbf{Set}$, and, therefore, it is κ-accessible by [**21**, Proposition 2.4.5], where κ can be taken as the maximum of the accessibility degree of $\mathcal{A}(X,-)$ and RU. □

COROLLARY 2.8. *If* G *is an accessible comonad on an accessible category* $\mathcal{C}$*, then:*

(1) *the category* $\mathcal{C}^G$ *of Eilenberg–Moore coalgebras is accessible;*
(2) *the forgetful functor* $U\colon \mathcal{C}^G \to \mathcal{C}$ *is accessible;*
(3) *the presentable* G*-coalgebras are those whose underlying object is presentable in* $\mathcal{C}$*. Furthermore, the presentability degree of a* G*-coalgebra* M *is at least the maximum of the presentability degree of* $U(M)$ *and the accessibility degree of* G*.*

Proof. The first two claims hold by [**21**, 5.1.6], while the last is an instance of Lemma 2.7, as comonadic functors are of codescent type. □

Compare the following result with [**25**, § 2].

PROPOSITION 2.9. *Suppose* $\mathcal{V}$ *is an accessible* (*respectively, locally presentable*) *monoidal category. Then both* $\mathbf{Mon}(\mathcal{V})$ *and* $\mathbf{Comon}(\mathcal{V})$ *are accessible* (*respectively, locally presentable*) *categories, and the respective forgetful functors are accessible.*

Proof. Both the category of monoids and comonoids can be constructed from $\mathcal{V}$ by using products, inserters and equifiers; see [**19**] for a description of these limits. Then, [**21**, 5.1.6] implies that the categories of monoids and comonoids, and the respective forgetful functors, are accessible. Now suppose that $\mathcal{V}$ is locally presentable. In that case, $\mathbf{Comon}(\mathcal{V})$ is cocomplete and $\mathbf{Mon}(\mathcal{V})$ is complete, and therefore both are locally presentable (see [**21**, 6.1.4]). □

As an application, consider the category $\mathbf{Comon}(\mathcal{V})$ for a locally presentable braided monoidal closed category $\mathcal{V}$. Then the functor $(-\otimes C)$ with domain a locally presentable category is cocontinuous, by the commutative diagram below, thus it has a right adjoint by Remark 2.5. The same argument holds for $(C\otimes -)$, so $\mathbf{Comon}(\mathcal{V})$ is a locally presentable monoidal biclosed category (see also [**25**, 3.2]).

$$\begin{tikzcd}
\mathbf{Comon}(\mathcal{V}) \arrow[r, "-\otimes C"] \arrow[d, "U"'] & \mathbf{Comon}(\mathcal{V}) \arrow[d, "U"] \\
\mathcal{V} \arrow[r, "-\otimes UC"] & \mathcal{V}
\end{tikzcd}$$

EXAMPLE 2.10. The category $\mathbf{gVect}_{\mathbb{Z}}$ of $\mathbb{Z}$*-graded* k-vector spaces, being equivalent to the category of functors from the discrete category $\mathbb{Z}$ into $\mathbf{Vect}$, is locally finitely presentable. Furthermore, it is locally finitely presentable as a monoidal category, with the tensor product $(V\otimes W)_n = \sum_{n=i+j} V_i \otimes W_j$ and unit I equal to k concentrated in degree 0. There is a symmetry on $\mathbf{gVect}_{\mathbb{Z}}$, given on homogeneous elements $x\in V_a$, $y\in W_b$ by $s_{V,W}(x\otimes y) = (-1)^{ab} y\otimes x$. The internal hom is $[V,W]_n = \prod_i \mathrm{Hom}_k(V_i, W_{i+n})$. The category $\mathbf{gVect}_{\mathbb{N}}$ of $\mathbb{N}$-graded k-vector spaces is a locally presentable monoidal subcategory of $\mathbf{gVect}_{\mathbb{Z}}$.

EXAMPLE 2.11. Let $\mathbf{dgVect}_{\mathbb{Z}}$ be the category of *chain complexes* of vector spaces, or *differential graded* vector spaces. This is a locally finitely presentable category; the finitely presentable objects are the bounded chain complexes of finite-dimensional vector spaces. There exists an obvious forgetful functor $\mathbf{dgVect}_{\mathbb{Z}} \to \mathbf{gVect}_{\mathbb{Z}}$ that forgets the differential, which preserves limits and colimits and is conservative. It is a classical fact that this is a symmetric monoidal closed category and the said forgetful functor is strict monoidal; in other words, if V and W are dg vector spaces, then their graded tensor product can be equipped with differentials, which are compatible with the relevant natural transformations: the associativity and unit constraints and the symmetry. Explicit formulas for these differentials can be found in any homological algebra textbook.

The full monoidal subcategory $\mathbf{dgVect}_{\mathbb{N}}$ of non-negatively graded chain complexes is locally presentable too.

2.3. *Enriched categories*

It might be helpful to recall the definitions of enriched categories, functors, and so on, that will be used in the article. We only give an outline; detailed definitions can be found in [**18**]. The base of enrichment will be a monoidal category $(\mathcal{V}, I, \otimes)$, that in many instances will be assumed to be braided, closed, cocomplete or even finitely presentable. A $\mathcal{V}$-category $\mathcal{C}$ consists of objects X, Y, etc., and objects $\mathcal{C}(X,Y)$ of $\mathcal{V}$, for each pair of object X, Y. It is equipped with composition morphisms $\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$ and identity morphisms $I \to \mathcal{C}(X,X)$ that satisfy associativity and identity axioms. A $\mathcal{V}$-functor $F\colon \mathcal{C} \to \mathcal{D}$ sends objects of $\mathcal{C}$ to objects of $\mathcal{D}$, and is given on enriched homs by morphisms $F_{X,Y}\colon \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ in $\mathcal{V}$, that are compatible with composition and identities. A $\mathcal{V}$-natural transformation τ from F to another $\mathcal{V}$-functor G consists of a family of morphisms $\tau_X\colon I \to \mathcal{D}(FX, GX)$ that satisfy naturality axioms.

When the monoidal category $\mathcal{V}$ has a braiding $c_{X,Y}\colon X \otimes Y \to Y \otimes X$, one can consider the tensor product of two $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$. This is a $\mathcal{V}$-category $\mathcal{C} \otimes \mathcal{D}$ with objects $\mathrm{ob}\mathcal{C} \times \mathrm{ob}\mathcal{D}$, and enriched homs

$$(\mathcal{C} \otimes \mathcal{D})\left((C,D),(C',D')\right) = \mathcal{C}(C,C') \otimes \mathcal{D}(D,D'). \tag{6}$$

The braiding is used to define the composition of $\mathcal{C} \otimes \mathcal{D}$ by

$$\begin{aligned}&\mathcal{C}(C',C'') \otimes \mathcal{D}(D',D'') \otimes \mathcal{C}(C,C') \otimes \mathcal{D}(D,D') \\ &\xrightarrow{1\otimes c\otimes 1} \mathcal{C}(C',C'') \otimes \mathcal{C}(C,C') \otimes \mathcal{D}(D',D'') \otimes \mathcal{D}(D,D') \longrightarrow \mathcal{C}(C,C'') \otimes \mathcal{D}(D,D''). \end{aligned}\tag{7}$$

2.4. *Kleisli categories*

In this section, we will describe some known facts regarding Kleisli categories for monoidal and enriched monads. We gather these facts here, in order to refer to them later.

In general, if (T, η, μ) is a monad on an ordinary category $\mathcal{V}$, its Kleisli category—denoted by $\mathcal{V}_{\mathsf{T}}$ or $\mathrm{Kl}(\mathsf{T})$—has the same objects as $\mathcal{V}$ and homs $\mathrm{Kl}(\mathsf{T})(X,Y) = \mathcal{V}(X, TY)$; the composition uses the multiplication μ of T and the identity morphism of an object X is the unit $\eta_X\colon X \to \mathsf{T}X$; for more details see [**20**, § VI.5]. There is a bijective on objects functor $F_{\mathsf{T}}\colon \mathcal{V} \to \mathrm{Kl}(\mathsf{T})$ that sends a morphism $f\colon X \to Y$ to $\eta_Y \cdot f$.

If $\mathcal{V}$ is a monoidal category and (T, η, μ) has a monoidal monad structure, that is, it is a (lax) monoidal endofunctor on $\mathcal{V}$ with η, μ monoidal, then $\mathrm{Kl}(\mathsf{T})$ carries a monoidal structure that makes F_{T} a strict monoidal functor; in other words, we can tensor objects of the Kleisli category as we do in $\mathcal{V}$. If $\mathcal{V}$ has a braiding c and the functor T is braided monoidal, then there exists a braiding on $\mathrm{Kl}(\mathsf{T})$ that makes F_{T} a braided monoidal functor.

We will now consider $\mathcal{V}$-enriched monads on a braided monoidal closed category $\mathcal{V}$. In other words, $\mathcal{V}$ is regarded as a $\mathcal{V}$-category, with enriched hom-objects $[A, B]$. A $\mathcal{V}$-monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathcal{V}$ consists of a endo-$\mathcal{V}$-functor T and unit η and multiplication μ that are $\mathcal{V}$-natural transformations, and that form a monad on the ordinary category $\mathcal{V}$. The Kleisli $\mathcal{V}$-category $\mathcal{V}_\mathsf{T}$ of T has the same objects as $\mathcal{V}$, and enriched homs $\mathcal{V}_\mathsf{T}(X, Y) = [X, \mathsf{T}Y]$. Composition and identities are given by

$$[Y, TZ] \otimes [X, TY] \xrightarrow{T\otimes 1} [TY, T^2Z] \otimes [X, TY] \xrightarrow{\text{comp}} [X, T^2Z] \xrightarrow{[X,\mu_Z]} [X, TZ] \tag{8}$$

$$I \xrightarrow{\text{id}} [X, X] \xrightarrow{[X,\eta_X]} [X, TX]. \tag{9}$$

On the other hand, the $\mathcal{V}$-category of T-algebras, denoted by $\mathcal{V}^\mathsf{T}$, has objects the usual T-algebras, and enriched homs $\mathcal{V}^T((A, a), (B, b))$ the equaliser of the morphisms

$$[A, B] \xrightarrow{T} [TA, TB] \xrightarrow{[TA,b]} [TA, B] \quad \text{and} \quad [A, B] \xrightarrow{[a,B]} [TA, B] \tag{10}$$

with composition induced by that of the $\mathcal{V}$-category $\mathcal{V}$. There is a full and faithful 'comparison' $\mathcal{V}$-functor

$$K : \mathcal{V}_\mathsf{T} \longrightarrow \mathcal{V}^\mathsf{T} \tag{11}$$

given on objects by $X \mapsto \mathsf{T}X$ and on homs by the isos $[X, TY] \cong \mathcal{V}^T(TX, TY)$ induced by the the morphisms

$$[X, TY] \xrightarrow{T} [TX, T^2Y] \xrightarrow{[TX,\mu_Y]} [TX, TY]. \tag{12}$$

As is the case for any $\mathcal{V}$-functor, K gives monoid morphisms between endo-homs

$$\mathcal{V}_T(X, X) = [X, TX] \cong \mathcal{V}^T(TX, TX) \rightarrowtail [TX, TX]; \tag{13}$$

the multiplication is composition, which in the case of $\mathcal{V}_T(X, X)$ was described in (8).

3. *Actions of monoidal categories and enrichment*

Recall that a *left action* of a monoidal category $\mathcal{V} = (\mathcal{V}, \otimes, I, a, l, r)$ on a category $\mathcal{D}$ is given by a functor $* : \mathcal{V} \times \mathcal{D} \to \mathcal{D}$, a natural isomorphism with components $\alpha_{XYD} : X * (Y * D) \xrightarrow{\sim} (X \otimes Y) * D$ and a natural isomorphism with components $\lambda_D : I * D \xrightarrow{\sim} D$, satisfying the commutativity of diagrams similar to those of a monoidal category; see [**13**, § 1] for a full description. Another way of describing a left action of $\mathcal{V}$ is by a strong monoidal functor $\mathcal{V} \to \mathrm{End}(\mathcal{D})$ into the strict monoidal category of endofunctors of $\mathcal{D}$, whose tensor product is given by composition.

The most important fact here for us, explained in detail in [**13**], is that to give a category $\mathcal{D}$ and a left action of a monoidal category $\mathcal{V}$ with a right adjoint for each $(- * D)$ is to give a $\mathcal{V}$-category $\mathcal{D}$.

Proposition 3.1. *Suppose* $* : \mathcal{V} \times \mathcal{D} \to \mathcal{D}$ *is a left action of the monoidal category* $\mathcal{V}$ *with the property that* $(- * D)$ *has a right adjoint* $H(D, -)$*, for each* $D \in \mathcal{D}$*;*

$$\mathcal{D}(X * D, E) \cong \mathcal{V}(X, H(D, E)). \tag{14}$$

Then, there exists a $\mathcal{V}$*-enriched category* $\underline{\mathcal{D}}$ *with underlying category* $\mathcal{D}$ *and hom-objects* $\underline{\mathcal{D}}(D, E) = H(D, E)$*. When* $\mathcal{V}$ *is left closed, this establishes an equivalence between left actions of* $\mathcal{V}$ *and tensored* $\mathcal{V}$*-categories.*

The proof of the existence of the composition $H(B,C) \otimes H(A,B) \to H(A,C)$ and the identity morphisms $I \to H(A,A)$ satisfying the usual axioms of enriched categories is easily deduced from the correspondence of arrows under the adjunction (14) and the action axioms.

Moreover, when the monoidal category $\mathcal{V}$ is symmetric, then the opposite of a $\mathcal{V}$-category can be defined in the usual way, so we have that $\mathcal{D}^{\mathrm{op}}$ is also enriched in $\mathcal{V}$, with the same objects and hom-objects $\mathcal{D}^{\mathrm{op}}(B,A) = \mathcal{D}(A,B)$. Note that if $\mathcal{V}$ is braided but not symmetric, there are two different choices of opposite $\mathcal{V}$-category, one using the braiding and the other using its inverse.

Remark 3.2. The statement of Proposition 3.1 mentions tensored $\mathcal{V}$-categories over a left closed monoidal category $\mathcal{V}$. These are $\mathcal{V}$-categories $\mathcal{C}$ for which $\mathcal{V}$-natural isomorphisms $\mathcal{C}(X * C, D) \cong [X, \mathcal{C}(C,D)]$, where $[-,-]$ is the left internal hom of $\mathcal{V}$, $X \in \mathcal{V}$ and $C, D \in \mathcal{C}$.

Example 3.3. In addition to the tensored $\mathcal{V}$-categories mentioned above, examples of actions of a monoidal left closed category $\mathcal{V}$ are provided by the cotensored $\mathcal{V}$-categories, or rather, by a $\mathcal{V}$-categories $\mathcal{A}$ with chosen cotensor products. Dually to tensors, a cotensor product of $A \in \mathcal{A}$ by $X \in \mathcal{V}$ is an object $\{X, A\} \in \mathcal{A}$ with a morphism $X \to \mathcal{A}(\{X,A\}, A)$ that induces isomorphisms $\mathcal{A}(B, \{X,A\}) \cong [X, \mathcal{A}(B,A)]$. Then, the canonical isomorphisms $\{X, \{Y, A\}\} \cong \{X \otimes Y, A\}$ and $\{I, A\} \cong A$ make the functor $\{-,-\}^{\mathrm{op}}$ into a left action of $\mathcal{V}$ on $\mathcal{A}_\circ^{\mathrm{op}}$, the opposite of the underlying category of $\mathcal{A}$. For example, the left internal hom $[-,-]^{\mathrm{op}}$ is a left action of $\mathcal{V}$ on $\mathcal{V}^{\mathrm{op}}$.

In Theorem 3.6 we shall give a monoidal version of Proposition 3.1, but before that we need the following easy theorem. First recall that given a braided monoidal category $\mathcal{V}$, a *$\mathcal{V}$-enriched monoidal structure* on a $\mathcal{V}$-category $\mathcal{A}$ consists of a $\mathcal{V}$-functor $\otimes \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, an object $I \in \mathcal{A}$ and $\mathcal{V}$-natural isomorphisms $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$, $I \otimes X \cong X \cong X \otimes I$, such that the underlying functor $\otimes_\circ$ together with I and these isomorphisms form a monoidal structure on the ordinary category $\mathcal{A}_\circ$. One says that $\mathcal{A}$ is a *monoidal $\mathcal{V}$-category*. It is not hard to see how this definition establishes the following equivalence.

Theorem 3.4. *Let $\mathcal{V}$ be a braided monoidal category. Suppose $\mathcal{A}$ is a $\mathcal{V}$-category equipped with a $\mathcal{V}$-functor $T \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and an object J. There is a bijection between:*

(1) *monoidal $\mathcal{V}$-category structures on $\mathcal{A}$ with tensor product T and unit J;*
(2) *extensions of $(\mathcal{A}_\circ, T_\circ, J)$ to a monoidal category such that the morphisms $T_{ABCD} \colon \mathcal{A}(A,C) \otimes \mathcal{A}(B,D) \to \mathcal{A}(T(A,B), T(C,D))$ and $\mathrm{id} \colon I \to \mathcal{A}(J,J)$ make the enriched-hom functor $\mathcal{A}(-,-) : \mathcal{A}_\circ^{\mathrm{op}} \times \mathcal{A}_\circ \to \mathcal{V}$ into a monoidal functor.*

Furthermore, if $\mathcal{A}_0$ is braided, with braiding c, the monoidal $\mathcal{V}$-category of (1) is braided with braiding c if and only if $\mathcal{A}(-,-)$ is a braided monoidal functor from $(\mathcal{A}_\circ^{\mathrm{op}} \times \mathcal{A}_\circ, c^{-1} \times c)$ to $(\mathcal{A}, c)$.

Definition 3.5. Let $* \colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ be a left action of the braided monoidal category $\mathcal{V}$. If $(\mathcal{A}, \diamond, J)$ is a monoidal category, by an *opmonoidal structure* on the action we shall mean an opmonoidal structure on the functor $*$, where its domain has the product monoidal structure, that makes the natural isomorphisms α and λ opmonoidal natural transformations. We speak of an opmonoidal action of $\mathcal{V}$.

In more explicit terms, an opmonoidal structure on the action $*$ consists of a morphism and a natural transformation

$$\xi_0 \colon I * J \longrightarrow J \quad \xi_{XYAB} \colon (X \otimes Y) * (A \diamond B) \longrightarrow (X * A) \diamond (Y * B) \tag{15}$$

that make the diagrams in (16)–(20) commute (the associativity constraints of both $\otimes$ and $\diamond$ are omitted); the first three diagrams exhibit $(*, \xi, \xi_0)$ as an opmonoidal functor, while the last four diagrams exhibit α and λ as opmonoidal transformations.

$$\begin{tikzcd}
(X \otimes Y \otimes Z) * (A \diamond B \diamond C) \arrow[r, "\xi_{X,Y\otimes Z,A,B\diamond C}"] \arrow[d, "\xi_{X\otimes Y,Z,A\diamond B,C}"'] & (X * A) \diamond ((Y \otimes Z) * (B \diamond C)) \arrow[d, "1\diamond\xi_{YZBC}"] \\
((X \otimes Y) * (A \diamond B)) \diamond (Z * C) \arrow[r, "\xi_{XYAB}\diamond 1"] & (X * A) \diamond (Y * B) \diamond (Z * C)
\end{tikzcd} \tag{16}$$

$$\begin{tikzcd}
(I \otimes X) * (J \diamond A) \arrow[r, "\cong"] \arrow[d, "\xi_{IXJA}"'] & X * A & (X \otimes I) * (A \diamond J) \arrow[r, "\cong"] \arrow[d, "\xi_{XIAJ}"'] & X * A \\
(I * J) \diamond (X * A) \arrow[r, "\xi_0\diamond 1"] & J \diamond (X * A) \arrow[u, "\cong"'] & (X * A) \diamond (I * J) \arrow[r, "1\diamond\xi_0"] & (X * A) \diamond J \arrow[u, "\cong"']
\end{tikzcd} \tag{17}$$

$$\begin{tikzcd}
(X \otimes X') * ((Y \otimes Y') * (A \diamond A')) \arrow[r, "\alpha"] \arrow[d, "1*\xi_{YY'AA'}"'] & (X \otimes X' \otimes Y \otimes Y') * (A \diamond A') \arrow[d, "(1\otimes c_{X'Y}\otimes 1)*1"] \\
(X \otimes X') * ((Y * A) \diamond (Y' * A')) \arrow[d, "\xi_{X,X',Y*A,Y'*A'}"'] & (X \otimes Y \otimes X' \otimes Y') * (A \diamond A') \arrow[d, "\xi_{X\otimes Y,X'\otimes Y',A,A'}"] \\
(X * (Y * A)) \diamond (X' * (Y' * A')) \arrow[r, "\alpha\diamond\alpha"] & ((X \otimes Y) * A) \diamond ((X' \otimes Y') * A')
\end{tikzcd} \tag{18}$$

$$\begin{tikzcd}
(I \otimes I) * J \arrow[r, "\xi_{IIJ}"] \arrow[d, "\cong"'] & I * (I * J) \arrow[d, "1*\xi_0"] \\
I * J \arrow[d, "\xi_0"'] & I * J \arrow[d, "\xi_0"] \\
J \arrow[r, equal] & J
\end{tikzcd} \tag{19}$$

$$\begin{tikzcd}
(I \otimes I) * (A \diamond B) \arrow[r, "\xi_{IIAB}"] \arrow[d, "\cong"'] & (I * A) \diamond (I * B) \arrow[d, "\lambda_A\diamond\lambda_B"] & I * J \arrow[r, "\lambda_J"] \arrow[d, "\xi_0"'] & J \arrow[d, equal] \\
I * (A \diamond B) \arrow[r, "\cong"] & A \diamond B & J \arrow[r, equal] & J
\end{tikzcd} \tag{20}$$

THEOREM 3.6. *Suppose given a left action $* \colon \mathcal{V} \times \mathcal{A} \to \mathcal{A}$ of a braided monoidal category $\mathcal{V}$ such that $(- * A)$ has a right adjoint for all A, and let $\underline{\mathcal{A}}$ be the associated $\mathcal{V}$-category. Then, each opmonoidal structure on the action induces a monoidal $\mathcal{V}$-category structure on $\underline{\mathcal{A}}$ with underlying monoidal category $\mathcal{A}$.*

Proof. We will first construct a functor $T \colon \underline{\mathcal{A}} \otimes \underline{\mathcal{A}} \to \underline{\mathcal{A}}$. On objects it will be given by $T(A, B) = A \diamond B$; on homs it is given by the morphisms $T_{ABCD} \colon \underline{\mathcal{A}}(A, C) \otimes \underline{\mathcal{A}}(B, D) \to \underline{\mathcal{A}}(A \diamond B, C \diamond D)$ that are transpose to the composition

$$(\underline{\mathcal{A}}(A, C) \otimes \underline{\mathcal{A}}(B, D)) * (A \diamond B) \xrightarrow{\xi} (\underline{\mathcal{A}}(A, C) * A) \diamond (\underline{\mathcal{A}}(B, D) * B) \xrightarrow{\varepsilon\diamond\varepsilon} C \diamond D. \tag{21}$$

The preservation of composition for the $\mathcal{V}$-functor T is expressed by the commutativity of the diagram in (22), which can be deduced from the commutativity of the diagram in (18) (expressing the monoidality of α) by setting $X = \underline{\mathcal{A}}(C, E)$, $X' = \underline{\mathcal{A}}(D, G)$, $Y = \underline{\mathcal{A}}(A, C)$, $Y' = \underline{\mathcal{A}}(B, D)$, $A = A \diamond B$ and $A' = E \diamond G$:

$$\begin{tikzcd}
\underline{\mathcal{A}}(C,E)\otimes\underline{\mathcal{A}}(D,G)\otimes\underline{\mathcal{A}}(A,C)\otimes\underline{\mathcal{A}}(B,D) \arrow[r, "T\otimes T"] \arrow[d, "1\otimes c\otimes 1"'] & \underline{\mathcal{A}}(C\diamond D,E\diamond G)\otimes\underline{\mathcal{A}}(A\diamond B,C\diamond D) \arrow[dd, "\mathrm{comp}"] \\
\underline{\mathcal{A}}(C,E)\otimes\underline{\mathcal{A}}(A,C)\otimes\underline{\mathcal{A}}(D,G)\otimes\underline{\mathcal{A}}(B,D) \arrow[d, "\mathrm{comp}\otimes\mathrm{comp}"'] & \\
\underline{\mathcal{A}}(A,E)\otimes\underline{\mathcal{A}}(B,G) \arrow[r, "T"] & \underline{\mathcal{A}}(A\diamond B,E\diamond G)
\end{tikzcd} \tag{22}$$

$$\begin{tikzcd}
\underline{\mathcal{A}}(A,A)\otimes\underline{\mathcal{A}}(B,B) \arrow[r, "T"] & \underline{\mathcal{A}}(A\diamond B,A\diamond B) \\
I \arrow[u, "\mathrm{id}\otimes\mathrm{id}"] \arrow[ur, "\mathrm{id}"'] &
\end{tikzcd} \tag{23}$$

The preservation of identities for the $\mathcal{V}$-functor T is the commutativity of the diagram in (23), which once translated under the adjunction $(-*(A\diamond B))\dashv\underline{\mathcal{A}}(A\diamond B,-)$, can be easily seen to hold by naturality of ξ.

We can now use Theorem 3.4 to complete the proof. We must show that the morphisms T_{ABCD} and $\mathrm{id}_J\colon I\to\underline{\mathcal{A}}(J,J)$ form a monoidal structure on functor $\underline{\mathcal{A}}(-,-)\colon\mathcal{A}^{\mathrm{op}}\times\mathcal{A}\to\mathcal{V}$. This is precisely the case, by Proposition 2.1, since T_{ABCD} and id_J are the transpose of the opmonoidal structure of the action $*$ under the parametrised adjunctions $(-*A)\dashv\underline{\mathcal{A}}(A,-)$. □

In the case of a right closed monoidal category $\mathcal{V}$ acting on itself via the tensor product, Theorem 3.6 says that $\mathcal{V}$ is a monoidal $\mathcal{V}$-category, provided that it is equipped with a braiding.

Example 3.7. If in Example 3.3 the monoidal right closed category $\mathcal{V}$ is braided, then $[-,-]^{\mathrm{op}}$ is an opmonoidal action of $\mathcal{V}$ on $\mathcal{V}^{\mathrm{op}}$. The opmonoidal structure is given by morphisms of the form (15) but in $\mathcal{V}^{\mathrm{op}}$. These are the canonical isomorphism $I\cong[I,I]$ and the morphism $[X,Y]\otimes[Z,W]\to[X\otimes Z,Y\otimes W]$ that makes $[-,-]$ a monoidal functor.

Definition 3.8. Suppose given an opmonoidal action as in Definition 3.5 and suppose that the monoidal category $\mathcal{A}$ braided. We say that the opmonoidal action is *braided* when the opmonoidal functor $*$ is so.

In more explicit terms, the opmonoidal action is braided when the following diagram commutes, where both the braiding of $\mathcal{V}$ and $\mathcal{A}$ are denoted by c:

$$\begin{tikzcd}
(X\otimes Y)*(A\diamond B) \arrow[r, "\xi"] \arrow[d, "c*c"'] & (X*A)\diamond(Y*B) \arrow[d, "c"] \\
(Y\otimes X)*(B\diamond A) \arrow[r, "\xi"] & (Y*B)\diamond(X*A)
\end{tikzcd} \tag{24}$$

Example 3.9. Continuing with Example 3.7, the action of $\mathcal{V}$ on $\mathcal{V}^{\mathrm{op}}$ given by the internal hom is braided if $\mathcal{V}$ is symmetric. The diagram (24) in this case looks as follows (where we use that $c^{-1}=c$):

$$\begin{tikzcd}
{[X,A]\otimes[Y,B]} \arrow[r] \arrow[d, "c"'] & {[X\otimes Y,A\otimes B]} \arrow[d, "{[c,c]}"] \\
{[Y,B]\otimes[X,A]} \arrow[r] & {[Y\otimes X,B\otimes A]}
\end{tikzcd} \tag{25}$$

THEOREM 3.10. *Suppose that in Theorem 3.6 $\mathcal{A}$ is braided. Then the opmonoidal action is braided if and only if the braiding of $\mathcal{A}$ is $\mathcal{V}$-natural. In this situation $\underline{\mathcal{A}}$ is braided.*

Proof. The commutativity of (24) is equivalent to the commutativity of

$$\begin{tikzcd} \underline{\mathcal{A}}(A,B)\otimes\underline{\mathcal{A}}(C,D) \arrow[r, "T_{ACBD}"] \arrow[d, "c"'] & \underline{\mathcal{A}}(A\diamond C, B\diamond D) \arrow[d, "\underline{\mathcal{A}}(c^{-1},c)"] \\ \underline{\mathcal{A}}(C,D)\otimes\underline{\mathcal{A}}(A,B) \arrow[r, "TCADB"] & \underline{\mathcal{A}}(C\diamond A, D\diamond B) \end{tikzcd} \tag{26}$$

where we used the notation of the proof of Theorem 3.6. This is the condition that $\mathcal{A}(-,-)$ is braided monoidal, required by Theorem 3.4. □

4. *The universal measuring comonoid*

The notion of the universal measuring coalgebra $P(A,B)$ over a field k appeared in Sweedler's book [**29**]. The elements of $P(A,B)$ can be thought of as generalised maps from A to B, and examples of this point of view are given in [**5**]. The natural isomorphism that defines the object $P(A,B)$ is

$$\mathbf{Alg}_k(A,\mathrm{Hom}_k(C,B))\cong\mathbf{Coalg}_k(C,P(A,B)). \tag{27}$$

Note that the plain algebra morphisms $A\to B$ correspond to the group-like elements of $P(A,B)$.

Our aim in this section is to prove the existence of $P(A,B)$ in a broader context, identifying the underlying categorical ideas. In that direction, consider an arbitrary braided monoidal closed category $\mathcal{V}$.

We remind the reader that in Section 2 we saw how the internal hom induces a functor $[-,-]\colon\mathbf{Comon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{Mon}(\mathcal{V})\to\mathbf{Mon}(\mathcal{V})$.

THEOREM 4.1. *Suppose that $\mathcal{V}$ is a locally presentable braided monoidal closed category. Then the functor $[-,B]^{\mathrm{op}}:\mathbf{Comon}(\mathcal{V})\to\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ has a right adjoint $P(-,B)$; that is, there is a natural isomorphism*

$$\mathbf{Mon}(\mathcal{V})(A,[C,B])\cong\mathbf{Comon}(\mathcal{V})(C,P(A,B)). \tag{28}$$

Proof. By Proposition 2.9, the category $\mathbf{Comon}(\mathcal{V})$ is locally presentable. The diagram

$$\begin{tikzcd} \mathbf{Comon}(\mathcal{V}) \arrow[r, "{[-,B]^{\mathrm{op}}}"] \arrow[d, "U"'] & \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \arrow[d, "V"] \\ \mathcal{V} \arrow[r, "{[-,U(B)]^{\mathrm{op}}}"] & \mathcal{V}^{\mathrm{op}} \end{tikzcd} \tag{29}$$

commutes, where U and V are the forgetful functors, and since $[-,U(B)]^{\mathrm{op}}$ is cocontinuous, so is the composition $[U-,U(B)]^{\mathrm{op}}$. Therefore, the functor at the top of the diagram is cocontinuous, since V creates colimits. The existence of the adjoint $P(-,B)$ now follows from the locally presentability of $\mathbf{Comon}(\mathcal{V})$ (Remark 2.5). □

The object $P(A,B)$ for monoids A and B is called the *universal measuring comonoid*, and the parametrised adjoint of $[-,-]^{\mathrm{op}}$, namely

$$P:\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{Mon}(\mathcal{V})\to\mathbf{Comon}(\mathcal{V}) \tag{30}$$

is called the *Sweedler hom* in [**4**].

Corollary 4.2. *The functor* $P(-,-)\colon \mathbf{Mon}(\mathcal{V})^{\mathrm{op}} \times \mathbf{Mon}(\mathcal{V}) \to \mathbf{Comon}(\mathcal{V})$ *is continuous in each variable.*

Proof. By definition, $P(-,B)$ is a right adjoint, thus continuous. In the other variable, $P(A,-)$ preserves limits if and only if $\mathbf{Comon}(\mathcal{V})(C,P(A,-))$ does for all C, if and only if $\mathbf{Mon}(\mathcal{V})(A,[C,-])$ does, by Theorem 4.1; this last condition is clearly true. □

Example 4.3. Since the category $\mathbf{Mod}_k$ of modules over a commutative ring k is locally presentable symmetric monoidal closed, we recover the classical situation: the existence, of each pair of k-algebras A, B, of a *universal measuring coalgebra* $P(A,B)$ with a natural isomorphism (28). See also [**24**, Proposition 4].

When k is a field, $P(A,k)$ is usually denoted by A° and called the *finite* or *Sweedler dual* of the k-algebra A. It can be presented as the subspace of A^*

$$A^\circ = \{\alpha \in A^* \,|\, \ker\alpha \text{ contains a cofinite ideal}\} = \{\alpha \in A^* \,|\, \dim(A \rightharpoonup \alpha) < \infty\}$$

where $\rightharpoonup$ denotes the left action of A on A^* given by $(a \rightharpoonup \alpha)(x) = \alpha(xa)$.

Example 4.4. The locally presentable braided monoidal closed category $\mathbf{gVect}_{\mathbb{N}}$ of $\mathbb{N}$-graded vector spaces was described in Example 2.10. Monoids and comonoids in $\mathbf{gVect}_{\mathbb{N}}$ are, respectively, *graded k-algebras* and *graded k-coalgebras*.

The full inclusion $(-)[0]\colon \mathbf{Vect} \to \mathbf{gVect}_{\mathbb{N}}$ that takes a space V to the graded space V concentrated in degree 0 is a left and a right adjoint to the functor that takes the homogeneous component of degree 0. All three functors are strong monoidal. These adjunctions induce adjunctions on the respective categories of monoids as the ones depicted vertically in the following diagram. Given a commutative algebra B, there are adjunctions depicted horizontally.

$$\begin{tikzcd}
\mathbf{gCoalg} \arrow[rr, shift left, "{[-,B[0]]}"] \arrow[rr, phantom, "\perp"] \arrow[d, shift left, "(-)_0"] \arrow[d, phantom, "\dashv"] && \mathbf{gAlg}^{\mathrm{op}} \arrow[ll, shift left, "{P(-,B[0])}"] \arrow[d, shift left, "(-)_0"] \arrow[d, phantom, "\dashv"] \\
\mathbf{Coalg} \arrow[u, shift left, "{(-)[0]}"] \arrow[rr, shift left, "{[-,B]}"] \arrow[rr, phantom, "\perp"] && \mathbf{Alg}^{\mathrm{op}} \arrow[u, shift left, "{(-)[0]}"] \arrow[ll, shift left, "{P(-,B)}"]
\end{tikzcd} \tag{31}$$

The square of left adjoints commute, since $[V,B[0]]_n = \prod_i [V_i, B[0]_{i+n}]$ is trivial unless $n=0$, in which case it is isomorphic to $[V_0,B]$. It follows that the square of right functors commute up to isomorphism, that is,

$$P(A,B[0])_0 \cong P(A_0,B). \tag{32}$$

Example 4.5. Recall from Example 2.11 the locally finitely presentable monoidal category $\mathbf{dgVect}_{\mathbb{Z}}$ of chain complexes. Monoids and comonoids in $\mathbf{dgVect}_{\mathbb{Z}}$ are usually called *dg algebras* and *dg coalgebras*, and the categories they form $\mathbf{dgAlg}_{\mathbb{Z}}$ and $\mathbf{dgCoalg}_{\mathbb{Z}}$. The universal measuring comonoid of two dg algebras is the Sweedler hom considered in [**4**, § 4.1.5].

5. *Enrichment of monoids in comonoids*

Now that we have established the existence of the universal measuring comonoid $P(A,B)$ under certain hypotheses, we may combine this construction with the theory of actions of monoidal categories of Section 3 in order to exhibit an enrichment of monoids over comonoids. In this section, $\mathcal{V}$ will denote a locally presentable braided monoidal closed category, with braiding

c. Recall that the internal hom functor $[-,-]$ is monoidal, and that the monoidal category of comonoids $\mathbf{Comon}(\mathcal{V})$ is symmetric when $\mathcal{V}$ is.

LEMMA 5.1. *The functor* $[-,-]^{\mathrm{op}}\colon \mathbf{Comon}(\mathcal{V})\times\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}\to\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ (4) *is an action of the monoidal category* $\mathbf{Comon}(\mathcal{V})$ *on* $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$*. If the braiding of* $\mathcal{V}$ *is a symmetry, then this is a braided opmonoidal action of the symmetric monoidal category of comonoids.*

Proof. The functor of the statement is obtained by taking comonoid categories on $[-,-]^{\mathrm{op}}\colon \mathcal{V}\times\mathcal{V}^{\mathrm{op}}\to\mathcal{V}^{\mathrm{op}}$. The latter is an opmonoidal left action of the braided monoidal $\mathcal{V}$ (Example 3.7) upon which $\mathbf{Mon}[-,-]^{\mathrm{op}}$ inherits the structure of a left action.

If $\mathcal{V}$ is symmetric, the internal hom is a braided opmonoidal action of $\mathcal{V}$ on $\mathcal{V}^{\mathrm{op}}$, by Example 3.9. Taking categories of monoids, we obtain a braided opmonoidal action of the symmetric category of comonoids on $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$. □

We can now apply Proposition 3.1, Theorem 3.6 and Theorem 3.10 to $\mathcal{C}=\mathbf{Comon}(\mathcal{V})$ and $\mathcal{A}=\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$. The functor $[-,B]^{\mathrm{op}}$ of Lemma 5.1 has a right adjoint $P(-,B)$ by Theorem 4.1, and $\mathbf{Comon}(\mathcal{V})$ is monoidal closed as explained in Section 2.2.

THEOREM 5.2. *Let* $\mathcal{V}$ *be a locally presentable braided monoidal closed category. Then:*

(1) *the category* $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ *is enriched in* $\mathbf{Comon}(\mathcal{V})$ *and tensored, with enriched hom objects* $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}(A,B)=P(B,A)$.

If $\mathcal{V}$ *is moreover symmetric, then:*

(2) $\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ *also carries a structure of a symmetric monoidal* $\mathbf{Comon}(\mathcal{V})$*-enriched category;*
(3) $\mathbf{Mon}(\mathcal{V})$ *is a symmetric monoidal and cotensored* $\mathbf{Comon}(\mathcal{V})$*-category, with hom objects* $\mathbf{Mon}(\mathcal{V})(A,B)=P(A,B)$.

Assume for the rest of the section that the braiding of $\mathcal{V}$ is a symmetry. By Lemma 2.3, the functor of Lemma 5.1 is a symmetric opmonoidal functor. Hence, by Proposition 2.1 we get the following result.

COROLLARY 5.3. *The Sweedler hom functor* $P:\mathbf{Mon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{Mon}(\mathcal{V})\to\mathbf{Comon}(\mathcal{V})$ *is a braided monoidal functor.*

Finally, since P is a monoidal functor, it induces a functor

$$\mathbf{Mon}P:\mathbf{Bimon}(\mathcal{V})^{\mathrm{op}}\times\mathbf{CommMon}(\mathcal{V})\to\mathbf{Bimon}(\mathcal{V}) \tag{33}$$

where $\mathbf{CommMon}(\mathcal{V})=\mathbf{Mon}(\mathbf{Mon}(\mathcal{V}))$ is the category of commutative monoids, and of course $\mathbf{Mon}(\mathbf{Comon}(\mathcal{V}))=\mathbf{Bimon}(\mathcal{V})$ is the category of bimonoids. This is still a braided monoidal functor by Remark 2.2 and so, since $\mathbf{Mon}(\mathbf{Bimon}(\mathcal{V}))=\mathbf{CommBimon}(\mathcal{V})$, we get the following result.

COROLLARY 5.4. *Suppose* $\mathcal{V}$ *is a locally presentable symmetric monoidal closed category. If* B *is a (cocommutative) bimonoid and* A *a commutative monoid, then* $P(B,A)$ *has a canonical structure of a (commutative) bimonoid. In particular, the finite dual* B° *is a (commutative) bimonoid.*

Note that the second part is also proved, in a much different way, in [**12**] for the case $\mathcal{V}=\mathbf{Mod}_R$.

Remark 5.5. When (B,ι,μ) is a commutative monoid in $\mathcal{V}$, $[-,B]\colon \mathbf{Comon}(\mathcal{V}) \to \mathbf{Mon}(\mathcal{V})^{\mathrm{op}}$ is an opmonoidal functor; its structure is given by the morphisms in $\mathcal{V}$

$$\chi\colon [A,B]\otimes[A',B] \longrightarrow [A\otimes A', B\otimes B] \xrightarrow{[1,\mu]} [A\otimes A',B] \quad \chi_0\colon I\cong[I,I]\xrightarrow{[I,\iota]}[I,B]. \tag{34}$$

If we denote the counit of the adjunction $[-,B]\dashv P(-,B)$ by the morphism in $\mathbf{Mon}(\mathcal{V})$

$$\eta_A\colon A\longrightarrow [P(A,B),B], \tag{35}$$

then the monoidal structure on the right adjoint $P(-,B)$ is given by the morphisms

$$\psi\colon P(A,B)\otimes P(A',B)\longrightarrow P(A\otimes A',B) \quad \psi_0\colon I\to P(I,B) \tag{36}$$

defined as the unique ones that make the following diagrams commute:

$$\begin{tikzcd} A\otimes A' \arrow[r, "\eta_{A\otimes A'}"] \arrow[d, "\eta_A\otimes\eta_{A'}"'] & {[P(A\otimes A',B),B]} \arrow[d, "{[\psi,B]}"] \\ {[P(A,B),B]\otimes[P(A',B),B]} \arrow[r, "\chi"] & {[P(A,B)\otimes P(A',B),B]} \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\eta_I"] \arrow[d, "\iota"'] & {[P(I,B),B]} \arrow[d, "{[\psi_0,B]}"] \\ B \arrow[r, "\sim"] & {[I,B]} \end{tikzcd} \tag{37}$$

6. *Comodules of universal measuring coalgebras*

Having established the enrichment of the category of monoids in the category of comonoids via the universal measuring comonoid, in this section we study these objects primarily from the point of view of their comodules or corepresentations, exhibiting further properties along the way.

6.1. *The finite dual as a subobject of a cofree comonoid*

If $\mathcal{V}$ is a locally presentable monoidal category, it is not hard to show that free monoids exist in $\mathcal{V}$, and then, $\mathbf{Mon}(\mathcal{V})$ becomes monadic over $\mathcal{V}$. We say only a few words about the proof. Since both $\mathbf{Mon}(\mathcal{V})$ and $\mathcal{V}$ are locally presentable (Proposition 2.9), it suffices to know that the forgetful functor from the former to the latter is continuous and accessible (by Remark 2.5); see also [**18**, Theorem 5.32] for a more general result. The fact that the forgetful functor preserves κ-filtered colimits, for some regular cardinal κ, can be easily verified using the fact that the tensor product of $\mathcal{V}$ does so. This concludes our sketch of a proof.

Easier still is to prove the fact that cofree comonoids exist in any locally presentable monoidal category $\mathcal{V}$; for, the forgetful functor from $\mathbf{Comon}(\mathcal{V})$ to $\mathcal{V}$ is cocontinuous, and thus a left adjoint again by Remark 2.5.

We shall denote the free monoid on $X\in\mathcal{V}$ by $T(X)$. As the notation suggests, the free monoid in the category of k-modules, for a commutative ring k, is the tensor algebra. The cofree comonoid on X we shall denote by $S(X)$.

In this section $\mathcal{V}$ will be a locally presentable *braided* monoidal closed category.

Lemma 6.1. *For any monoid B and any object $X\in\mathcal{V}$, $P(T(X),B)\cong S([X,B])$. In particular, $T(X)^\circ\cong S([X,I])$.*

Proof. Consider the commutative diagram (29). All four functors have a right adjoint, thus the diagram formed by the right adjoints commutes up to natural isomorphism, whose component at X has domain and codomain those of the statement. □

Let V be the forgetful $\mathbf{Mon}(\mathcal{V}) \to \mathcal{V}$. The functor $P(-,B)$ sends colimits in $\mathbf{Mon}(\mathcal{V})$ to limits in $\mathbf{Comon}(\mathcal{V})$ by adjointness. In particular, it sends the canonical diagram $T^2V(A) \rightrightarrows TV(A) \to A$ that exhibits a monoid A as coequaliser of free monoids, into an equaliser

$$P(A,B) \rightarrowtail S[V(A),V(B)] \rightrightarrows S[VT(A),V(B)]. \tag{38}$$

In the case when $\mathcal{V}$ is the category of k-vector spaces and $B=k$, this equaliser exhibits A° as a subcoalgebra of the cofree coalgebra on A^*. Composing with the counit $S \Rightarrow 1$ of the cofree coalgebra comonad, we obtain a morphism

$$A^\circ \longrightarrow A^* \tag{39}$$

that is the classical injection of the finite dual into the dual space [**29**].

6.2. *Coendomorphism comonoids*

Recall from the background Section 2 the notion of a dual object. The *coendomorphism comonoid* of an object X with left dual $X^\vee$ is the object $X^\vee \otimes X$, with comultiplication $X \otimes \mathrm{coev} \otimes X$ and counit ev. We shall denote it by $\mathrm{coend}(X)$. These comonoids are useful to us because C-comodule structures $X \to X \otimes C$ are in bijection with comonoid morphisms $\mathrm{coend}(X) \to C$. In particular, the coendomorphism coalgebras offer a reinterpretation of the so-called *fundamental theorem of coalgebras* below.

Recall that a set of objects $\mathcal{G} \subset \mathrm{ob}\,\mathcal{C}$ is strongly generating if, the functors $\{\mathcal{C}(G,-)\colon \mathcal{C} \to \mathbf{Set}\}_{G\in\mathcal{G}}$ are jointly conservative, that is, if a morphism f is invertible whenever $\mathcal{C}(G,f)$ is an invertible for all $G \in \mathcal{G}$. See [**18**, § 3.6].

LEMMA 6.2. *When $\mathcal{V}$ is the category of k-vector spaces, the family of coendomorphism coalgebras $\{\mathrm{coend}(k^n)\}_{n\geqslant 1}$ is strongly generating in $\mathbf{Coalg}_k$.*

Proof. If X is a finite-dimensional C-comodule, the image of the associated $X^* \otimes X \to C$ is called the *coefficient space* or *coalgebra of coefficients* of X, denoted by $\mathrm{cf}(X)$. It is the smallest subcoalgebra of C for which X is a comodule; see [**11**, § 1.2].

By the fundamental theorem of coalgebras [**29**, Theorem. 2.2.1], C is union of finite-dimensional subcoalgebras. It is not hard to see that, if $D \subset C$ is a finite dimensional subcoalgebra regarded as a C-comodule, then $\mathrm{cf}(D) = D$ (for, evaluating $D^* \otimes D \to C$ on $\varepsilon_D \otimes d$ gives back d). Therefore, the morphism of coalgebras $\sum_D \mathrm{coend}(D) \to C$ induced by the morphisms $\mathrm{coend}(D) \to C$, for each finite dimensional subcoalgebra $D \subset C$, is surjective. Hence, the following morphism is surjective (where $S \cdot E$, for a set S and a coalgebra E, denotes the copower, that is, the coproduct of S-copies of E).

$$\sum_n \mathbf{Coalg}_k(\mathrm{coend}(k^n), C) \cdot \mathrm{coend}(k^n) \longrightarrow C \tag{40}$$

In particular, (40) is an extremal epimorphism, that is, it does not factor through any non-trivial subobject of C; for more information see the paragraph previous to [**15**, Proposition 4.6], or [**17**, § 8.7]. This is equivalent to saying that the coalgebras $\mathrm{coend}(k^n)$ form a strong generator (see [**17**, § 8.7]). □

6.3. *Comodules over the universal measuring coalgebra*

Recall from Section 2.4, the Kleisli construction for an enriched monad on a braided monoidal closed $\mathcal{V}$. We will be interested in the enriched monad $T = (-\otimes B)$ induced by tensoring with a monoid B; in this case we will abbreviate the categories of Kleisli and of Eilenberg–Moore algebras by $\mathcal{V}_B$ and $\mathcal{V}^B$. The former always has tensor products by objects of $\mathcal{V}$ (in the sense of [**18**, § 3.7]), since the universal $\mathcal{V} \to \mathcal{V}_B$ is a left adjoint; the tensor product of $X \in \mathcal{V}_B$ by

$Z \in \mathcal{V}$ is $Z \otimes X$. As is always the case, the base category $\mathcal{V}$ acts on $\mathcal{V}_B$ on the left by tensor products. The $\mathcal{V}$-monad $(A \otimes -)$ extends to a $\mathcal{V}$-monad on $\mathcal{V}_B$ and lifts to a $\mathcal{V}$-monad on $\mathcal{V}^B$ thanks to the isomorphism $(A \otimes X) \otimes B \cong A \otimes (X \otimes B)$.

Proposition 6.3. *Let A, B be two monoids in the locally presentable braided monoidal closed category $\mathcal{V}$ and X an object. There is a bijection between:*

(1) *algebra structures on X, for the monad $(A \otimes -)$ on $\mathcal{V}_B$;*
(2) *monoid morphisms $A \to \mathcal{V}_B(X, X) = [X, X \otimes B]$;*
(3) *algebra structures on $X \otimes B$, for the monad $(A \otimes -)$ on $\mathcal{V}^B$;*
(4) *monoid morphisms $A \to \mathcal{V}^B(X \otimes B, X \otimes B)$.*

If X has a dual, then the above data are equivalent to:

(5) *right $P(A, B)$-comodule structures on X.*

It may be instructive to spell out the properties that a morphism $A \otimes X \to X$ has to satisfy in order to be an algebra structure on $X \in \mathcal{V}_B$ for the monad $(A \otimes -)$, as in the item 1 of the above proposition. It is a morphism $\psi\colon A \otimes X \to X \otimes B$ in $\mathcal{V}$ that makes the following pair of diagrams commute:

$$\begin{tikzcd} A \otimes A \otimes X \arrow[r, "1\otimes\psi"] \arrow[d, "\mu\otimes 1"'] & A \otimes X \otimes B \arrow[r, "\psi\otimes 1"] & X \otimes B \otimes B \arrow[d, "1\otimes\mu"] \\ A \otimes X \arrow[rr, "\psi"] & & X \otimes B \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, equal] \arrow[d, "\eta\otimes 1"'] & X \arrow[d, "1\otimes\eta"] \\ A \otimes X \arrow[r, "\psi"] & X \otimes B \end{tikzcd} \tag{41}$$

Proof. Morphisms $\xi\colon A \otimes X \to X$ in $\mathcal{V}_B$ are in bijection with morphisms $\hat{\xi}\colon A \to \mathcal{V}_T(X, X)$ in $\mathcal{V}$, by the universal property of the tensor product with objects of $\mathcal{V}$. Under this correspondence, ξ is an algebra structure for X if and only if $\hat{\xi}$ is a monoid morphism in $\mathcal{V}$, where the multiplication in its codomain is composition. This proves the equivalence of (1) and (2). The equivalence between (3) and (4) holds for precisely the same reason, while the equivalence of (2) and (4) is a consequence of the full and faithful comparison $\mathcal{V}$-functor $\mathcal{V}_B \rightarrowtail \mathcal{V}^B$.

If X has a (left) dual $X^\vee$, the isomorphism $[X, X \otimes B] \cong [X^\vee \otimes X, B]$ becomes an isomorphism of monoids when the domain has the composition of $\mathcal{V}_B$ as multiplication and the codomain has the convolution multiplication induced by the comonoid $\mathrm{coend}(X) = X^\vee \otimes X$ as in Section 6.2, and the monoid B. Thus, a monoid morphism as in (2) can equally be given by a monoid morphism $A \to [\mathrm{coend}(X), B]$, and therefore by a comonoid morphism $\mathrm{coend}(X) \to P(A, B)$. This corresponds to a morphism $X \to X \otimes P(A, B)$ satisfying the comodule axioms. □

Definition 6.4. Given two monoids A and B, the category ${}^A\mathcal{V}_B$ has objects pairs (X, ψ), where $X \in \mathcal{V}$ and $\psi\colon A \otimes X \to X \otimes B$ satisfies the two axioms depicted in the previous paragraph; it has morphisms $(X, \psi) \to (X', \psi')$ morphisms $f\colon X \to X'$ in $\mathcal{V}$ that satisfy $(f \otimes B) \cdot \psi = \psi' \cdot (A \otimes f)$. Composition and identities are the obvious ones, so there is a faithful forgetful functor ${}^A\mathcal{V}_B \to \mathcal{V}$.

The category just defined fits in the following pullback diagram, where $(\mathcal{V}_B)^{(A\otimes -)}$ is the category of algebras of the monad $(A \otimes -)$ on $\mathcal{V}_B$, and the bottom arrow is the universal Kleisli functor:

$$\begin{tikzcd} {}^{A}\mathcal{V}_B \arrow[r] \arrow[d] & (\mathcal{V}_B)^{(A\otimes -)} \arrow[d] \\ \mathcal{V} \arrow[r] & \mathcal{V}_B \end{tikzcd} \tag{42}$$

If we recall the notion of a dualizable object from the background Section 2, we obtain the following result.

Corollary 6.5. *There is an isomorphism between the categories of dualizable right $P(A,B)$-comodules and that of dualizable objects of ${}^{A}\mathcal{V}_B$; furthermore, the isomorphism commutes with the respective forgetful functors into $\mathcal{V}$.*

The isomorphism of the previous corollary is given on objects by Proposition 6.3. The rest of the details are left to the reader. When $B = I$ we have:

Corollary 6.6. *For a monoid A in $\mathcal{V}$, the category of dualizable right A°-comodules is isomorphic to the category of dualizable left A-modules.*

Proof. Setting $B = I$ in Proposition 6.3, the Kleisli $\mathcal{V}$-category $\mathcal{V}_B$ becomes just $\mathcal{V}$, and the data in the item (1) of the said proposition just an A-module structure on X. $\square$

In the example when $\mathcal{V}$ is the category of vector spaces, Corollary 6.5 gives an alternative description of the category of finite-dimensional right $P(A,B)$-comodules, for any pair of algebras A, B.

Corollary 6.7. *If A, B are algebras over a field k, then*

$$P(A,B) \cong \int^{(X,\psi)} X^* \otimes X \tag{43}$$

where (X,ψ) runs over all the objects of ${}^{A}\mathcal{V}_B$ with $\dim_k X < \infty$.

Proof. The forgetful functor $({}^{A}\mathcal{V}_B)_d \to \mathcal{V}$ from the category of dualizable objects of ${}^{A}\mathcal{V}_B$ is, up to composing with an isomorphism, the forgetful functor from the category of dualizable $P(A,B)$-comodules. Then, the coalgebra can be reconstructed by the coend (43); the ideas behind this reconstruction go back to [**27**], but see for example [**28**] for a paper where coends are explicitly used. $\square$

The corollary above holds for more general categories $\mathcal{V}$, as shown in [**22**], but we do not pursue that point.

Corollary 6.8. *There is a bijection between right $P(A,B)$-comodule structures on k^n and algebra morphisms $A \to \mathrm{M}_{n\times n}(B)$. There is a bijection between isomorphism classes of n-dimensional $P(A,B)$-comodules and the quotient of the set $\mathbf{Alg}_k(A, \mathrm{M}_{n\times n}(B))$ by the action of $\mathrm{GL}_n(k)$ on $\mathrm{M}_{n\times n}(B)$ by conjugation.*

Proof. The result is an easy consequence of Proposition 6.3. The canonical isomorphism between $[k^n, k^n \otimes B]$ and $B^{n\times n}$ is compatible with the Kleisli multiplication on the former and the matrix multiplication on the latter. Given two algebra morphisms $\sigma, \tau \colon A \to [k^n, k^n \otimes B]$, an invertible matrix $M \in \mathrm{GL}_n(k)$ represents an isomorphism of $P(A,B)$-comodules if and only if $(M \otimes B)\cdot\sigma(a) = \tau(a)\cdot M$, which in terms of $\mathrm{M}_{n\times n}(B)$ means that $M\sigma(a) = \tau(a)M$, for all $a \in A$. $\square$

The following examples for $\mathcal{V} = \mathbf{Vect}_k$ provide applications of the measuring coalgebra corepresentations point of view.

EXAMPLE 6.9. Given a k-algebra A, there is an isomorphism of algebras $A^\circ \cong k$ if and only if A satisfies:

(1) it has an augmentation $A \to k$, that is, k is an A-module;
(2) all the finite-dimensional modules are direct sums of the module k.

This is a consequence of Corollary 6.5. For, $A^\circ \cong k$ if and only if the forgetful functor from the category of finite-dimensional A°-comodules into the category of finite dimensional vector spaces is an isomorphism. But this category is isomorphic to the category of finite-dimensional A-modules.

An example is the group algebra $k[G]$ for a infinite simple group of cardinality larger than that of the field k; any finite-dimensional representation of $k[G]$ is given by a group morphism $G \to \mathrm{Aut}_k(k^n)$, which cannot be injective by the cardinality assumptions, thus it must be trivial by simplicity of G. An example of such a group G is $\mathrm{PSL}(2, K)$ for an infinite field $k \subset K$ of cardinality larger than that of k. This example was introduced in [**6**, Lemma 2.7].

EXAMPLE 6.10. The coalgebra A° can be zero, as pointed out in [**29**, p. 114], for example, if A is an infinite-dimensional division k-algebra. It can be instructive to deduce this from the universal property of the finite dual. The set $\mathbf{Alg}_k(A, C^*)$ is empty for all non-zero finite-dimensional coalgebras C. Therefore, $\mathbf{Coalg}_k(C, A^\circ)$ has this same property, and the functions

$$\mathbf{Coalg}_k(C, 0) \longrightarrow \mathbf{Coalg}_k(C, A^\circ) \tag{44}$$

induced by the unique morphism of coalgebras $0 \to A^\circ$ are isomorphisms, for C of finite dimension. We conclude that $0 \to A^\circ$ is an isomorphism, by Lemma 6.2.

EXAMPLE 6.11. If B has an augmentation $\varepsilon \colon B \to k$, there is an induced coalgebra morphism $P(A, \varepsilon) \colon P(A, B) \to P(A, k) = A^\circ$. In these circumstances, the equality $P(A, \varepsilon) \cdot P(A, \iota) = P(A, \varepsilon \cdot \iota) = P(A, 1_k) = 1_{A^\circ}$, induces functors on the categories of comodules

$$1 = \left(\mathbf{Comod}(A^\circ) \xrightarrow{P(A,\iota)_*} \mathbf{Comod}(P(A, B)) \xrightarrow{P(A,\varepsilon)_*} \mathbf{Comod}(A^\circ) \right) \tag{45}$$

that exhibit the category of A°-comodules as a retract of that of $P(A, B)$-comodules. These functors are given by corestriction of scalars, so they commute with the respective forgetful functors into $\mathbf{Vect}_k$, and are conservative.

An A°-comodule X is simple if and only if $P(A, \iota)_*(X)$ is a simple $P(A, B)$-comodule. The proof of this claim is elementary. Both functors in (45) preserve monomorphisms and are conservative, so they induce a retraction

$$1 = (\mathrm{Sub}(X) \longrightarrow \mathrm{Sub}(P(A, \iota)_*(X)) \longrightarrow \mathrm{Sub}(X)) \tag{46}$$

where both functions reflect equalities of comparable subobjects (that is, if $S \subseteq T$ are sent to the same subobject, then $S = T$). Therefore, $\mathrm{Sub}(X)$ has only bottom and top element if and only if $\mathrm{Sub}(P(A, \iota)_*(X))$ satisfies the same property.

As a consequence, $P(A, B)$ has simple comodules of dimension n if A has simple modules of dimension n. For example, if B is augmented (for example, $B = k[G]$ for a monoid G), then $P(\mathrm{M}_{n\times n}(k), B)$ has simple comodules of dimension n.

Another example is $P(U(\mathfrak{sl}(2, \mathbb{C})), B)$, which we show to be infinite-dimensional. By the above comments, this coalgebra has $P(U(\mathfrak{sl}(2, \mathbb{C})), k) = U(\mathfrak{sl}(2, \mathbb{C}))^\circ$ as a retraction. The finite-dimensional comodules over the latter coalgebra can be identified with finite-dimensional

$\mathfrak{sl}(2,\mathbb{C})$-representations; in particular, $U(\mathfrak{sl}(2,\mathbb{C}))^\circ$ has simple comodules of all dimensions, and therefore it is an infinite-dimensional coalgebra. This last claim can be deduced from [**1**, Corollary 4.5], which exhibits a bijection between isomorphism classes of simple comodules and simple subcoalgebras of a given coalgebra; therefore $U(\mathfrak{sl}(2,\mathbb{C}))^\circ$ has infinitely many non-isomorphic simple subcoalgebras, and hence it is infinite-dimensional.

6.4. *Tambara's coendomorphism algebra*

Tambara introduced in [**30**] an algebra $a(A,B)$ for each pair of algebras A, B over a field k, called the *coendomorphism algebra*, with the property that there is a bijection

$$\mathbf{Alg}_k(a(A,B),A') \cong \mathbf{Alg}_k(B, A\otimes A') \tag{47}$$

natural in C. The $a(A,B)$-modules are described in [**30**, § 2] in a way similar to our Proposition 6.3. More precisely, finite dimensional $a(A,B)$-modules can be identified with finite dimensional $P(B,A)$-comodules.

Proposition 6.12. *For all algebras A and B over a field k, there is a canonical isomorphism $a(A,B)^\circ \cong P(B,A)$.*

Proof. There is a function, natural in $C \in \mathbf{Coalg}_k$, which, by Yoneda's lemma is induced by a unique morphism of coalgebras $f\colon a(A,B)^\circ \to P(B,A)$.

$$\begin{aligned}\mathbf{Coalg}_k(C, a(A,B)^\circ) &\cong \mathbf{Alg}_k(a(A,B), C^*)\\ &\cong \mathbf{Alg}_k(B, A\otimes C^*) \longrightarrow \mathbf{Alg}_k(B,[C,A])\\ &\cong \mathbf{Coalg}_k(C, P(B,A))\end{aligned} \tag{48}$$

Here we used that the canonical inclusion $A\otimes C^* \hookrightarrow [C,A]$ is a morphism of algebras, as it can be readily verified. We can now use that this inclusion is an isomorphism if C is finite-dimensional, so the function (48) is an isomorphism in that case. Using the fact that finite-dimensional coalgebras are strong generating (Lemma 6.2), we deduce that f is an isomorphism. □

7. *Monoidal structures*

There are two natural ways in which the universal measuring comonoid acquires a bimonoid structure, and two ways in which the category of dualizable comodules acquires a monoidal structure. In this section we take these two ways in turn, and give an explicit description of the associated monoidal structures.

First, we have seen in Corollary 5.4 that, when $\mathcal{V}$ is a symmetric monoidal closed category, the comonoid $P(A,B)$ has a bimonoid structure if A is a bimonoid and B a commutative monoid. Then, the category $\mathbf{Comod}_d(P(A,B))$ of dualizable right $P(A,B)$-comodules has a monoidal structure, that can be transferred to the equivalent category $({}^A\mathcal{V}_B)_d$ of dualizable objects of ${}^A\mathcal{V}_B$, see Definition 6.4. The resulting monoidal structure on $({}^A\mathcal{V}_B)_d$ is given in the following way.

Corollary 7.1. *Given a bimonoid (A,μ,ι) and a commutative monoid (B,Δ,ϵ) in a symmetric monoidal closed locally presentable category $\mathcal{V}$, the isomorphism between $\mathbf{Comod}_d(P(A,B))$ and $({}^A\mathcal{V}_B)_d$ is a monoidal isomorphism when we equip:*

1. *$\mathbf{Comod}_d(P(A,B))$ with the monoidal structure associated to the induced bimonoid structure on $P(A,B)$;*

(2) $({}^A\mathcal{V}_B)_d$ *with the monoidal structure defined as follows:*
   (a) *if* (X, φ) *and* (Y, ψ) *are two objects, their tensor product is* $X \otimes Y$ *equipped with*

$$AXY \xrightarrow{\Delta XY} AAXY \xrightarrow{AcY} AXAY \xrightarrow{\varphi\psi} XBYB \xrightarrow{XcB} XYBB \xrightarrow{XY\mu} XYB$$

   where $\otimes$ *is omitted,*
   (b) *the monoidal unit is* I *equipped with*

$$A \otimes I \xrightarrow{\varepsilon \otimes I} I \otimes I \xrightarrow{I \otimes \iota} I \otimes B,$$

   (c) *the forgetful functor* $({}^A\mathcal{V}_B)_d \to \mathcal{V}$ *is strict monoidal.*

The second way in which $P(A, B)$ has a bimonoid structure is when $A = B$. The multiplication $P(A, A)^{\otimes 2} \to P(A, A)$ is the morphism of comonoids that corresponds to

$$A \xrightarrow{\eta} [P(A, A), A] \xrightarrow{[1,\eta]} [P(A, A), [P(A, A), A]] \cong [P(A, A)^{\otimes 2}, A] \qquad (49)$$

where η denotes the unit of the adjunction between $P(-, A)$ and $[-, A]$. The unit $I \to P(A, A)$ is the morphism of coalgebras that corresponds to the identity $A \to A$.

COROLLARY 7.2. *Given a monoid* A *in a locally presentable monoidal category* $\mathcal{V}$*, the isomorphism between* $\mathbf{Comod}_d(P(A, A))$ *and* $({}^A\mathcal{V}_B)_d$ *becomes an isomorphism of monoidal categories when we equip:*

(1) $\mathbf{Comod}_d(P(A, A))$ *with the monoidal structure associated to the bimonoid structure on* $P(A, A)$*;*
(2) $({}^A\mathcal{V}_B)_d$ *with the monoidal structure defined as follows:*
   (a) *if* (X, φ) *and* (Y, ψ) *are two of its objects, their tensor product is* $X \otimes Y$ *equipped with*

$$A \otimes X \otimes Y \xrightarrow{\varphi \otimes Y} X \otimes A \otimes Y \xrightarrow{X \otimes \psi} X \otimes Y \otimes A,$$

   (b) *the monoidal unit is* I *equipped with* $A \otimes I \cong A \cong I \otimes A$*,*
   (c) *the forgetful functor* $({}^A\mathcal{V}_B)_d \to \mathcal{V}$ *is strict monoidal.*

8. *Universal measuring coalgebras and cocommutativity*

We now return to the more general case of monoids and comonoids in a symmetric monoidal closed category $\mathcal{V}$. Recall from Section 2.1 the opposite (co)monoids.

LEMMA 8.1. *If* $\mathcal{V}$ *is symmetric monoidal closed, there is a natural isomorphism* $P(A, B)^{\mathrm{cop}} \cong P(A^{\mathrm{op}}, B^{\mathrm{op}})$.

Proof. First, we show that the monoid $[C^{\mathrm{cop}}, B]$ equals $[C, B^{\mathrm{op}}]^{\mathrm{op}}$, by showing that the multiplications ν and ν' of these monoids—which coincide as objects in $\mathcal{V}$—are equal. The multiplication ν corresponds under the tensor–hom adjunction to

$$[C, B]^{\otimes 2} \otimes C \xrightarrow{1 \otimes 1 \otimes (c \cdot \Delta)} [C, B]^{\otimes 2} \otimes C^{\otimes 2} \xrightarrow{1 \otimes c \otimes 1} ([C, B] \otimes C)^{\otimes 2} \xrightarrow{\mathrm{ev}^{\otimes 2}} B^{\otimes 2} \xrightarrow{\mu} B, \qquad (50)$$

where c denotes the braiding, while ν' corresponds to

$$[C, B]^{\otimes 2} \otimes C \xrightarrow{c \otimes \Delta} [C, B]^{\otimes 2} \otimes C^{\otimes 2} \xrightarrow{1 \otimes c \otimes 1} ([C, B] \otimes C)^{\otimes 2} \xrightarrow{\mathrm{ev}^{\otimes 2}} B^{\otimes 2} \xrightarrow{\mu \cdot c} B. \qquad (51)$$

Verifying that both composite morphisms are equal provided that the braiding c which is a symmetry is now routine.

We complete the proof by exhibiting the following string of natural isomorphisms

$$\begin{aligned}\mathcal{C}(C, P(A,B)^{\mathrm{cop}}) &\cong \mathcal{C}(C^{\mathrm{cop}}, P(A,B)) \cong \mathcal{A}(A,[C^{\mathrm{cop}},B]) \cong \mathcal{A}(A,[C,B^{\mathrm{op}}]^{\mathrm{op}}) \\ &\cong \mathcal{A}(A^{\mathrm{op}},[C,B^{\mathrm{op}}]) \cong \mathcal{C}(C,P(A^{\mathrm{op}},B^{\mathrm{op}}))\end{aligned} \tag{52}$$

where we abbreviated $\mathcal{C} = \mathbf{Comon}(\mathcal{V})$ and $\mathcal{A} = \mathbf{Mon}(\mathcal{V})$. □

COROLLARY 8.2. *In the situation of Lemma 8.1, $P(A,B)$ is a cocommutative comonoid provided that A and B are commutative monoids. In particular, A° is cocommutative if A is commutative.*

9. *Universal measuring comonoids of cocommutative Hopf monoids*

In the classical case of k-vector spaces, the finite dual $A^\circ = P(A,k)$ of a k-algebra A is constructed as a subspace of the linear dual A^* (39), and this is used to endow A° with an antipode if A has an antipode s. The argument consists of showing that, if $\alpha \in A^\circ \subset A^*$, the functional $\alpha \cdot s$ also belongs to A°, so the linear map given by precomposing with the antipode s restricts to A°. Exactly the same argument is carried over to the case of a Noetherian commutative ring k in [**2**], with the additional hypothesis that A° should be a pure sub-k-module of k^A. In this section we prove that all restrictions on the base commutative ring k can be lifted, as long as the Hopf algebra A is cocommutative. More precisely, we prove:

THEOREM 9.1. *If H is a cocommutative Hopf monoid in a locally presentable symmetric monoidal closed category $\mathcal{V}$, then $P(H,B)$ is a Hopf monoid, for any commutative monoid B.*

Proof. The cocommutativity of H will be used in the fact that the comultiplication $\Delta\colon H \to H \otimes H$ is a morphism of comonoids. Denote by s the antipode of H; it is a monoid morphism $H^{\mathrm{op}} \to H$, so $P(s,B)$ is a comonoid morphism $P(H,B) \to P(H^{\mathrm{op}},B) \cong P(H,B)^{\mathrm{cop}}$, by Lemma 8.1 and commutativity of B. We will show that the underlying arrow of $P(s,B)$ in $\mathcal{V}$ is an antipode for the bimonoid $P(H,B)$.

In Remark 5.5 we exhibit the relationship between the opmonoidal structure of $[-,B]$ and the monoidal structure of $P(-,B)$, via the unit of the adjunction. In this proof we shall use the same notations as in the said remark.

To keep the notation simple, we shall denote all the multiplications by μ when no confusion is possible. The multiplication of $P(H,B)$ arises from the monoidal structure of $P(-,B)$ and the comonoid structure of H. Explicitly, it is the composition $P(\Delta,B)\cdot\psi_{H,H}\colon P(H,B)^{\otimes 2} \to P(H,B)$. It is easy to verify that the corresponding morphism of monoids $H \to [P(H,B)^{\otimes 2},B]$ is

$$[\psi_{H,H},B]\cdot\eta_{H\otimes H}\cdot\Delta = \chi_{P(H,B),P(H,B)}\cdot(\eta_H\otimes\eta_H)\cdot\Delta \tag{53}$$

where the equality uses one of the diagrams displayed in (37).

Denote the antipode of H by s. We are to show the following equality:

$$\begin{aligned}&\left(P(H,B) \xrightarrow{\Delta} P(H,B)^{\otimes 2} \xrightarrow{P(s,B)\otimes 1} P(H,B)^{\otimes 2} \xrightarrow{\mu} P(H,B)\right) \\ &= \left(P(H,B) \xrightarrow{P(\iota,B)} P(I,B) \cong I \xrightarrow{P(\varepsilon,B)} P(H,B)\right);\end{aligned} \tag{54}$$

in order to do so, we shall show that the two compositions have equal transposes under $[-,B] \dashv P(-,B)$. These transposes are calculated by first applying $[-,B]$ and then pre-composing with the unit η_H.

We have already calculated the transpose of μ (53), from where it follows that the transpose of $\mu \cdot (P(s,B)\otimes 1)\cdot \Delta$ is the first composition in the following chain of equalities.

$$\begin{aligned}
&[\Delta, B]\cdot [P(s,B)\otimes 1, B]\cdot \chi_{P(H,B),P(H,B)}\cdot (\eta_H\otimes\eta_H)\cdot \Delta_H\\
&\quad = [\Delta, B]\cdot \chi_{P(H,B),P(H,B)}\cdot ([P(s,1),B]\otimes 1)\cdot (\eta_H\otimes\eta_H)\cdot\Delta_H\\
&\quad = [\Delta, B]\cdot \chi_{P(H,B),P(H,B)}\cdot (\eta_H\otimes\eta_H)\cdot (s\otimes H)\cdot\Delta_H\\
&\quad = \mu_{[P(H,B),B]}\cdot (\eta_H\otimes\eta_H)\cdot (s\otimes H)\cdot\Delta_H\\
&\quad = \eta_H\cdot\mu_H\cdot (s\otimes H)\cdot\Delta_H = \eta_H\cdot\iota_H\cdot\varepsilon_H
\end{aligned}\tag{55}$$

The first equality uses the naturality of χ, the second uses the naturality of η, the third holds since $\mu_{[P(H,B),B]} = [\Delta, B]\cdot\chi_{P(H,B),P(H,B)}$ is the convolution product of $[P(H,B),B]$; the fourth equality is the fact that η_H is a monoid morphism, and the last is one of the two antipode axioms.

On the other hand, the transpose of $P(\varepsilon_H, B)\cdot P(\iota_H, B)$ is precisely $\eta_H\cdot\iota_H\cdot\varepsilon_H$, by naturality of η. Therefore, we have proved the equality (54). The other antipode axiom for $P(s,B)$ is symmetric to the one just verified, and holds by the same argument, concluding the proof. □

Example 9.2. Let k be a commutative ring and H be any cocommutative Hopf k-algebra. Then $P(H,k) = H^\circ$ is a Hopf algebra. In this general case, there is no obvious reason why H° should be a sub-k-module of H^*.

Example 9.3. This is a good place to examine the meaning of the results so far when the base category $\mathcal{V}$ is the category **Set** of sets, with its monoidal structure given by cartesian product. Each set has a unique (cocommutative) comonoid structure (where the multiplication is the diagonal function), that is, the forgetful functor **Comon**(**Set**) $\to$ **Set** is an isomorphism. The universal measuring set $P(A,B)$ of a pair of monoids A and B is the set **Mon**(**Set**)(A,B) of monoid morphisms $A\to B$.

Any monoid A is automatically a cocommutative bimonoid. If B is a commutative monoid, point-wise multiplication endows **Mon**(**Set**)(A,B) with a monoid structure; compare with Corollary 5.4. A Hopf algebra H in **Set** is just a group; the antipode $s\colon H\to H$ is given by $s(x) = x^{-1}$. Theorem 9.1, then, says that **Mon**(**Set**)(H,B) is a group if B is commutative and H is a group. The inverse of a monoid map is, of course, $(f^{-1})(x) = f(x^{-1})$.

10. *Universal measuring comonoids of Hopf monoids*

Having shown that the universal measuring comonoid $P(A,B)$ is a Hopf monoid when A is a cocommutative Hopf monoid and B is a commutative monoid, we now investigate the case of a general, not necessarily cocommutative, Hopf monoid A. In order to do so, we first need some basic notions and facts about Hopf monads and Hopf comonads and Hopf monoids. The main result of the section, Theorem 10.11, is powerful enough to encompass the examples of vector spaces and dg vector spaces.

10.1. *Hopf monads*

In this section we briefly recall the notion of Hopf monad. More details can be found in [**7**]. Let $\mathcal{C}$ be a monoidal category and $\mathsf{T} = (T,\eta,\mu)$ a monad on it. An *opmonoidal structure* on T consists of a natural transformation $T_{2,X,Y}\colon T(X\otimes Y)\to T(X)\otimes T(Y)$ and a morphism $T_0\colon T(I)\to I$ satisfying various axioms that make the following result of [**23**] hold: the category

$\mathcal{C}^T$ of Eilenberg–Moore algebras has a monoidal structure that makes the forgetful functor into $\mathcal{C}$ strict monoidal.

We will later be interested in the case of the monad $T = (A \otimes -)$ induced by a bimonoid A in a braided tensor category $\mathcal{C}$. The opmonoidal structure is given by

$$T_{2,X,Y} \colon A \otimes X \otimes Y \xrightarrow{\Delta \otimes 1 \otimes 1} A \otimes A \otimes X \otimes Y \xrightarrow{1 \otimes c_{A,X} \otimes 1} A \otimes X \otimes A \otimes Y \tag{56}$$

and $T_0 = \varepsilon \otimes 1 \colon A \otimes I \to I \otimes I \cong I$.

Given an opmonoidal monad T as above, its left and right *fusion operators* or *Hopf maps* are the displayed compositions:

$$H^{\ell}_{X,Y} \colon T(X \otimes TY) \xrightarrow{T_{2,X,TY}} TX \otimes T^2 Y \xrightarrow{1 \otimes \mu_Y} TX \otimes TY \tag{57}$$

$$H^{r}_{X,Y} \colon T(TX \otimes Y) \xrightarrow{T_{2,TX,Y}} T^2 X \otimes TY \xrightarrow{\mu_X \otimes 1} TX \otimes TY. \tag{58}$$

The opmonoidal monad T is *left* (*respectively, right*) *Hopf* if H^{ℓ} (respectively, H^r) is invertible.

One of the main results of [7] states that, if $\mathcal{C}$ is left (respectively right) closed, T is left (respectively, right) Hopf if and only if the monoidal category $\mathcal{C}^T$ is left (respectively, right) closed and the forgetful functor is strong closed (that is, it preserves internal homs up to isomorphism).

10.2. *Hopf comonads*

In the interest of completeness, and since [7] gives full descriptions only for the case of monads, we shall provide some details about the theory of Hopf comonads. One difference with the case of monads is that, although there is an abundance of examples of closed categories, even the basic examples of the category of sets or the category of vector spaces are not coclosed categories (categories whose opposite categories are closed). In examples, when tensoring with an object M has a left adjoint, it does so because M has a dual. Below we briefly treat the relationship between the Hopf condition for comonads and the existence of duals.

A *monoidal structure* on a comonad $\mathsf{G} = (G, \varepsilon, \delta)$ consists of natural transformations $G_{2,X,Y} \colon GX \otimes GY \to G(X \otimes Y)$ and a morphism $G_0 \colon I \to G(I)$ satisfying certain axioms that imply that its category $\mathcal{C}^G$ of Eilenberg–Moore coalgebras is monoidal and the forgetful functor $U \colon \mathcal{C}^G \to \mathcal{C}$ is strict monoidal.

Given a monoidal comonad as in the previous paragraph, the right and left fusion operators are defined in the following way:

$$H^{r}_{X,Y} \colon G(X) \otimes G(Y) \xrightarrow{1 \otimes \delta_Y} G(X) \otimes G^2(Y) \xrightarrow{G_{2,X,G(Y)}} G(X \otimes G(Y)) \tag{59}$$

$$H^{\ell}_{X,Y} \colon G(X) \otimes G(Y) \xrightarrow{\delta_X \otimes 1} G^2(X) \otimes G(Y) \xrightarrow{G_{2,G(X),Y}} G(G(X) \otimes Y) \tag{60}$$

One says that the comonad G is *right* (*respectively, left*) *Hopf* if H^r (respectively, H^{ℓ}) is invertible.

Similarly, there are morphisms as displayed below, natural in G-coalgebras (M, χ) and $X \in \mathcal{C}$:

$$\bar{H}^{r}_{X,M} \colon G(X) \otimes M \xrightarrow{1 \otimes \chi} G(X) \otimes G(M) \xrightarrow{G_{2,X,M}} G(X \otimes M) \tag{61}$$

$$\bar{H}^{\ell}_{M,Y} \colon M \otimes G(Y) \xrightarrow{\chi \otimes 1} G(M) \otimes G(Y) \xrightarrow{G_{2,M,Y}} G(M \otimes Y). \tag{62}$$

Clearly, $\bar{H}^r$ is invertible if and only if H^r is invertible; for, each G-coalgebra is a U-split equaliser of cofree coalgebras [20, § VI].

Let (M,χ) be a G-coalgebra and consider the situation when $(-\otimes M)$ has a left adjoint L; typically, L is given by tensoring with a right dual of M.

$$\begin{tikzcd} \mathcal{C}^G \arrow[r, shift left, "-\otimes(M,\chi)"] \arrow[d, "U"'] \arrow[r, phantom, "\top"] & \mathcal{C}^G \arrow[l, shift left, dotted, "\hat{L}"] \arrow[d, "U"] \\ \mathcal{C} \arrow[r, shift left, "-\otimes M"] \arrow[r, phantom, "\top"] & \mathcal{C} \arrow[l, shift left, "L"] \end{tikzcd} \tag{63}$$

A *lifting* of the adjunction $L \dashv (-\otimes M)$ to $\mathcal{C}^G$ is a left adjoint to $\hat{L} \dashv (-\otimes(M,\chi))$ that makes (U,U) a strict morphism of adjunctions; this means that the square formed by the left adjoints commutes and the unit and counit of the respective adjunctions are compatible with U in an obvious way (see [**20**, § IV.7]).

Lemma 10.1. *The adjunction $L \dashv (-\otimes M)$ as above lifts to $\mathcal{C}^G$, for all* G*-coalgebras (M,χ), if and only if* G *is right Hopf. Symmetrically, an adjunction $L \dashv (M\otimes -)$ lifts to $\mathcal{C}^G$, for all* G*-coalgebras (M,χ), if and only if* G *is left Hopf.*

Proof. This is dual to part of [**7**, Theorem 3.6]; in fact, it is dual to Theorem 3.13 together with Example 3.12 of op. cit. □

Remark 10.2. Let X be an object in the monoidal category $\mathcal{A}$, and suppose given an adjunction $(L,(-\otimes X),\eta,\varepsilon)$. Consider the canonical left action of the monoidal category $\mathcal{A}$ on itself, and note that the right adjoint is a strong morphism with respect to it, with structure given by the associativity and unit constraints

$$A\otimes(B\otimes X)\cong(A\otimes B)\otimes X \tag{64}$$

and $I\otimes X\cong X$. By doctrinal adjunction [**16**], the left adjoint L carries a unique opmorphism structure that makes η and ε compatible with the action. Then X has a right dual if and only if the opmorphism L is a strong morphism; in which case, the right dual is $L(I)$.

Lemma 10.3. *Suppose given a monoidal comonad on the monoidal category $\mathcal{C}$ and adjunctions as in (63), so that (U,U) is a strict morphism of adjunctions. Then (M,χ) has a right dual in $\mathcal{C}^G$ provided that M has a right dual in $\mathcal{C}$.*

Proof. Let $M^\vee$ be the right dual of M. By Remark 10.2, the left adjoint $\hat{L}$ is an opmorphism with respect to the left action of $\mathcal{C}^G$ on itself, with structure given by morphisms $\lambda\colon \hat{L}((N,\nu)\otimes(N',\nu'))\to(N,\nu)\otimes\hat{L}(N',\nu')$ whose image under U are the isomorphisms $(N\otimes N')\otimes M^\vee\cong N\otimes(N'\otimes M^\vee)$. Thus λ is an isomorphism, and $\hat{L}$ is isomorphic to $(-\otimes\hat{L}(I))$, so (M,χ) has a right adjoint. □

Proposition 10.4. *Let (M,χ) be a* G*-coalgebra. If* G *is right (respectively, left) Hopf, then (M,χ) has a right (respectively, left) dual in $\mathcal{C}^G$ provided that M have a right (respectively, left) dual in $\mathcal{C}$.*

Proof. Suppose that $M^\vee$ is a right dual to M, so $L=(-\otimes M^\vee)$ is a left adjoint to $(-\otimes M)$. This adjunction lifts to an adjunction $\hat{L}\dashv(-\otimes(M,\chi))$ on $\mathcal{C}^G$ by Lemma 10.1. The result can now be deduced from Lemma 10.3. □

Lemma 10.5. *Let* G *be a κ-accessible monoidal comonad on an accessible monoidal category $\mathcal{V}$. Then $\mathcal{V}^G$ is an accessible monoidal category, and locally presentable if $\mathcal{V}$ is so. Dualizable* G*-coalgebras are κ-presentable.*

Proof. It was mentioned in Corollary 2.8 that $\mathcal{V}^G$ is an accessible category, with accessible forgetful functor U to $\mathcal{V}$. It remains to be shown that the functor $((M,\chi)\otimes -)$ is accessible for any G-coalgebra (M,χ), and similarly tensoring on the other side. This is a consequence of [**21**, Proposition 2.4.10], since U is conservative and accessible, and $U((M,\chi)\otimes -)$ is the accessible $M\otimes U(-)$. Therefore $\mathcal{V}^G$ is accessible monoidal as in Definition 2.6.

The hypotheses that G is κ-accessible and that $\mathcal{V}$ is accessible as a monoidal category tell us that $\mathcal{V}$ is κ-accessible and its unit object I is κ-presentable. If M is an object of $\mathcal{V}$ with a left dual, then $\mathcal{V}(M,-)\cong\mathcal{V}(I,-\otimes M^\vee)$ is accessible, so M is κ-presentable. If (M,χ) is a G-coalgebra, then (M,χ) is presentable with presentability degree at least that of M and that of G, that is, at least κ, by Corollary 2.8. □

PROPOSITION 10.6. *Let* G *be a* κ*-accessible monoidal comonad on the accessible monoidal category* $\mathcal{V}$*. Assume that each* G*-coalgebra is a* κ*-filtered colimit of right* (*respectively, left*) *dualizable* G*-coalgebras. Then* G *is right* (*respectively, left*) *Hopf if and only if each right* (*respectively, left*) *dualizable* G*-coalgebra has a right* (*respectively, left*) *dual in* $\mathcal{C}^G$.

Proof. We briefly deal with the case of right dualizable G-coalgebras. The direct implication holds by Proposition 10.4. For the converse, dualizable objects of $\mathcal{V}^G$ are κ-presentable, by Lemma 10.5. By the κ-accessibility of G, the domain and codomain of $\bar{H}^r$ (see (61)) preserve κ-filtered colimits, so $\bar{H}^r$ is an isomorphism if it is so on dualizable objects. □

10.3. *Hopf (co)monads induced by Hopf monoids*

Let $\mathcal{C}$ be a braided monoidal category and $(H,\Delta,\varepsilon,\mu,\iota)$ bimonoid in it. There are eight questions that we may naturally ask ourselves about H:

Is the monad $(H\otimes -)$ left or right Hopf?
Is the monad $(-\otimes H)$ left or right Hopf?
Is the comonad $(H\otimes -)$ left or right Hopf?
Is the comonad $(-\otimes H)$ left or right Hopf?

In this section, we give the answers to the above questions in terms of the Hopf maps of H. There are four such operators—see Sections 10.1 and 10.2—depicted in diagrams (72)–(75). We call the first two maps *fusion operators* and the last two *opfusion operators*. The contents of this section derive from [**7**, Lemma 5.1]; our contribution consists in the inclusion of the dual statements—that is, those for right Hopf monads—and a condensed proof.

PROPOSITION 10.7. (1) *The following are equivalent for a bimonoid* H*.*
- (a) H *is a Hopf monoid.*
- (b) *The two fusion operators of diagrams* (72)–(75) *are invertible.*
- (c) *Any one of the two fusion operators in diagrams* (72)–(75) *is invertible.*
- (d) *The opmonoidal monad* $(H\otimes -)$ *is left Hopf.*
- (e) *The opmonoidal monad* $(-\otimes H)$ *is right Hopf.*
- (f) *The monoidal comonad* $(H\otimes -)$ *is right Hopf.*
- (g) *The monoidal comonad* $(-\otimes H)$ *is left Hopf.*

(2) *The following are equivalent for a bimonoid* H*:*
- (h) H *is an op-Hopf monoid.*
- (i) *The two opfusion operators of diagrams* (72)–(75) *are invertible.*
- (j) *Any one of the two opfusion operators in diagrams* (72)–(75) *is invertible.*
- (k) *The opmonoidal monad* $(H\otimes -)$ *is right Hopf.*
- (l) *The opmonoidal monad* $(-\otimes H)$ *is left Hopf.*
- (m) *The monoidal comonad* $(H\otimes -)$ *is left Hopf.*
- (n) *The monoidal comonad* $(-\otimes H)$ *is right Hopf.*

Proof. We prove the equivalences of the statements in (1), leaving the proof of the equivalences in (2) for the reader. Consider the function

$$\Phi\colon \mathcal{V}(H,H) \longrightarrow \mathcal{V}(H^{\otimes 2},H^{\otimes 2}),\quad f\mapsto (H\otimes\mu)\cdot(H\otimes f\otimes H)\cdot(\Delta\otimes H). \tag{65}$$

It is easy to verify that Φ is a monoid morphism if the domain has the convolution product induced by the bimonoid H, and the codomain has the product given by composition. In fact, Φ is an isomorphism onto the monoid consisting of those endomorphisms that are simultaneously: endomorphisms of left H-comodules on the cofree left H-comodule $H^{\otimes 2}$; and, endomorphisms of free right H-modules on the free left H-module $H^{\otimes 2}$. From these considerations it follows that 1_H has a convolution inverse, that is, there exists an antipode, if and only if $\Phi(1_H)$, the first fusion operator in diagrams (72)–(75) is invertible.

Now consider the function

$$\Phi'\colon \mathcal{V}(H,H) \longrightarrow \mathcal{V}(H^{\otimes 2},H^{\otimes 2}),\quad f\mapsto (\mu\otimes H)\cdot(H\otimes f\otimes H)\cdot(H\otimes\Delta). \tag{66}$$

Again, it is easy to verify that Φ' is an anti-morphism of monoids when the domain is equipped with convolution and the codomain with composition. Furthermore, it is an isomorphism onto the submonoid of those endomorphisms of $H^{\otimes 2}$ that are simultaneously: right H-comodule endomorphisms on the cofree H-comodule $H^{\otimes 2}$; and, left H-module endomorphisms on the free left H-module $H^{\otimes 2}$. Therefore, there exists an antipode for H if and only if $\Phi'(1_H)$, which is the second fusion operator in diagrams (72)–(75), is invertible. These first two paragraphs of the proof show the equivalence among the conditions (a), (b) and (c).

The left fusion operator of the opmonoidal monad $(H\otimes -)$, the right fusion operator of the opmonoidal monad $(-\otimes H)$, the right fusion operator of the monoidal comonad $(H\otimes -)$ and the left fusion operator of the monoidal comonad $(-\otimes H)$, have components depicted in the respective order in diagrams (67)–(70):

$$HXHY \xrightarrow{\Delta XHY} HHXHY \xrightarrow{Hc_{H,X}HY} HXHHY \xrightarrow{HX\mu Y} HXHY \tag{67}$$

$$XHYH \xrightarrow{XHY\Delta} XHYHH \xrightarrow{XHc_{YH}H} XHHYH \xrightarrow{X\mu YH} XHYH \tag{68}$$

$$HXHY \xrightarrow{HX\Delta Y} HXHHY \xrightarrow{Hc_{X,H}HY} HHXHY \xrightarrow{\mu XHY} HXHY \tag{69}$$

$$XHYH \xrightarrow{X\Delta YH} XHHYH \xrightarrow{XHc_{H,Y}H} XHYHH \xrightarrow{XHY\mu} XHYH. \tag{70}$$

Setting $X=Y=I$ in these fusion operators we obtain, respectively, h, h', h' and h. In fact, the general components can be easily obtained from h and h' by tensoring with X and Y and composing with the braiding; for example, the first composite is obtained as

$$(H\otimes c_{H,X}\otimes Y)\cdot(h\otimes X\otimes Y)\cdot(H\otimes c^{-1}_{X,H}\otimes Y). \tag{71}$$

Therefore, any one of the four fusion operators depicted above is invertible if and only if either h or h' is invertible, completing the proof of 1.

$$h\colon H^{\otimes 2} \xrightarrow{\Delta\otimes 1} H^{\otimes 3} \xrightarrow{1\otimes\mu} H^{\otimes 2} \tag{72}$$

$$h'\colon H^{\otimes 2} \xrightarrow{1\otimes\Delta} H^{\otimes 3} \xrightarrow{\mu\otimes 1} H^{\otimes 2} \tag{73}$$

$$\bar{h}\colon H^{\otimes 2} \xrightarrow{\Delta\otimes 1} H^{\otimes 3} \xrightarrow{1\otimes c_{H,H}} H^{\otimes 3} \xrightarrow{\mu\otimes 1} H^{\otimes 2} \tag{74}$$

$$\bar{h}'\colon H^{\otimes 2} \xrightarrow{1\otimes\Delta} H^{\otimes 3} \xrightarrow{c_{H,H}\otimes 1} H^{\otimes 3} \xrightarrow{1\otimes\mu} H^{\otimes 2}. \tag{75}$$

$\square$

10.4. *Universal measuring comonoids and Hopf monoids*

Recall from Section 2.4 the Kleisli category for a monoidal monad on a monoidal category, and from Definition 6.4 the category ${}^A\mathcal{V}_B$, for a pair of monoids A, B: it is the pullback (42) of the forgetful $(\mathcal{V}_B)^{(A\otimes -)} \to \mathcal{V}_B$ along the universal Kleisli functor $F_B\colon \mathcal{V} \to \mathcal{V}_B$. Here $\mathcal{V}_B$ is the Kleisli category of the monad $(-\otimes B)$. The monad $(-\otimes B)$ is monoidal if the monoid B is commutative. Furthermore, it is an easy calculation to verify that if $\mathcal{V}$ is a *symmetric* monoidal category, then $(-\otimes B)$ is a braided monoidal functor when B is commutative, see Lemma 2.3; then $\mathcal{V}_B$ is a symmetric monoidal category.

Proposition 10.8. *Let $\mathcal{V}$ be a symmetric monoidal closed category and B a commutative monoid in it. Suppose that A is a Hopf (respectively, op-Hopf) monoid in $\mathcal{V}$. Then ${}^A\mathcal{V}_B$ is monoidal and an object $(X,\psi) \in {}^A\mathcal{V}_B$ has a left (respectively, right) dual if and only if X has a dual in $\mathcal{V}$.*

Proof. The category $\mathcal{V}_B$ is symmetric monoidal and the Kleisli functor $F_B\colon \mathcal{V} \to \mathcal{V}_B$ is braided and strict monoidal; see Section 2.4. Thus, $A = F_B(A)$ is a Hopf (respectively, op-Hopf) monoid in $\mathcal{V}_B$, so the category $(\mathcal{V}_B)^{(A\otimes -)}$ of left A-modules in $\mathcal{V}_B$ is monoidal and the forgetful functor $(\mathcal{V}_B)^{(A\otimes -)} \to \mathcal{V}_B$ strict monoidal. Then, the pullback ${}^A\mathcal{V}_B$ defined in (42) is a monoidal category and the forgetful ${}^A\mathcal{V}_B \to \mathcal{V}$ is strict monoidal. Suppose that an object (X,ψ) of ${}^A\mathcal{V}_B$ is dualizable, that is, X has a dual in $\mathcal{V}$. Then $F_B(X)$ has a dual in $\mathcal{V}_B$, by the strict monoidality of F_B, so the projection (X,φ) of (X,ψ) to $(\mathcal{V}_B)^{(A\otimes -)}$ is dualizable. By the hypothesis of being Hopf (respectively, op-Hopf), we have that (X,φ) has a left (respectively, right) dual and the forgetful functor into $\mathcal{V}_B$ preserves evaluation and coevaluation. It follows that $(X,\psi) \in {}^A\mathcal{V}_B$ has a left (respectively, right) dual by the definition of ${}^A\mathcal{V}_B$ as a pullback. □

Definition 10.9. A braided monoidal category $\mathcal{V}$ is said to satisfy the *fundamental theorem of comodules* if, for each comonoid C in $\mathcal{V}$, each C-comodule is a filtered colimit of dualizable C-comodules.

Remark 10.10. The previous definition elicits a number of comments. First, it seems possible to drop the assumption that the monoidal category be braided, distinguishing between left and right dualizable objects. We prefer to keep the definition more readable by retaining the braiding assumption.

Secondly, [**22**] says that $\mathcal{V}$ satisfies the fundamental theorem of comodules if, for each comonoid C, each C-comodule is filtered colimit of dualizable strong subobjects. We do not require the colimit to be one of subobjects.

Thirdly, one might think that there is a certain ambiguity in our definition with respect to left and right C-comodules. It is not a real one, however, since left C-comodules are right comodules over the opposite comonoid.

Theorem 10.11. *Let $\mathcal{V}$ be a locally presentable symmetric monoidal closed category that satisfies the fundamental theorem of comodules, A a Hopf (respectively, op-Hopf) monoid and B a commutative monoid. Then $P(A,B)$ is a Hopf (respectively, op-Hopf) monoid.*

Proof. The category of dualizable right $P(A,B)$-comodules is monoidally isomorphic over $\mathcal{V}$ to the category of dualizable objects in ${}^A\mathcal{V}_B$, by Corollary 6.5 and Corollary 7.1, and the latter category is left (respectively, right) autonomous by Proposition 10.8. So, any dualizable comodule has a left (respectively, right) dual. By the fundamental theorem of comodules, each $P(A,B)$-comodule is a colimit of dualizable ones, so the comonad $(-\otimes P(A,B))$ is left (respectively, right) Hopf by Proposition 10.6. This is equivalent to saying that $P(A,B)$ is a Hopf (respectively, op-Hopf) monoid, by Proposition 10.7. □

EXAMPLE 10.12. If A is a Hopf algebra over a field k, then $P(A,B)$ is a Hopf algebra for any commutative k-algebra B. Let A^{op} be the bialgebra obtained by taking the opposite multiplication but leaving the comultiplication intact. If A^{op} is a Hopf algebra, then $P(A,B)$ is op-Hopf. The example of graded (co)algebras is explored in the next section.

11. *Example: graded (co)algebras*

Recall from Example 2.10 that the category $\mathbf{gVect}_{\mathbb{Z}}$ of $\mathbb{Z}$-graded vector spaces is a locally finitely presentable symmetric monoidal closed category. In what follows, graded (co)algebra (Example 4.4) and graded (co)module mean (co)monoid and (co)module in the said monoidal category.

LEMMA 11.1. *Let M be a (right) graded comodule over a graded coalgebra C. Any homogeneous finite-dimensional space of M is contained in a finite-dimensional sub graded comodule.*

Proof. The proof is identical to that of [**10**, Lemma 1.1], except that we admit negative grading. □

The above lemma immediately yields:

COROLLARY 11.2. *The category $\mathbf{gVect}_{\mathbb{Z}}$ of graded vector spaces satisfies the fundamental theorem of comodules.*

Proof. The category of graded vector spaces has an internal hom given by

$$\mathrm{Hom}(X,Y)_n = \prod_{i\in\mathbb{Z}}[X_i, Y_{i+n}] \tag{76}$$

and unit object $I = k[0]$ the base field k concentrated on degree 0, as mentioned in Example 2.10. We have to show that finite-dimensional graded spaces have a dual object in $\mathbf{gVect}_{\mathbb{Z}}$; for this suppose that X is finite-dimensional, that is, it is 0 except in finitely many degrees, say between $-n$ and n. If X had a dual, it should be $\mathrm{Hom}(X, k[0])$

$$\mathrm{Hom}(X, k[0])_m = [X_{-m}, k]. \tag{77}$$

By general considerations on duals and internal homs, X has a dual if and only if the comparison morphism $\mathrm{Hom}(X,k[0]) \otimes Y \to \mathrm{Hom}(X,Y)$, with m-component

$$\sum_{i=-n}^{n} [X_{-i}, k] \otimes Y_{m-i} \longrightarrow \prod_{j=-n}^{n} [X_j, Y_{j+m}] \tag{78}$$

is an isomorphism. Since the product in the codomain is finite, it can be replaced by a sum, and reindexing, it is isomorphic to $\sum_{\ell=-n}^{n}[X_{-\ell}, Y_{m-\ell}]$. It is now easy to see that (78) is the sum of the isomorphisms of the type $[V,k]\otimes W \cong [V,W]$ for V a finite-dimensional vector space. □

REMARK 11.3. In the above corollary, it was important that the grading is over the group of integers. For example, in the category of non-negatively ($\mathbb{N}$-) graded spaces, very few objects have a dual: they are all concentrated in degree 0.

We can now describe the result of applying Theorem 10.11 to the base category $\mathcal{V} = \mathbf{gVect}_{\mathbb{Z}}$.

Proposition 11.4. *Let H be a $\mathbb{Z}$-graded bialgebra and B a commutative $\mathbb{Z}$-graded algebra. Then $P(H, B)$ is a $\mathbb{Z}$-graded bialgebra. If H is a Hopf (respectively, op-Hopf) graded algebra, then $P(H, B)$ is Hopf (respectively, op-Hopf) too.*

If, instead of $\mathbb{Z}$-graded spaces, we wanted to work with $\mathbb{N}$-graded spaces, and obstacle presents itself: $\mathbf{gVect}_{\mathbb{N}}$ does not have enough objects with duals to satisfy the fundamental theorem of comodules—Definition 10.9. We can say something, however, if we admit the restriction to (graded) (co)commutative algebras.

Proposition 11.5. *Let H be a cocommutative $\mathbb{N}$-graded Hopf algebra and B a commutative $\mathbb{N}$-graded algebra. Then $P(H, B)$ is a $\mathbb{N}$-graded Hopf algebra.*

The proof of the proposition is an application of Theorem 9.1.

An $\mathbb{N}$-graded vector space is *connected* if its component of degree 0 is one-dimensional. It is well known that connected $\mathbb{N}$-graded bialgebras automatically are Hopf algebras. Even if one is only interested in connected spaces, Proposition 11.5 is not redundant, as $P(H, B)$ may not be connected even when H and B are so. For example, a morphism of $\mathbb{N}$-graded coalgebras $k[0] \to C$ is equivalently given by an element $g \in C_0$,that is, a group-like element of C, that is, $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$. If C is connected, there is at most one such element, as the restriction of $\varepsilon \colon C \to k[0]$ to degree 0 is an isomorphism $C_0 \cong k$. Therefore, if $P(A, B)$ is connected, then there exists at most one $\mathbb{N}$-graded morphism of algebras $A \to B$, by the definition of $P(A, B)$. An example where this does not happen, and therefore where $P(A, B)$ is not connected, is that of $k = \mathbb{F}_2$, the field of characteristic 2 (so $-1 = 1$ and graded (co)commutativity is just ordinary (co)commutativity), and $A = \mathbb{F}_2[x]$ is the polynomial algebra with the usual Hopf algebra structure, whose cocomultiplication is given by $\Delta(x) = 1 \otimes x + x \otimes 1$. For any connected $\mathbb{F}_2$-algebra B, a morphism of graded algebras $\mathbb{F}_2[x] \to B$ is defined by a unique element of B_1. In this way, any B for which $B_1 \neq 0$ provides an example in which $P(\mathbb{F}_2[x], B)$ is not connected.

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