The Forgotten Turing

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October 16, 2011

In fond memory of Robin Oliver Gandy: 1919-1995

Introduction

Alan Turing is remembered for many things. He is widely known as code breaker and cryptographer, and as - at least in some sense - inventor of the computer. He is if anything even more famous as the father of Artificial Intelligence. Beyond that he was a mathematician, mathematical logician and pioneer in the study of morphogenesis.

Logicians remember Turing for his celebrated Entscheidungsproblem paper [Turing 1937], for work on the λ-calculus [Turing 1937a] and if they are cognoscenti for his second great paper [Turing 1939] in the Proceedings of the London Mathematical Society. All that work was completed by 1940. It is not widely appreciated that Turing's interest in logic continued to the end of his life. His later interest in the Theory of Types, a central area in the foundations of mathematics, is largely forgotten. That interest has had more influence than is evident. I am going to tell the story: it is an odd one.

The one and only student

My DPhil supervisor at Oxford was Robin Gandy: he was the only person to take a doctorate under Turing's supervision, and also one of his closest friends.

While Turing had only one student, Gandy had many. I was one of the the cohort from the early 1970s. Gandy rather liked the thought that his students were intellectual grandchildren of Turing. We mostly cared rather less about that, but from Gandy's reminiscences we all caught a glimpse of Turing through the eyes of someone who had known him very well. Turing met Gandy in 1940 at a party in King's College, Cambridge. Gandy was a third year student, while Turing, who had intermitted his Fellowship of the College at the start of the war, was already engaged in his celebrated work at Bletchley Park. During the war, Gandy became a friend, and then much later Turing's student. By the end of Turing's life Gandy had begun his academic career as a Lecturer in Applied Mathematics at Leicester. You can read about their unique relationship in the Turing biography [Hodges 1983] by Andrew Hodges. Here I shall focus on Turing's influence on Gandy as his PhD supervisor.

It seems entirely fitting that Turing's student should have become a well-respected mathematician in his own right. But in fact, Gandy's intellectual development was not straightforward. I believe that without Turing's decisive intervention Gandy would never have become a serious logician. Turing's grandchildren owe him a great debt.

Memories

When I started research with Robin Gandy in 1971, I was a typical disoriented student of the early 70s, with long curly black hair, and no real sense of mathematical direction. Gandy on the other hand was a leading UK logician. He had made substantial contributions to the subject[1] and important results carried his name. As Reader in Mathematical Logic at Oxford, he was with Michael Dummett running the new course in Mathematics and Philosophy. He was a Fellow of Wolfson College where he lived happily until his retirement. There should have been an enormous distance between us but somehow there was not.

All Gandy's students experienced his extraordinary character. He was not what most people think of as an academic. He did not seem quite serious. He was amused and amusing, with great enthusiasm for life. He was most definitely loud: you always knew where he was down the pub. By the time I knew him he had put aside his famous motorbike and leathers, but he remained a remarkable and handsome figure. He was sufficiently unselfcon-

[1] His obituary in the Bulletin of Symbolic Logic [Moschovakis-Yates 1996] gives an account of some high points of what was a very distinguished career.

scious that at times he lectured with shirt visibly tucked into underpants. He certainly did not aspire to glamour in the usual sense, but there was a glamour of personality about him. He enjoyed being a little outrageous: he would mischievously recall that he had once been a very pretty boy - and he was amused occasionally to be rewarded with a shocked reaction. But I think that even when he was young, personality must have outweighed looks.

I believe that with Gandy the dashing legend and extravagant personality obscured his real and unusual warmth. Speaking at a memorial meeting in 1995, Dummett said that Gandy liked people the way some people like cats. As with cats, people liked Gandy in return. (And yes, Gandy also liked cats.) Gandy had a special generosity of spirit, which I suppose is what attracted Turing initially.

It would be hard to do anyone justice in a quick description like this, but I have so far omitted one significant side to Gandy. It does not quite fit the rest; he was passionate about mathematical logic. Certainly he was as far from the dry as dust logician of common parlance as one could imagine. How did he come to be a logician?

Early years

Gandy worked on radar during the war and was posted to Hanslope Park where Turing went to work after his time at Bletchley Park. For a time the two of them shared a house together with Gandy's cat Timothy. Gandy returned to King's College from war service in 1945 for a fourth year, and in 1946 took Part III of the Mathematical Tripos, with Distinction. He spent the next few years thinking about the foundations of physics and working towards a Fellowship. In 1949, after what seems like a reasonable period, he applied for an internal Research Fellowship at King. This required him to submit a dissertation.

Gandy's dissertation was entitled 'Some Considerations concerning the Logical Structure underlying Fundamental Theories in Theoretical Physics'. It seems King's did not keep a copy, but I am grateful to Patricia McGuire, the Archivist, for locating the three expert reports which were considered by the Electors to Fellowships. Turing, who was by then working in Manchester, was one of the experts. The others were Frank Smithies of St.John's College and Richard Braithwaite of King's, Knightbridge Professor of Moral Philosophy. We can tell from the reports that Gandy's dissertation was about how scientific theory is related to empirical observation. Gandy's approach was to consider the design of a machine - like a Turing machine but seemingly more complicated - to derive scientific hypotheses from data.

Turing took great care with his report. The general assessment is supplemented by three pages of detailed criticism and commentary. The final section of the assessment reads as follows.

> A less pretentious approach would have made it possible to cover much more ground. This might have been done by the method of example and analogy. Examples are given at some points and form some of the best parts of the thesis. The detailed criticism are numerous, but their number reflects as much on the reviewer's industry as on the author's shortcomings. The majority of papers of this nature are too flimsy to stand up for criticism. I believe that in a year's time Mr. Gandy should be able to produce something worthy of a Fellowship.

Of the others Smithies was sympathetic to Gandy's ambition, but felt it unrealised while Braithwaite was sceptical. Gandy was not elected.

Turing thought that Gandy should have been able to produce something worthy of a Fellowship in a further year. Gandy applied again in 1950 with a dissertation with the more straightforward title 'The Foundations of Physics'. Again the dissertation is lost, but Patricia McGuire has found the reports.

One again was written by Turing. It begins bluntly.

> I am very disappointed in this thesis.

It continues as follows.

> The writer has a good imagination and good ideas but he has failed to put them across because of poor technique and taking much too little time over the actual writing of the thesis. He has very rightly decided that symbolic logic is the right medium for these very general considerations, but unfortunately he does not really know enough symbolic logic to carry the programme through successfully. His ideas on the subject of 'groups of indifference' are very stimulating and I shall be most interested to hear whether anything comes of them in the end. But they are certainly not sound as expounded at present, and it does not seem possible to put it right by merely trivial alterations.

That is damning enough but the rest of Turing's report justifies his negative view by an illustrative analysis of a single half-page passage. Turing identifies a range of problems, some arising from lack of clarity of exposition and some definite errors in logic and mathematics. The second reviewer, Max Newman, though less incisive, is no more supportive. Again Gandy's application for a Fellowship was unsuccessful.

There is an interesting contrast between Turing's two reports. The first is completely dispassionate. There is no hint that Turing even knows the author of the dissertation. The second is very different. The disappointment sounds personal and it is evidently on the basis of personal knowledge that Turing writes about the time spent and the lack of knowledge of symbolic logic. The friendship has not compromised the judgement of the dissertation which is almost brutal; but the frustration Turing felt about his friend is clear.

Let us take stock at this point. In 1950 Gandy was already 30. He had written two dissertations but seemingly without what now would be regarded as research supervision. He may know what a Turing machine is, but he has almost no technical proficiency in logic. The trajectory of Gandy's intellectual development appears very unpromising. How did he become a logician at all, let alone the very distinguished logician of later years?

Student and supervisor

Something important happened in the next few years. At the end of 1952 Gandy completed a PhD dissertation [Gandy 1953]. It is in two parts, and the first and more substantial of these, on the Theory of Types, constitutes Gandy's first steps in mathematical logic.

Andrew Hodges [**?**] records simply that around 1950 Turing became Gandy's supervisor. It is not clear what the arrangement amounted to but it seems natural to suppose that it was instigated by Turing following the second unsuccessful Fellowship dissertation. Turing became responsible for arranging the oral examination and, as explained in [Hodges 1983], had difficulties doing so. But the oral must have taken place by the next summer as the dissertation was deposited in the University Library in July 1953.

The influence of a supervisor on a PhD dissertation is seldom clear. I start with what Gandy says. He closes his introduction with the following acknowledgement.

> Finally I must try and show the extent of my debt to A. M. Turing. He first called my somewhat unwilling attention to the system of Church and the importance of the deduction theorem. Much of the work on permutations and invariance and on the form of theories was done in conjunction with him. Without his encouragement I should long ago have given way to despair; without his criticism my ideas would have remained shallow and obscure.

That is surely heartfelt and goes beyond the usual words of thanks to a supervisor; but in terms of content it is far from telling the full story.

So what is in the dissertation? Given Gandy's earlier failures, the title 'On axiomatic systems in mathematics and theories of physics' is worrying: it suggests more of the same. The second part is indeed a further attempt to describe the foundations of physics[2] in logical terms. It does not get far, and there are oddities. The section on the deduction theorem, to which Gandy refers, contains reflections on meaning, with no obvious relation to the theorem or the rest of the thesis. But the sections on structure and theories show a good grasp of logical fundamentals and my guess is that overall the second part of the PhD dissertation represents a substantial advance over the 1950 Fellowship dissertation. However the first part on mathematical logic is at a quite different level.

I'll give a brief overview of it. By the system of Church Gandy means Church's Theory of Types [Church 1940]. He presents it together with his own variant of it and proves their equivalence. There is a novelty stemming from Turing [Turing 1948]: types come with a distinguished default value[3], which informally Turing called 'nonsense elements'. The invariance under permutations, to which Gandy, forms a substantial section. Presumably it derives from the groups of indifference of the 1950 Fellowship dissertation. There is a concrete application: the only definable individual is the nonsense element. There are technically proficient sections on what we would now call a notion of inner model, and on the definition of truth for sentences of restricted complexity. There is one further section, the third, called 'Virtual

[2]Gandy's interest in physics never left him and happily he did eventually succeed in making a serious contribution. His late paper [Gandy 1980] on physics and mechanism is still much discussed.

[3]In the language of modern computer science this amounts to raising and exception.

Types'. In the context of the PhD dissertation it does not stand out: nothing is done with the main construction. But both its intrinsic intellectual significance and its importance for our story makes it quite special.

Gandy thanks Turing for drawing his reluctant attention to Church's Theory of Types. There is a suggestion there that Turing's involvement was substantial, but there is no way of knowing the extent of it. All that is clear is that in a few years under Turing's supervision Gandy became a serious logician.

Chinese translation

I would like to break away from the main story for a moment to say something about the construction of Virtual Types. I don't want to explain the mathematics, but I would like to give sufficient flavour of the idea to place it in within Turing's intellectual concerns.

Imagine as English speakers that we are interested in translation from English into Chinese[4]. We are given some machines that claim to perform this task, and we want to assess them. We have a co-operative Chinese speaker to help us, but we have nobody who speaks both languages. The task looks hopeless. We can't tell if the machines give true translations. But there are two things we can test - their consistency, and their extensional equality.

What is consistency? Let's take one machine to begin with. We provide it with two sentences which mean the same thing, let's say, 'The cat sat on the mat' and 'The mat was sat on by the cat'. We feed these into our machine and get two translations out. Then we give the translations to our Chinese speaker. He can't tell us whether they mean 'The cat sat on the mat' but he can tell us whether they mean the same thing as each other. They might both mean 'The moon is made of green cheese,' but that doesn't matter. As long as the machine takes two sentences that mean the same thing as each other and gives two translations that mean the same thing as each other, it is consistent.

Now let us take two machines. We've tested to see if they're consistent. Now we want to see if they're extensionally equal. This simply means that we feed 'The cat sat on the mat,' into one machine and 'The mat was sat on

[4]The echo of a famous debate is conscious, but I am not getting into all that

by the cat,' into the other[5] show the translations to the Chinese speaker and find that they mean the same thing.

All the basic ideas for handling Virtual Types appear in this fancy about translation. We look at operations on relevant data (English sentences). We look at those which take equivalent data (synonymous English sentences) to equivalent data (synonymous Chinese sentences). Operations are equivalent when their outputs on the same data (English sentence) are equivalent (synonymous). That's pretty much all there is to it. In technical terms I have just described the inductive extension of partial equivalence relations to function spaces, or in Gandy's terms the definition of the function space of Virtual Types.

My purpose in this is to draw attention to the similarity between the considerations arising with Virtual Types and the issues involved in the Turing Test. They focus on input-output behaviour. In the Turing Test, If the response of a machine is indistinguishable from that of a person then we regard them as equivalent, each as conscious as the other. The flavour of Virtual Types is very much part of Turing's world.

Genesis of an idea

I'd now like to go back to the the story of Gandy's intellectual development. His first published papers [Gandy 1956, Gandy 1959] were on the relative consistency of the axiom of extensionality. I'll just call this the consistency result. There is no sign of it in the PhD dissertation [Gandy 1953], but it uses the inductive construction from the section on Virtual Types. The connection is this. Most mathematicians think expensionally and as a foundation for mathematics, Church's Theory of Types already has an axiom of extensionality built in. The idea of Virtual Types is to create new types with their appropriate extensional equality. Technically one is taking 'quotients by partial equivalence relations'. Gandy's insight was that if you started with a system without extensionality you could use the same idea to construct new types for which the axiom of extensionality holds. That gives what is typical of logic, a metamathematical theorem: the consistency result. This concrete application of the inductive construction from [Gandy 1953] is important.

[5] We are dealing with equality and we have consistency, so we could in principle feed both 'The cat sat on the mat'. But the formulation I give is right in generalisations.

As I shall indicate, it can be regarded as the ancestor of many further ones.

Andrew Hodges [Hodges 1983] records that Gandy visited Turing ten days before Turing's death and that they discussed Type Theory. Gandy's consistency result is not in the PhD dissertation, and his paper [Gandy 1956] was only received in July 1955, a year after Turing's death. However I am certain that the consistency result was known at the time of the 1954 visit, and for two reasons. The first is straightforward: in effect Gandy told me so himself.

My DPhil thesis was on what Stephen Kleene [Kleene 1959] had called the countable functionals and Georg Kreisel [Kreisel 1959] the continuous functionals. These two original approaches to this higher type structure are quite different but both use the Virtual Types technique from [Gandy 1953] and [Gandy 1956]. In those days I was not much interested by intellectual antecedents, but Gandy did once talk with me of the connection between his early paper and the work in which I was interested. It was not one of our usual detached conversations. I recall from it that Gandy was evidently very proud of his early work, and that he wanted me to understand that Turing had been impressed by it and had praised it. I am confident that Gandy specifically mentioned the paper[6] in the conversation. What I remember most of all is that Turing's liking the paper mattered very much to Gandy.

Foresight and hindsight

When I had the conversation with Gandy, I did not see what the fuss was about. The idea of the construction[7] of Virtual Types seemed so simple. Could Turing really have been impressed by it? I once raised the question with Kreisel who teased me by observing elliptically that Turing was unusually talented and there was no telling what his views may have been. At the time I took this to mean that Turing might have hidden his true opinion. But we know that Turing could be painfully honest, and this reading does not seem quite right.

With hindsight I can see that my early sense that there was not much in the idea was mistaken. The idea of the paper may be simple but it has wide

[6] In fact I had no knowledge of Gandy's PhD dissertation until I started writing this piece.

[7] I did not read Gandy's paper and did not refer to the connection in my thesis. So this is by way of setting the record straight.

application. Gandy's own extension to set theory [Gandy 1959] is delicate as Scott [Scott 1962] observed; and Scott's analysis leads very naturally to the Scott-Solovay formulation of Boolean-valued models. In proof theory the same idea occurs, and when extended to an impredicative setting gives a method associated with Tait and Girard: that of 'reducibility candidates'. The method was soon adapted for other purposes in theoretical computer science where it is known as Plotkin's logical relations. There are many other appearances of the idea within abstract mathematics. The modern terminology is partial equivalence relations[8] or subquotients. Recently extensionality is back on the agenda as a consequence of Voevodsky's Univalence Axiom for Homotopy Type Theory. There are hopes to adapt the construction for use in that area. One should not underestimate simple ideas; those with wide application have a special place in mathematics. So Turing's liking Gandy's paper seems to show great foresight. But there is more to the story than that.

Turing and Type Theory

Turing's influence as Gandy's supervisor relates specifically to Type Theory, and I started this paper with the thought that Turing's interest in the area is largely forgotten. It is time to say something about that interest. There are three published papers, all appearing in the Journal of Symbolic Logic. They have hardly had the impact of Turing's other work. But Turing took the trouble to write them and he did not after all write that much. What do the papers amount to?

The first [Newman-Turing 1942] was written with Max Newman about a year after the appearance of Church's formulation of his Simple Theory of Types. Church's theory has a rather subtle axiom of infinity[9] the type of individuals and the paper shows how to derive from it the same formulation of infinity for all types involving the type of individuals. That is by way of being a sanity check: it is not desperately difficult if one keeps ones head. But we might just note the early appearance of an induction over the collection

[8] Searching the web for partial equivalence relation generates more than two and a half million hits.

[9] It is not important for us, but the essential idea is this. One is not given a handle on the type ι of individuals, so one deals with the type $(\iota \to \iota) \to (\iota \to \iota)$ of so-called Church numerals over it. One asks that it be infinite.

of types. It is the demands of an inductive construction which drives the definition of Virtual types.

The second paper [Turing 1942] appeared soon afterwards and is of quite different character. Russell and Whitehead and later for example both Quine and Curry used dots rather than brackets as a form of punctuation. For Turing this amounts to the use of conventions improving readability. He describes conventions, extending those of Curry and gives a precise treatment of the evident issues of disambiguation.

Turing's third paper [Turing 1948] appeared after the end of the war. The practical issue it considers is the use of Type Theory informally and so without strict regard for the typing rules. The idea is to get some of the benefits of set theory, and Turing considers explicitly a cumulative hierarchy based on individuals. Abstractly one can think of what he does as a kind of reverse engineering of a strictly typed system towards a more type free system. In the looser system some expression are interpretable and some not. Turing's paper may be forgotten, but the issue of the relationship between formal and informal mathematical practice remains very much alive.

Turing's intellectual taste

Before coming to the final point of the story I want to say something about Turing's intellectual attitude. The papers and records of talks making up his Collected Works are very varied. The pure mathematics and the contributions to logic are outweighed by the machine intelligence and morphogenesis. The Turing mythology is tied up with Bletchley Park. That story, alongside the later proposals for real computing machines, stresses the distinctly practical side to Turing's understanding of mechanism and computation. One might imagine that Turing had little interest in developments in pure mathematics in the post war period, and little time generally for abstract mathematics. But he was surely aware of them.

Just before the tribute to Turing's influence which I quoted earlier, Gandy acknowledges other intellectual influences.

> The debt which I owe to Bourbaki[10] and to Philip Hall[11] for the

[10]Bourbaki is the name of the now famous group centred in France which was establishing a new vision of abstract mathematics.

[11]Philip Hall was the leading UK algebraist of the time. He was a Fellow of King's, but

> development of abstract structure theory is obvious; what is new here is perhaps the technique of extending the usual definitions to objects of arbitrarily high type. Similarly my debt to Klein and Weyl will be apparent. From the many writers on mathematical and natural philosophy who have influenced me, I single out Poincaré, Russell and Ramsey.

The influences are by no means easy to see, and I think that the passage must reflect not simply Gandy's interest but Turing's as well. Both of them appear well aware of the importance of developments in pure mathematics.

From computability to morphogenesis, Turing had an unusual instinct for really fundamental questions; and like any good mathematician he sought definitive answers. He could find them: the Entscheidungsproblem paper [Turing 1937] provides such an answer to the fundamental question - what is it for a function to be computable? But to understand the history of Turing's involvement in Type Theory, we need to appreciate something more. I think that it is captured at the end of the letter Gandy wrote to Max Newman after Turing's death.

> I thought you hit the nail on the head in the Guardian[12]; the mark of his particular genius was that however abstract the topic he always had absolutely concrete examples in mind; and this, of course, was why he found a lot of contemporary mathematics unsympathetic - he did not like developing abstract concepts merely for their own sake.

The never written paper

I now return to Turing's second paper [Turing 1942] on Type Theory. From a modern perspective, the work involved is a necessary precursor to the effective implementation of a formal language. Turing's intention seems to have been more immediate: he says that he will use the conventions in forthcoming papers. These never appeared. Presumably war work took over. But one planned title which Turing mentions is striking: 'The theory of virtual types'.

Gandy's mention of him is not college piety. He had wide interests in abstract mathematics. For example he owned a copy of the thesis of the French logician Herbrand.

[12]Newman had written Turing's obituary in what was then the Manchester Guardian.

When I read that I gulped; and I went back and looked more closely at Gandy's PhD dissertation. I quote now from the end of the section there on Virtual Types.

> So far as I know the idea of introducing virtual types is due to A. M. Turing; (see footnote in Newman and Turing (1)[13]). He has not published his version and I do not know to what extent the version given here is in agreement with his.

What should we make of that? Well first we are better placed than was Gandy when he wrote. We now have long experience with the basic construction and its many applications. It is what mathematicians call canonical: there is only one way to proceed. Were there something else to do we would have seen it by now. I have no doubt Turing's version would have been the same as Gandy's.

So then what? Well supervisors often have ideas which they have to a greater or lesser extent thought through, and which they suggest to a student leaving it to the student to work out the details. Presumably something of the kind happened in this case. Given the tell-tale footnote Gandy had to be aware that Turing had considered the question of introducing Virtual Types or (as we would not say) subquotients. Turing with delicacy, discretion, reticence - who knows exactly what - left the matter of the agreement between the two approaches lie. Nobody wanted to pursue the matter. It happens more often than one might think.

So why had Turing never written up his own work on Virtual Types? It feels to me like a reflection of his intellectual taste. He understood what to do in the 1940s but I imagine that he thought it simple and not that important. He did not have a concrete example of its use to give it value. That I believe came later.

The following seems to me the likely run of events. Up until 1950, Gandy was pursuing his original interests in physics, and Turing's influence was the casual interest of a friend. After Gandy's 1950 Fellowship application had failed Turing became Gandy's effective supervisor. I imagine this to have been instigated by Turing on the grounds that Gandy needed to develop his logic to support his views on the foundations of physics. Turing encouraged Gandy to work in Type Theory, guiding him to produce his own system and

[13] Gandy's reference is wrong: Turing's footnote is in [Turing 1942].

working with him on the permutation ideas from Gandy's 1950 dissertation. My guess is that Turing's hand shows in the definite concrete application. Turing described the idea of Virtual Types; and was happy to find that once Gandy appreciated the problem he found the fundamental induction step himself. The rest of the logic material of the PhD dissertation involves adapting to Type Theory existing ideas; it looks like the result of routine research supervision. All this took place over a couple of years, and I suppose that from time to time Gandy and Turing also discussed then current ideas about structure and the like, with applications to physics in mind. In late 1952 Gandy wrote up, perhaps leaving too little time for the full working out of his ideas on physics, or perhaps leaving things sketchy on Turing's advice. Under Turing's guidance the emphasis is very much on logic.

There is usually a fallow period between submission and examination of a PhD dissertation. Gandy doubtless continued to read about and work on logic and physics, but I believe that the next event in the story happened, after the oral, in the second half of 1953. I believe that Gandy himself discovered the application of the Virtual Types idea to the consistency of extensionality. That concrete application of the idea changed everything. Turing's old idea had become the definitive solution to a problem, thereby establishing its significance. And perhaps more importantly it resolved the difficulty about Turing's unwritten version: Gandy had the application (the consistency result) and legitimate ownership of the material. It is easy to imagine Turing's pleasure in saying to Gandy that he must write up the result for publication.

Earlier I said that I had two reasons for believing that the consistency result was obtained before Turing's death. The second one is just this. There is no way to make sense of Turing'ss approval of Gandy's work, the approval which meant so much to Gandy, otherwise. The construction as it appears in the PhD dissertation was apparent to Turing in 1942. It is not psychologically realistic to suppose that Turing convincingly praised something he had thought of himself many years before. There had to be something new as the focus of approval. This must have been the new metamathematical application to the consistency result. I am convinced that this was Gandy's own idea[14] and I am convinced that this sign of a completely new insight is

[14] Circumstantial evidence supports this. In all his papers, Turing following Church considers systems in which extensionality is an axiom. The system without extensionality

what really appealed to Turing.

It is a remarkable story. Turing's interest in Type Theory lasted from his reading Church [Church 1940] to the end of his life. He never published the construction which was to prove to be his most influential idea in the area. Towards the end of his life when his main interests were in areas other than logic, he taught the subject to his friend and student Gandy. Gandy found the crucial application which established the significance of the idea and so it entered the literature.

Turing's legacy

Turing can hardly have supposed that by the time of his centenary he would be recognised as a national war hero, but he must have known that he left an intellectual legacy. The Turing Machine and Turing Test are the familiar aspects of that. Together with other more specialised scientific contributions, recognised in their own area, this legacy is being celebrated during the centenary year. I hope I have shown here that the use of partial equivalence relations stemming from the idea of Virtual Types is part of his intellectual estate. It comes to us via Turing's student Gandy, but it is part of the inheritance none the less.

I also want to draw attention to a less obvious legacy. Gandy was Turing's friend before he was Turing's student. The history of Gandy's Fellowship applications suggests that Turing had the frustrating sense that his friend had great potential likely never to be realised. Taking Gandy on as a student was a serious project of Turing's last years, and a successful one. Contrary to what would have been reasonable expectations in 1950, Turing turned Gandy into a mathematical logician. Turing was not to see logic become the love of Gandy's life; but if my reading of the intellectual history is correct, he had the satisfaction of seeing Gandy come good as a logician with his own independent ideas.

Turing left his mathematical books and papers to Gandy in a will dated 11th February 1954. This seems to me to a concrete sign of his pleasure in Gandy's intellectual development. But the real legacy to Gandy was his becoming a logician. That legacy is passed on, to Gandy's students and to their students and so into the future of logic. In this story there are two

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is never entertained.

legacies, the human influence along with the intellectual; and both stem from the forgotten Turing, the Type Theorist.

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