Under consideration for publication in Math. Struct. in Comp. Science

Some Reasons for Generalizing Domain Theory

Martin Hyland

Received 9 October 2009

One natural way to generalize Domain Theory is to replace partially ordered sets by categories. This kind of generalization has recently found application in the study of concurrency. An outline is given of the elegant mathematical foundations which have been developed. This is specialized to give a construction of cartesian closed categories of domains, which throws light on standard presentations of Domain Theory.

Introduction

This paper is one of what I hope will be a series written in conscious homage to Kreisel's paper (Kreisel 1971). That paper was prepared for the European Meeting of the ASL in Manchester 1969, was widely aired at the time and was finally published in the proceedings in 1971. It is a survey of the state of Recursion Theory and its generalizations around 1970. At one level it is a remarkable attempt to influence the development of logic, providing concrete proposals and problems as well as more general reflections on future directions. At times, for example in connection with the theory of admissible sets, the discussion foreshadows developments whose significance was only later fully appreciated. At other points, for example in connection with axiomatics, an approach is promoted which has been explored extensively but has in my view been less successful. (It is of course a positive feature that suggestions can even now be regarded as contentious.) However one can read the paper on another level as providing a case study of the value of generalization in a specific area of mathematics. This reading was the inspiration for the talk which I gave in the Domains IX Workshop at the University of Sussex.

Aims of generalization

The intellectual range of Kreisel's paper is such that any overview is out of the question. However in addition to a wealth of technical observations and suggestions the paper contains a series of compelling reflections on the value of generalization based on the then available experience with Generalized Recursion Theory. Kreisel distinguishes four main aims of the generalization of recursion theory, each loosely associated with specific generalizations. The aims are the following.

(a) *Advancing other parts of logic and mathematics.* Today this seems to be the most successful of the aims. The associated generalizations are those inspired by the model theoretic ideas of explicit and implicit definability. Here one finds Kreisal's key idea, that in certain generalizations one generalizes not only the obvious notions (recursive, semi-recursive) but also the notion of finiteness (typically replaced by a notion of definability). This played out to good effect in what were then the developing theories of admissible sets and of inductive definitions. Generalizations in which the idea of finiteness is itself generalized have worn well. Moreover the basic insight has wider application. Finiteness manifested in compact spaces and proper maps plays an analogous role in algebraic geometry and in various natural formulations of constructive topology.

(b) *Understanding the mathematical character of ordinary recursion theory.* This aim seems to me less successful than the first, but it is hard to explain why. The theories first considered under this heading are higher type recursion theories. Kleene's recursion theory on total functionals of finite type has perhaps become a minority interest notwithstanding recent work of John Longley (Longley 2007). But Kreisel starts with Platek's recursion theory on partial functionals of finite type (Platek 1966) and this is close to being a study of PCF in a now less familiar context. Since continuous (or perhaps better finitely based) models of PCF are an established part of the computer science culture, one can see that this line of enquiry has proved valuable. However it does not seem to have provided much fresh insight into ordinary recursion theory. The same could be said about axiomatic recursion theory which Kreisel considers both in this context and in the context of theories of self-application (see below).

(c) *Analysis of the general concept of computation.* This is a lively area today, and those involved one way or another in the current hot topic of quantum computation might profit from reading Kreisel's cool observations on the general notion computation and its relation to the physical. Alongside considerations of this by now familiar kind, there is a discussion of the idea of extensions of Church's thesis to arbitrary structures. I suppose that the connection is intentional, but the issues of explicit and invariant definability which Kreisel considers may not be so familiar. I think that a pity.

(d) *Other uses.* This is not as it appears a catch-all category. The thought seems to be that there are uses which emerge from reflection on recursion theory in a general sense. Kreisel talks also of incidental uses. He picks out two.

(i) *A formal theory of self application.* Kreisel focuses on logical systems which permit self-application. This general line has been pursued energetically by others though not with outcomes envisaged by Kreisel. Curiously one section on incidental uses closes with a very brief discussion of Scott's then recently introduced D_∞ model of the lambda calculus. Thus Kreisel's paper contains a glimpse of Domain Theory, a topic which has arguably proved more widely influential than anything else in the paper. Kreisel is particularly struck by what he takes to be the genuinely mathematical nature of the theory, the issue addressed in this paper.

(ii) *Axiomatic analysis of kinds of evidence.* This is accompanied by reflections on generalizations of recursion theory to various (typically enumerated) structures. Here I sense a prejudice of the time. Kreisel was himself acutely aware of the issues concerning choice of data which are needed to make sense of and then exploit a constructive point of view. But there is absolutely no encouragement to think about kinds of evidence in that spirit. I am naturally sensitive to this. In (Hyland 1982) I hoped inter alia to promulgate the view that recursive mathematics and in particular recursive analysis has a natural home in the effective topos. The book (van Oosten 2008) contains a more extensive account.

It would not be sensible to write a survey of generalizations of Domain Theory as Kreisel did for Recursion Theory. That would be an enormous task. However I believe that Kreisel's idea to reflect on lessons learnt from generalizations is very pertinent to Domain Theory. When I gave the talk on which this paper is based I envisaged at most this paper. But I have since come to see a number of further issues which are illuminated by recent expereience. So I have taken the risk of announcing this as one of a series. It may help put that in context if I briefly try to match Kreisel's aims with some aims for the generalization of Domain Theory in Computer Science.

(a) *Advancing other parts of computer science.* The primary or original aim of Domain Theory was to provide semantics for higher-order functional programming languages, and some generalizations are driven by that. However the categorical generalizations from which I start are in themselves motivated by another area of Computer Science viz the study of concurrent systems. (Cattani and Winskel 2005) gives both the basic idea and technical developments. So the aim of advancing other parts of the subject is more easily successful than it was for Recursion Theory.

(b) *Understanding the mathematical character of Scott's domain theory.* Ideas about domains were originally presented in terms of topology or of lattices: that is the starting point of the classic paper Scott's Continuous Lattices (Scott 1976). So the mathematical character of Domain Theory seems evident, to the extent that one important focus of research since the early days has been to reduce the mathematical requirements of the theory. (A delightful new approach was presented by Dana Scott at the Domains IX Workshop. I hope that it will be made readily available soon.) However recent experience with generalizations suggests a different way into the traditional theory and its applications. I sketch this unashamedly mathematical point of view in Section 4.

(c) *Analysis of a general notion of approximation.* Domain Theory can be based on a notion of approximation of information. The Information Systems approach to the subject makes that explicit. However generalizations of the notion of domain show clearly that there are a number of interesting notions of approximation. I shall not be pursuing this here.

(d) *Other uses.* Domain Theory has stimulated various incidental lines of enquiry. I mention two.

(i) *Models of the lambda calculus.* The main modern tools are the filter models. The classic papers are (Barendregt et al 1983) and (Coppo et al 1984). See (Berline 2000) for a survey of models of this general kind. Filter models are related to domains, but the connections are not spelt out in the literature. The paper (Hyland et al 2006) claims that Engeler models should not be considered as domains, and initiates a programme to explore the nature of filter models. I hope to return to this.

(ii) *Analysis.* Prima facie Domain Theory deals with issues of approximation, the territory of analysis. There is a tradition of work in this direction (Edalat 1997) under the banner Exact Real Analysis, and Abstract Stone Duality has recently provided another approach: (Bauer and Taylor 2009) contains a detailed account of recent thinking. However the impact of these ideas on the theory and practice of computational or numerical analysis has up to now been disappointing.

There is a good deal of material here which should illuminate the mathematical background to Domain Theory. In this paper I hope to show that one style of generalization leads naturally to new understanding of the basic theory. But I see opportunities to push this moral further.

Generalizing Domains

By a domain, at least in the context of the series of Domains Workshops, one generally means a Scott Domain, that is an algebraic, bounded complete cpo. (Recall if need be that by an accident of terminology a cpo is a directed complete partial order with least element). The category of Scott domains and (Scott) continuous or directed-sup preserving maps is a convenient setting for a considerable range of semantic purposes. Scott domains are closely related to algebraic lattices: any Scott closed subset of an algebraic lattice is a Scott domain, while adding a necessarily open point as top element to a Scott domain gives an algebraic lattice. For reasons of mathematical simplicity I shall concentrate on algebraic lattices. and leave for another occasion the question of extending the ideas to incorporate Scott domains.

The standard notion of Scott domain can be generalized in many ways.

— One can keep the same topological or ordered setting, and extend the objects to be considered. The best known example is the category of SFP domains first introduced by Plotkin in (Plotkin 1976).
— There are notions of domain in which the order is not one of increase of extensional information. Examples are stable domains and strongly stable domains. The idea originates with Berry (Berry 1979). People sometimes talk of refinements of the notion of domain. Linear Logic (Girard 1987) has its roots in a model of this kind. As explained in (Taylor 1990), Stable Domain Theory has interesting connections with the work of Diers (Diers 1980) in categorical algebra.
— There are notions of domain with more than one order, typically both a Scott order and a stable order. This idea also appears in Berry's thesis (Berry 1979), and was then analysed in terms of event structures by Winskel (Winskel 1980). The subject was explored further in (Winskel 1994) and the connection with Linear Logic was spelled out in (Plotkin and Winskel 1994) and then in greater detail in (Curien, Plotkin, Winskel 2000).
— Axiomatic Domain Theory provides axiomatizations of categories of domains. (Fiore, Plotkin, Power 1997) gives a non-standard example of a category satisfying a sensible axiomatization. The subject is related to Synthetic Domain Theory (Hyland 1991; Taylor 1991; Reus and Streicher 1999). (Fiore and Plotkin 1997) gives one angle on the connection. These approaches have to some extent been subsumed by Paul Taylor's Abstract Stone Duality (Taylor 2000; Taylor 2002a; Taylor 2002b), a further generalization.

There is a good deal to be learnt from each of these generalizations. However in this paper I shall consider generalizations in a different spirit from any of the above, the generalization from domains as posets to domains as categories. This is a very natural generalization, and one with a long history. The first person to have taken the idea seriously seems to have been Daniel Lehmann who gave a categorical version of the Smyth power domain (Lehmann 1976). Some time after Lehmann's work and independently of it a number of people looked at generalizations of domains for their own sake. The most systematic was Paul Taylor, who in the late 1980s gave categorical generalizations of much of the material of his thesis (Taylor 1986), and also of much analogous Stable Domain Theory. This work is largely unpublished, but some draft papers are available on Taylor's website. Other work of this period includes (Coquand, Gunther, Winskel 1989) in the stable tradition and (Hyland and Pitts 1989) for the categorification of algebraic lattices.

More recently Glynn Winskel has been promoting the use of categorifications of Domain Theory and in particular the use of presheaf categories in the study of concurrency. There is a tenuous link with Lehmann's original idea. Abramsky (Abramsky 1983) shows that Lehmann's categorified (Smyth) powerdomain is a completion under indexed products, while presheaves are completions under colimits. For an extensive treatment of Winskel's general ideas and some deep results on bisimulation see (Cattani and Winskel 2005). The mathematical background to that paper is explained in the paper (Fiore et al) in preparation, and it is that background which inspires this paper.

Mathematical Understanding

50 years on, Kreisel's sections on the mathematical understanding of Recursion Theory no longer appear as compelling as they once did. But the idea that one plausible aim of generalization is that it can lead to an improvement of mathematical understanding still seems right. In this paper I describe in outline a case study.

What is Domain Theory? Or perhaps better, what are domains and how best to treat them? I intend the questions not in the epistemological sense - how best to teach Domain Theory to beginning computer scientists. Rather I intend the logical or better conceptual sense - how best to see domains within the wider mathematical universe. That is not a new question: in the early days one wondered whether it was best to think of domains as particular kinds of lattice or particular kinds of space. My aim is to address from one particular direction the general question of where (categories of) domains come from.

Outline of the paper

The main section of this paper is Section 4 in which I sketch the approach to Domain Theory which results from reflection on recent experience with categorical generalizations. For simplicity I treat the case of algebraic lattices, but I put that theory in a general categorical context. I treat only traditional issues (cartesian closure and fixed points) and ignore recent developments (differential structure).

I start by giving some categorical background in Section 1. Then by way of a warm-up Section 2 gives a sketch of the main features of the Relational Model for Linear Logic from a categorical point of view. One can regard recent work giving higher dimensional generalizations of Domain Theory as coming from a categorification of the Relational Model, and Section 3 describes some of this work. It is a mere sketch intended simply to make plausible the claim that the new mathematical analysis of Domain Theory itself is inspired by the recent generalizations. Hence I have tried to keep my main Section 4 independent of higher-dimensional category theory. I close the paper with suggestions as to lessons to be learnt.

Acknowledgements

I formulated the material on the Relational Model for the Workshop on Domain Theory which was held to mark Dana Scott's 70th Birthday in Copenhagen, July 2002. I am grateful to the organisers, Lars Birkedal and Giuseppe Rosolini for initial stimulation. Then I tried to place Engeler models in an appropriate mathematical context. I hope soon to return to that and the more general issue of filter models for the lambda calculus, and so complete the project initiated in (Hyland et al 2006); but I here take the opportunity to thank my coauthors on that paper.

I am grateful to Bernhard Reus both generally as organizer of the stimulating Domains IX Workshop, and specifically for suggesting that I use recent experience with higher dimensional models (as represented by (Fiore et al 2008) and (Fiore et al)) as the basis for a talk. This paper is the result of his suggestion. I am further grateful to referees for perceptive comments, useful suggestions, and for their tolerance of a poorly prepared first draft.

Finally I have already drawn attention to Paul Taylor's substantial research on Domain Theory and its categorification, but here I also acknowledge the use of his diagram macros.

1. Categorical background

I am going to give a rational reconstruction of Domain Theory from what is, I believe, an unfamiliar point of view, but one in which I hope to interest those concerned with the theory and applications of domains. As a result I have been exercised as to what to expect from the reader. It seems reasonable to assume that any reader of this paper will know the basics of the classical treatment of Domain Theory if not from original papers then from some textbook such as Winskel's attractive and accessible account (Winskel 1993). So very likely they will know enough Category Theory to understand what is meant by a category of domains being cartesian closed. But it does not seem wise to rely on much more.

I cannot provide all background for beginners in Category Theory. They should find the introductory text (Awodey 2006) helpful: it will get them as far as the treatment of algebra via monads. Mac Lane's book (Mac Lane 1971) remains a standard reference and gives relevant material not only on monads but also on monoidal categories. One can find there also the basic fact that filtered colimits commute with finite limits in the category **Set** of sets and functions. Other material which is needed for the story seems less easily accessible and less familiar and I give some sketches in this section.

1.1. *Distributive laws*

This paper rests in the first place on standard observations about distributive laws between monads. For a fuller account and general background on monads see (Barr and Wells 1984).

Theorem 1.1. Let P and M be monads on a category $\mathbf{C}$. Then the following data are equivalent.

1 A lifting of the monad P to the category M-**Alg** of M-algebras.
2 A distributive law $\lambda : MP \to PM$ in the sense of Beck (Beck 1967).
3 An extension of M to the Kleisli category $\mathbf{Kl}(P)$ of P.

The significance of this theorem is the following. Suppose that one wishes to extend a monad M to a Kleisli category $\mathbf{Kl}(P)$. That seems complicated, but a lift of the monad P to the category M-**Alg** can be found by routine considerations. So one can apply the Theorem.

The equivalence of 1 and 2 above is standard (it appears in (Barr and Wells 1984) for example) and the full result must in some sense be folklore. However its significance does not seem to be well known, and the best I can do by way of a reference is (Hyland et al 2006).

1.2. *Commutative and monoidal monads*

I make use of some facts about commutative monads which are not readily available. Most of what I need is in the original series of papers (Kock 1970; Kock 1971a; Kock 1971b; Kock 1972) by Anders Kock. I am not aware of the material being presented as a whole. A 2-dimensional version of the theory is given in (Hyland and Power 2002). It raises serious questions of coherence, but the order enriched version which I focus on later in the paper is much easier.

Take a symmetric monoidal category $\mathcal{L}$. I shall write $\times$ for the monoidal structure with unit 1, as I only use the notions below in case the base category is cartesian closed. I write t and t^* as in (Hyland and Power 2002) for versions of the strength of a strong monad T on $\mathcal{L}$. A *commutative monad* T on $\mathcal{L}$ is a strong monad such that the diagram

$$\begin{array}{ccccc} TA \times TB & \xrightarrow{t^*} & T(TA \times B) & \xrightarrow{T(t)} & T^2(A \times B) \\ \Big\downarrow t & & & & \Big\downarrow \mu_{A\times B} \\ T(A \times TB) & \xrightarrow[Tt^*]{} & T^2(A \times B) & \xrightarrow[\mu_{A\times B}]{} & T(A \times B) \end{array}$$

commutes. This commutative diagram expresses the property that the algebraic operations of T commute with each other. With T commutative we have a unique choice of natural map $TA \times TB \to T(A \times B)$ and one sees readily that it together with the evident map $1 \to T1$ give T the structure of a monoidal monad. Conversely given a monoidal monad one easily recovers a strength making the monad commutative.

If T is a commutative monad on a symmetric monoidal closed category with equalizers, then one can readily equip the category T-Alg with a (symmetric) closed structure in the sense of Eilenberg and Kelly (Eilenberg and Kelly 1966). (In my view the natural formulation is that T-Alg is a closed symmetric multicategory. The significance of this notion is indicated in (Hyland and Power 2002) but not made fully explicit there.) I briefly explain how to obtain the closed structure. For T-algebras $\mathcal{A} = (a : TA \to A)$ and $\mathcal{B} = (b : TB \to B)$, the function space $[\mathcal{A}, \mathcal{B}]$ is given by T-algebra structure on the equalizer of maps $[a, B]$ and $[TA, b].T_{A,B}$ from $[A, B]$ to $[TA, B]$. It is easy then to check that raising to the power of the algebra $\mathcal{A}$ preserves limits so that in good cases one gets a corresponding symmetric monoidal structure. I need only the simplest case.

Proposition 1.2. If $\mathcal{L}$ is locally finitely presentable, and T is finitary commutative monad then T-Alg is symmetric monoidal closed.

Proof. The category T-Alg is complete and one constructs the closed structure easily from equalisers as above. But T-Alg is also cocomplete (see (Adámek and Rosický 1994) for example) and so one can take coequalizers in T-Alg to get a good universal notion of tensor product. □

Remark One can readily extract from (Adámek and Rosický 1994) the fact that that one has the same result for any accessible monad on a locally presentable category. And one needs much less: $\mathcal{L}$ is finitely complete suffices for the closed structure, while for the tensor product one needs finite colimits and some directed colimits which the commutative monad T preserves. (I do not know sensible minimal conditions for the tensor product.)

As I said above I shall use this theory in the case when the base category is cartesian closed. Generally one has monoidal but not cartesian closed structure on the category T-Alg of T-algebras, but the case when T-Alg is cartesian closed is also part of the story. A commutative monad on a category with products is cartesian closed just when T preserves products in the sense that the evident map $T(A \times B) \to TA \times TB$ is inverse to the monoidal structure $TA \times TB \to T(A \times B)$, and the unique map $T1 \to 1$ is inverse to the monoidal structure $1 \to T1$. The following is very easy.

Proposition 1.3. If T is a cartesian closed monad on a cartesian closed category with equalizers, then T-Alg is itself cartesian closed.

1.3. *Categorical models of linear logic*

I shall give a brief outline of the basic categorical perspective on models of Girard's Linear Logic (Girard 1987). Categorical models are models in the spirit of categorical proof theory. (See (Hyland 2002) for other aspects of the subject.) Early treatments of the notion of categorical model stressing computational aspects are in (Benton et al 1993a) and (Benton et al 1993b). Categorical models are studied further in (Hyland and Schalk 2003), and I use the perspective of that paper here.

Definition 1.4. A *(categorical) model of intuitionistic linear logic* consists of a category $\mathcal{L}$ which (i) is equipped with symmetric monoidal closed structure, (ii) has finite products and (iii) is equipped with a linear exponential comonad $(!, \varepsilon, \delta)$.

I adopt the usual linear logic notation, writing $\otimes$ for the tensor of the monoidal structure and $\multimap$ for the linear function space. Details of the notion of linear exponential comonad can be found in (Hyland and Schalk 2003). The main point of the definition comes from the fact that the Kleisli category $\mathbf{Kl}(!)$ for the comonad ! is cartesian closed. It is worth recalling the proof.

Proposition 1.5. Let $\mathcal{L}$ be a model of intuitionistic linear logic. Then the Kleisli category $\mathcal{C} = \mathbf{Kl}(!)$ is cartesian closed.

Proof. It is clear that products in $\mathcal{L}$ give products in $\mathcal{C}$. The existence of an adjoint to product follows from the following.

$$\begin{aligned}
\mathcal{C}(X \times Y, Z) &= \mathcal{L}(!(X \times Y), Z) \\
&\cong \mathcal{L}(!X \otimes !Y, Z) \\
&\cong \mathcal{L}(!X, !Y \multimap Z) \\
&= \mathcal{C}(X, !Y \multimap Z)
\end{aligned}$$

Thus to get a function space $A \Rightarrow B$ in $\mathcal{C}$ it suffices to take $(A \Rightarrow B) = (!A \multimap B)$. □

The equation $(A \Rightarrow B) = (!A \multimap B)$, often called the Girard translation from its first appearance in (Girard 1987), analyses intuitionistic implication into a combination of a resource modality ! and a linear (resource sensitive) implication. This view is the root of Linear Logic.

1.4. *Linear Logic and Domain Theory*

In view of the enormous impact of Linear Logic in the last twenty years, I want to warn against any too facile use of it in the context of Domain Theory, and with that aim I make the following observations.

Observation 1. *Any cartesian closed category $\mathcal{C}$ can be obtained from a model of classical Linear Logic, by means of the Girard translation.*
For there is a well known $*$-autonomous structure on $\mathcal{L} = \mathcal{C} \times \mathcal{C}^{op}$, see (Hyland and Schalk 2003). On $\mathcal{L}$ there is a simple comonad ! taking (U, X) to $(U, 1)$. (This comonad lies behind modified realizability.) Then since $\mathcal{L}((U, 1), (V.Y)) \cong \mathcal{C}(U, V)$ one sees at once that $\mathbf{Kl}(!)$ is equivalent to $\mathcal{C}$.
Observation 2 *Let* **Dom** *be the category of Scott domains and Scott continuous maps and* **SDom** *the category of Scott domains and strict maps. Then* **SDom** *is a symmetric monoidal closed category and equipped with a lift comonad gives a model of Intuitionistic Linear Logic. The Kleisli category is isomorphic to* **Dom**.
For lifting gives a commutative monoid on the category **Dom** of Scott domains and all objects are uniquely algebras for the monad. (Categorically, the uniqueness comes from the fact that the lifting monad is lax idempotent.) Now **SDom** is the category

of algebras for the monad and so it is symmetric monoidal closed by Section 1.2. The standard adjunction produces a comonad on **SDom** and the identification of the Kleisli category is immediate.

Observation 3 *Let* **Dom** *be the category of Scott domains and Scott continuous maps and* **LDom** *the category of Scott domains and linear maps (that is maps preserving all sups). Then there is a comonad on* **LDom** *with* **Dom** *as Kleisli category. But* **LDom** *is not a model of Linear Logic.*

A Scott domain A has a subposet A_0 of compact elements; and A is the directed completion $D(A_0)$ of A_0. Let $S(A_0)$ be the free completion of A_0 by finite bounded sups. We see that linear maps $D(S(A_0)) \to B$ correspond to Scott continuous maps $A \to B$. So we appear to have a case of the Girard translation. However there is a real problem with the linear function space. For an algebraic lattice A, the space $A \multimap 2$ of sup-preserving maps will be A^{op}. But that need not be a domain. So function spaces do not exist in general, and the Girard translation cannot be used. (For completeness I give a concrete example where A^{op} is not a domain. Take A to be the lattice of open sets of $2^{\mathbb{N}}$, that is of Cantor space. The compact open sets form a basis for the topology so A is algebraic. But in A^{op}, the lattice of closed sets, the points are minimal elements above $\bot$; and they are evidently not compact elements of that lattice.)

There are moral lessons in these observations. The first suggests that one should look critically at any attempt to derive any particular cartesian closed category from a model of Linear Logic. The attempt looks peculiarly idiotic in the case of the cartesian closed category **Set** of sets and functions. The second shows that the mere fact that a cartesian closed category arises from a useful model of Linear Logic does not in itself provide understanding of that category. For with lifting the model of Linear Logic seems to be derived from the category of domains. The third observation is tantalising. In the absence of the Girard translation there is no particular advantage to an approach to Scott continuous maps via linear maps. Rather the existence of **LDom** and of the comonad on it are facts which themselves need explaining.

2. The Relational Model

I start by reviewing some ideas which I first presented at the Workshop on Domain Theory for Dana Scott's 70th Birthday in Copenhagen, July 2002. In the early 1970s it seemed as if Domain Theory had been invented primarily in order to enable one to construct genuinely mathematical models of the pure lambda calculus. But knowing what we do now, Domain Theory is not at all the most natural and obvious place to find such models. Rather the easiest construction takes place using the Relational Model, the fundamental degenerate model for Linear Logic. I give a brief account here.

2.1. *The Model of Linear Logic*

To understand the categorical generalizations it is good to place the Relational Model in a mathematical context.

Multiplicative and additive structure The category **Rel** of sets and relations is the Kleisli category of the power set monad P on the category **Set** of sets and functions. One can identify P-algebras as $\bigvee$-complete lattices. P is a manifestly commutative and so (as mentioned above) a monoidal monad.

In view of Kock (Kock 1971a) and the Remark in Section 1.2, one hopes that the category P-**Alg** of P-algebras has a monoidal structure (which it does) and that it restricts to give a monoidal structure given by product on the Kleisli category **Rel**. With this structure **Rel** is compact closed so $*$-autonomous in a degenerate fashion. Also **Rel** has finite biproducts given by coproduct of sets. (This is inevitable. **Set** and so the Kleisli category **Rel** have products and by an observation of Robin Houston (Housten 2008) products in compact closed categories are automatically biproducts.)

Exponential structure There is a simple way to equip **Rel** with a linear exponential comonad. One takes the multiset monad M on **Set** whose algebras are commutative monoids, extends it to a monad on **Rel** and then uses the (degenerate) duality on **Rel** to obtain a comonad. A brief explanation of the extension is in order. Observe again that P is a commutative monad. Any commutative monad lifts easily to the category of commutative monoids, that is to M-**Alg**. (There is an important reason behind this: the notion of commutative monoid is what we would now usually call operadic as it is the monad generated by a symmetric operad.) It follows from the basic facts about distributive laws, that we get an extension of M to a monad which we still call M on **Rel** $=$ **Kl**(P). (This observation is old. It is a throw away remark in (Eilenberg and Wright 1967), where operadic theories are reasonably enough called linear.) Finally to get the comonad M^* (here it does seem worth avoiding confusion) we dualize using the opposite in the sense of relations: if $r : A \mathrel{-\!\!\!+\!\!\!\to} B$ then $r^* : B \mathrel{-\!\!\!+\!\!\!\to} A$. On objects and maps $M^* = M$, and the comonad structure in **Rel** is the opposite of the monad structure.

2.2. *The Kleisli category*

From a model of linear logic we get a cartesian closed category by taking the Kleisli category of the linear exponential comonad. So we have a category **Kl**(M^*). The maps $r : A \mathrel{-\!\!\bullet\!\!\to} B$ in it are relations $r : M(A) \mathrel{-\!\!\!+\!\!\!\to} B$. One way to think of such a relation is that if xrb then the tokens in the multiset x trigger the token b. Here I use a language suggestive of the idea of a Petri net. Since composition of maps reflects this natural intuition, I like to think of this further Kleisli category as the category of Petri relations and shall write **PRel** $=$ **Kl**(M^*). I give a brief account of properties of **PRel** the category of Petri relations.

Cartesian closure and points

Proposition 2.1. The category **PRel** is cartesian closed: the terminal object is 0, the product of A and B is $A + B$, and the space $B \Rightarrow C$ of functions is $M(B) \times C$.

Proof. Cartesian closure is a direct consequence of the Girard translation 1.5. □

Since $M(0) \cong 1$ in **Set**, the collection $\mathbf{PRel}(0, A)$ of points of A can be identified with the power set $P(A)$. It is easy to check that if $r : MA \relbar\!\!\!\mapstochar\!\!\rightarrow B$ is a map $A \relbar\!\!\bullet\!\!\rightarrow B$ in **Prel** then $r_* = \mathbf{PRel}(0, r) : P(A) \to P(B)$ acts on $x \in P(A)$ as follows.

$$r_*(x) = \{ b \in B \mid \exists \mathbf{a} \in Mx.\, \mathbf{a} r b \}$$

It is immediate that r_* preserves directed sups, that is, it is Scott continuous. All Scott continuous maps occur. If $f : P(A) \to P(B)$ is Scott continuous then there is a unique maximal but many minimal relations $r : MA \relbar\!\!\!\mapstochar\!\!\rightarrow B$ such that $r_* = f$

Proposition 2.2. The category **PRel** does not have enough points.

Proof. Observe that $\mathbf{PRel}(0, 1) \cong P(1) \cong 2$. For finite posets, order-preserving maps are automatically Scott continuous, and there are just three order-preserving maps $2 \to 2$. On the other hand $M(1) \cong \mathbb{N}$ in **Set**. Thus $\mathbf{PRel}(1, 1) \cong \mathbf{Rel}(\mathbb{N}, 1) \cong P(\mathbb{N})$ is infinite. □

The fact that a category does not have enough points causes unnecessary anxiety to many experts in lambda calculus. I shall try to dispel these concerns in a companion paper.

Fixed points I give a crude hands on account of fixed points of maps. We are given a parametrised map $A + C \relbar\!\!\bullet\!\!\rightarrow A$ in **PRel** and seek a parametrized fixed point $C \relbar\!\!\bullet\!\!\rightarrow A$. That is given $M(A) \times M(C) \relbar\!\!\!\mapstochar\!\!\rightarrow A$ we seek a suitable $M(C) \relbar\!\!\!\mapstochar\!\!\rightarrow A$. This will be determined elementwise over $M(C)$ and so it suffices to take $f : M(A) \relbar\!\!\!\mapstochar\!\!\rightarrow A$ and find a fixed point $x : 1 \relbar\!\!\!\mapstochar\!\!\rightarrow A$. But $\mathbf{PRel(0, f)} : \mathbf{PRel}(0, A) \to \mathbf{PRel}(0, A)$ is a Scott continuous map $P(A) \to P(A)$ on the power set of A and it suffices to take the least fixed point. The standard good properties of this fixed point are immediate.

Proposition 2.3. The category **PRel** has parametrized fixed points.

For programming language semantics an even more significant feature of Domain Theory is the possibility of finding fixed points for a wide range of operators on domains themselves. For Scott's 70th birthday I illustrated this for the Relational Model by describing how to find models of the lambda calculus in $\mathbf{Kl}(\hat{M})$. The function space $A \Rightarrow B$ is given by $M(A) \times B$. So the task is to find a set D with a retraction to $M(D) \times D$ in $\mathbf{Kl}(\hat{M})$. Now any injective map $C \to D$ in **Set** induces an evident retraction from D to C in **Rel** and hence a fortiori in $\mathbf{Kl}(\hat{M})$. But the problem in **Set** is trivial: for any countable set D, the set $M(D) \times D$ is countable so one easily obtains models of the lambda calculus, indeed models of the $\beta\eta$ calculus if one takes a bijection. Of course this crude approach gives no control over the equalities between lambda terms in the model, but one can remedy that. For example one has the obvious analogue of Engeler models (Engeler 1981). Old tools from (Hyland 1976) can be readily adapted to analyze the equality in the naturally occuring examples. I hope to explain that in a further paper.

Conclusions It is clear that the Relational Model **Rel** and its induced cartesian closed closed category **PRel** have very rich structure. In particular **PRel** has all the structure which one usually requires for basic domain theoretic programming semantics. One should not flinch at its failing to have enough points. (Indeed Game Semantics makes a virtue of this aspect of its modelling of programme languages.) However, notwithstanding the fact that the action of maps in **PRel** on points is Scott continuous, what one has in the Relational Model is not a form of Domain Theory. Perhaps more significantly it does not look at all like a substitute for Domain Theory: the simple explanation of an information ordering in Domain Theory is lacking. It may be hard to pin down what all that amounts to, but it seems that there must be more to Domain Theory than simply providing cartesian closed categories with good fixed point properties.

3. Kleisli Bicategories

In this section I sketch material from (Fiore et al) sufficient to provide a background to recent generalizations of Domain Theory, and in particular to Winskel's approach (Cattani and Winskel 2005) to concurrency. One angle on this material is that it is a generalization to the 2-dimensional level of the analysis just given of the Relational Model.

3.1. *Profunctors*

The paper (Fiore et al) introduces the notion of a Kleisli structure. This is essentially a slight generalization of the 2-dimensional notion of pseudo-monad, which is sensitive to issues of size. The most familiar example of a Kleisli structure is the *presheaf Kleisli structure* arising from the presheaf construction. The data for this Kleisli structure is as follows.

— One takes $\mathbf{Cat} \hookrightarrow \mathbf{CAT}$, the inclusion of the 2-category of small categories into that of locally small categories. To a small category A, assign $\mathcal{P}(A) =_{\text{def}} [A^{op}, \mathbf{Set}]$, the locally small category of presheaves over A; and for each A take the usual Yoneda embedding $y_A : A \to \mathcal{P}(A)$.
— For each functor $f : A \to \mathcal{P}(B)$, take $f^\sharp : \mathcal{P}(A) \to \mathcal{P}(B)$ to be its left Kan extension along the Yoneda embedding (Mac Lane 1971). This data is structured by families of invertible 2-cells

$$\eta_f : f \to f^\sharp\, y_A \qquad \kappa_A : (y_A)^\sharp \to 1_{TA} \qquad \kappa_{g,f} : (g^\sharp\, f)^\sharp \to g^\sharp\, f^\sharp$$

which I do not make explicit here. The 2-cells satisfy natural unit and pentagon coherence conditions as explained in (Fiore et al).

The existence of this Kleisli structure follows from the classic identification of the presheaf category $\mathcal{P}(A)$ as the free closure of A under small colimits (Im and Kelly 1986). This also is explained in (Fiore et al).

The notion of Kleisli structure is a variant of the notion of monad for which the Kleisli construction of a bicategory of free algebras makes immediate sense. The Kleisli bicategory $\mathbf{Kl}(\mathcal{P})$ obtained from the presheaf Kleisli structure $\mathcal{P}$ can be identified with the standard bicategory **Prof** of profunctors. The 0-cells are the small categories. For small categories A and B the Kleisli construction gives

$$\mathbf{Kl}(\mathcal{P})[A, B] = \mathbf{CAT}[A, \mathcal{P}(B)]$$

and since $\mathcal{P}(B) =_{\text{def}} [B^{op}, \mathbf{Set}]$, this is isomorphic by exponential transpose to

$$\mathbf{Prof}[A, B] = \mathbf{CAT}[B^{op} \times A, \mathbf{Set}]\,.$$

A profunctor from A to B is with these conventions a functor $f : B^{op} \times A \to \mathbf{Set}$, which I write $f(b, a)$. There is an evident analogy with the relational model where elements of $\mathbf{Rel}(A, B)$ can be identified with maps $f : B \times A \to 2$. And one should think of $\mathbf{Prof}[A, B]$ as the category of set-valued relations between the categories A and B.

The identity profunctor from A to A is the hom-functor $A(-, -) : A^{op} \times A \to \mathbf{Set}$, the exponential transpose of the Yoneda embedding y_A. Composition of profunctors is given by a coend formula: for profunctors $f : B^{op} \times A \to \mathbf{Set}$ and $g : C^{op} \times B \to \mathbf{Set}$, the composite profunctor $g \cdot f : C^{op} \times A \to \mathbf{Set}$ is given by the formula

$$(g \cdot f)(c, a) = \int^{b} g(c, b) \times f(b, a)\,.$$

This identifies **Prof** with $\mathbf{Kl}(\mathcal{P})$ the Kleisli bictategory of the presheaf Kleisli structure, and in particular gives a full proof of the basic fact.

Theorem 3.1. Prof carries the structure of a bicategory.

It is a claim of (Fiore et al) that the approach via Kleisli structures is a good way to establish the existence of this leading example of a bicategory.

3.2. *Distributivity*

The paper (Fiore et al) provides a theory of liftings and extensions in the bicategorical context which is a kind of categorification of that explained in Section 1.1. The generalizations of Domain Theory exploited by Winskel stem from special cases of this general theory, and I give a brief overview of them.

3.2.1. *Filtered and finite colimits* In a suitable sense $\mathcal{P}(A)$ is the closure of the small category A under small colimits. Another classic construction is the closure of a small category under filtered colimits, often called the Ind-completion. Again this gives a Kleisli structure on $\mathbf{Cat} \hookrightarrow \mathbf{CAT}$. To a small category A, now assign $\mathcal{D}(A)$, which one can take to be the full subcategory of $\mathcal{P}(A)$ on filtered colimits of representables. The Yoneda embedding factors through the inclusion of $\mathcal{D}(A)$ in $\mathcal{P}(A)$ to give a Yoneda embedding $y_A : A \to \mathcal{D}(A)$. Also $\mathcal{D}(B)$ is *Ind-complete*, in the sense that it is closed under filtered colimits. Hence for $f : A \to \mathcal{D}(B)$, one can take $f^\sharp : \mathcal{D}(A) \to \mathcal{D}(B)$ to be a standard choice of left Kan extension of f along the Yoneda embedding. Again there is 2-dimensional structure to explain, but that restricts from the structure in the preseaf Kleisli structure in a straightforward fashion. The result is the *Ind-completion Kleisli structure*.

Alongside the Ind-completion Kleisli structure it is natural to consider also the 2-monad $\mathcal{C}$ for categories with finite colimits. (As a 2-monad on **CAT** it restricts to a 2-monad on **Cat**.) One can freely add all colimits to a small category by adding first finite colimits and then filtered colimits. So it is natural to suppose that the presheaf

Kleisli structure $\mathcal{P}$ is the composite $\mathcal{DC}$. So one hopes that the Kleisli structure $\mathcal{D}$ lifts to C-Alg or equivalently that the 2-monad C extends to the Kleisli category $\mathbf{Kl}(\mathcal{D})$.

Arguments establishing the hoped for distributivity are already in the original treatment (Grothendieck, Artin, Verdier 1972). An accessible account is in (Johnstone 1982), and I give a quick sketch. First, if $\mathbf{A}$ is a category with finite colimits, then $D(\mathbf{A})$ has finite colimits (and so is cocomplete), and moreover the Yoneda functor $\mathbf{A} \to \mathcal{D}(\mathbf{A})$ preserves finite colimits. Further if $f : \mathbf{A} \to \mathcal{X}$ is a colimit preserving functor to a cocomplete category $\mathcal{X}$ then the left Kan extension $f^\sharp : \mathcal{D}(\mathbf{A}) \to \mathcal{X}$ also preserves them. This shows that $\mathcal{D}$ lifts to $\mathcal{C}$-Alg, and so inter alia gives us a composite Kleisli structure $\mathcal{DC}$. The identification of that with the presheaf structure $\mathcal{P}$ is straightforward.

There is another aspect of the Ind-completion which is important for Domain Theory. Filtered colimits are exactly the small colimits which commute with all finite limits in the category **Set** of sets. Again this classic result goes back to the preliminary generalities in SGA 4 (Grothendieck, Artin, Verdier 1972). It is treated fully in Mac Lane (Mac Lane 1971). It follows from this characterization that in case a category A has finite colimits, the Ind-completion $\mathcal{D}(A)$ can be identified as the full subcategory $\mathbf{Lex}(A^{op}, \mathbf{Set})$ of the presheaf category $\mathcal{P}(A)$ consisting of the finite limit preserving functors.

3.2.2. *Symmetric monoidal categories* The presheaf Kleisli structure is a 2-dimensional analogue of the power set monad on **Set**, and the corresponding Kleisli bicategory **Prof** is an analogue of the Kleisli category **Rel**. There are many monads which extend from **Cat** to **Prof**, but the closest analogue to the multiset monad which we extended from **Set** to **Rel** is the 2-monad Σ for symmetric (strict) monoidal categories. The multiset monad is the monad for commutative monoids, and symmetric monoidal categories are weak commutative monoids in the 2-category of categories. (As an aside, I mention that this is more than a simple categorification. The set of finite multisets on a set A can be obtained by taking the free symmetric monoidal category on A as a discrete category, and then taking its set of connected components. This is the basis for an illuminating approach to finitary polynomial functors.) The extension of the 2-monad Σ for symmetric monoidal categories to **Prof** follows from the lifting of the Kleisli structure $\mathcal{P}$ to Σ-Alg. The fundamental technical fact is the lifting to monoidal categories using the Day tensor product. This is the essential content of the paper (Im and Kelly 1986). (Of course the case of symmetric monoidal categories follows easily as symmetry is inherited by the Day tensor product.)

The extension of Σ to **Prof** is a pseudo-monad, and just as in the case of M on **Rel** one can dualise now to get a pseudo-comonad $\Sigma^\star$ on **Prof**. As explained in (Fiore et al 2008), the Kleisli bicategory **Esp** (for 'espèces de structures') of this pseudo-comonad is the bicategory in which Joyal's theory of species has a natural home: the category of general species, without finiteness restriction, is $\mathbf{Esp}(1, 1)$ where 1 is the terminal category.

Cartesian closure and points **PRel** is cartesian closed by a direct application of the Girard translation of Linear Logic. To handle the bicategorical situation one needs a

2-dimensional version of that, but the general theory is not exciting and the result can be extracted easily enough. The following fact is explained in detail in (Fiore et al 2008).

Proposition 3.2. The bicategory **Esp** is cartesian closed.

The question of points in the bicategory **Esp** is curious, and the difficulty does not parallel that in **PRel**.

Proposition 3.3. The bicategory **Esp** does not generally have enough points. However if one restricts the objects from small categories to groupoids one does have enough points.

Proof. A single non-invertible arrow creates a problem for points. But if one deals only with groupoids, the maps are the determined by analytic functors in the obvious sense extending the notion in (Joyal 1981). □

Fixed points The situation is very similar to that for **PRel**, with easy 2-dimensional modifications. It has been considered by Ryu Hasegawa (Hasegawa 2002).

Proposition 3.4. The bicategory **Esp** has parametrized fixed points (in the obvious up-to-isomorphism sense).

Hasegawa (Hasegawa 2002) makes a connection between these fixed points and the bicategorical trace on **Prof** arising from the compact closed structure, but his analysis of the Lagrange-Good inversion formula is not well understood.

In addition one can solve a wide range of domain equations within **Esp**. For straightforward (functorial) domain equations this is also considered in (Hasegawa 2002). But one can do more and as was the case for **PRel** the situation is very similar to that for Domain Theory.

3.2.3. *Categories with finite limits* Finally in this section I describe the example which when specialised from categories to posets provides the approach to Domain Theory of the next section.

Let L be the 2-monad on **CAT** restricting to **Cat** for categories with finite limits. The Kleisli structure $\mathcal{P}$ lifts to the 2-category L-Alg of L-algebras by reason of the following observations.

— Any presheaf category $\mathcal{P}(A)$ has finite limits, and if A has finite limits then the Yoneda $y_A : A \to \mathcal{P}(A)$ preserves finite limits.
— If A and B have finite limits and the functor $f : A \to \mathcal{P}(B)$ preserves finite limits, then the left Kan extension $f^\sharp : \mathcal{P}(A) \to \mathcal{P}(B)$ preserves finite limits.

This immediately gives a lift of the presheaf Kleisli structure $\mathcal{P}$ to L-Alg and hence an extension of L to a pseudo-monad on **Prof**.

Again one can dualise to get a pseudo-comonad L^* on **Prof**. It does not seem worth naming the resulting Kleisli bicategory $\mathbf{Kl}(L^*)$, though an order-enriched version of it plays a big role in the next section.

Cartesian closure and points As before, using a 2-dimensional version of the Girard translation, one readily gets cartesian closure.

Proposition 3.5. The bicategory $\mathbf{Kl}(L^*)$ is cartesian closed.

The question of points in the bicategory $\mathbf{Kl}(L^*)$ is relatively straighforward. A profunctor $L^*(A) \to\!\!\!\!\!+ B$ corresponds to a functor $L^*(A) \to \mathcal{P}(B)$. But $L^*(A)$ is the closure of A under finite colimits. So by Section 3.2.1, such a functor corresponds to a filtered colimit preserving functor $\mathcal{P}(A) \to \mathcal{P}(B)$. That gives the action on points of a map in $\mathbf{Kl}(L^*)$, and the following is then obvious

Proposition 3.6. The bicategory $\mathbf{Kl}(L^*)$ does have enough points.

Fixed points For fixed points the situation is very similar to that for **PRel** and **Esp**.

Proposition 3.7. The bicategory $\mathbf{Kl}(L^*)$ has parametrized fixed points (in the obvious up-to-isomorphism sense).

Again one can also solve a wide range of domain equations within $\mathbf{Kl}(L^*)$ and the situation is similar to that for Domain Theory.

4. Ordered sets

The background to the generalized domain theory sketched in the previous section requires some non-trivial higher dimensional category theory, and will not be to the taste of all readers. However restricting back from categories to posets gives a version which can be understood at the level of ordinary categories. In this section I present this version. For this paper one can think of what results as using experience of a generalization to provide a fresh approach to traditional Domain Theory. This in its turn should inform developments in the higher dimensional setting, but that is for another time.

4.1. *Posets and Linear Systems*

We start with the category **Pos** of posets and order preserving maps. For X a poset, let $P(X) = [X^{op}, 2]$ be the poset of down-closed subsets of X ordered by inclusion. We have a Yoneda embedding $y : X \to P(X)$ in **Pos** given by

$$y(a) = {\downarrow}(a) = \{b \mid b \leq a\}\,.$$

$P(X)$ is the free $\bigvee$-complete poset generated by the poset X in the obvious sense that any functor $f : X \to C$ where C is $\bigvee$-complete factors uniquely through the Yoneda as

$$X \xrightarrow{\;y_X\;} P(X) \xrightarrow{\;f^\sharp\;} C$$

where the left Kan extension $f^\sharp$ is defined for $r \in P(X)$ by

$$f^\sharp(r) = \bigvee\{f(a) | a \in r\}\,.$$

It follows that P has the structure of a monad on **Pos** with unit the Yoneda embedding $y : X \to P(X)$, and multiplication $P(P(X)) \to P(X)$ given by union. This corresponds to the presheaf Kleisli structure of the previous section.

Now consider what corresponds to the bicategory of profunctors. It will be the Kleisli category of P. I adopt Glynn Winskel's nomenclature as for example in (Winskel and Zappa Nardelli 2004) and call this category **Lin** the category of linear systems. A map from X to Y in **Lin** is a functor $X \to P(Y)$ in **Pos** and so can represented via the transpose $X \times Y^{op} \to 2$ as a relation $r : X \nrightarrow Y$ such that

$$a' \geq a \text{ and } a\, r\, b \text{ and } b \geq b' \text{ implies } a'\, r\, b' .$$

Here for clarity I write r as an infix relation.

4.2. *Directed and finite sups*

Consider the analogue of the decomposition of the presheaf Kleisli structure. Let D be the monad for directed completion and S the monad for $\vee$-semilattices (that is for finite sup completion). The following observations are all easy. First if A is a $\vee$-semilattice then so is $D(A)$ (so in fact $D(A)$ is $\bigvee$-complete). Secondly the Yoneda embedding $A \to D(A)$ preserves finite sups. Finally if $f : A \to L$ is a finite sup preserving map from a $\vee$-semilattice A to a suplattice L, then the left Kan extension $D(A) \to L$ also preserves finite sups.

It follows that the monad D on **Pos** lifts to one on S-Alg the category of $\vee$-semilattices and so S extends from **Pos** to the Kleisli category **Kl**(D). There is a distributive law $SD \to DS$ and DS is a monad which one can readily identify with the monad P from the previous section.

4.3. *Extending a monad*

I turn now to the question of when one can extend a monad on **Pos** to one on **Lin**. With the simplest form of domain theory in view I concentrate on the monad M for meet semi-lattices. As a functor M is given by the collection of finitely generated up-closed subsets, ordered by $\supseteq$, that is by the opposite of inclusion. A routine simplification of the arguments of (Fiore et al) shows that this monad extends. For the benefit of those ill at ease with the relevant category theory, it seems worth going through what that amounts to in this easy special case.

By the general theory of distributive laws to give an extension of M to the Kleisli category **Lin** is to give a lifting of the monad P to the category M-Alg of M-algebras or meet semilattices. I follow the line of analysis of (Fiore et al) and then explain why it follows that one does indeed get the required lifting. The analysis involves the following points.

1 If X is a $\wedge$-semilattice then so is $P(X)$. This is evident: as $P(X)$ is $\bigvee$-complete it is $\bigwedge$-complete and so $\wedge$-complete. Moreover if X is a $\wedge$-semilattice then the Yoneda $y : X \to P(X)$ preserves the $\wedge$-semilattice structure. This is a straighforward check

$$y(T) = \downarrow(T) = X = T \quad \text{the top element in } PX$$

$$y(a \wedge b) = \downarrow (a \wedge b) = \downarrow (a) \cap \downarrow (b) = y(a) \cap y(b) \quad \text{in } PX$$

2 For X and Y $\wedge$-semilattices and $f : X \to P(Y)$ a map of $\wedge$-semilattices, the left Kan extension $f^\sharp : P(X) \to P(Y)$ is also a map of $\wedge$-semilattices. Recall that for $r \in P(X)$ one has $f^\sharp(r) = \bigcup\{f(a)|a \in r\}$. To see that $f^\sharp$ preserves $\top$ observe that

$$f^\sharp(\top) = f^\sharp(X) = \bigcup\{f(a)|a \in X\} = f(\top) = \top$$

To see that $f^\sharp$ preserves $\cap$ observe that

$$\begin{aligned} f^\sharp(r) \cap f^\sharp(s) &= \bigcup\{f(a)|a \in r\} \cap \bigcup\{f(b)|b \in s\} \\ &= \bigcup\{f(a) \cap f(b)|a \in r,\ b \in s\} \\ &= \bigvee\{f(a \cap b)|a \in r,\ b \in s\} \\ &= \bigvee\{f(c)|c \in r \cap s\} \\ &= f^\sharp(r \cap s)\,. \end{aligned}$$

These points are enough to provide a lift of P to a monad $\tilde{P}$ say on M-Alg. The action of $\tilde{P}$ on objects is given by point 1 above: we take A to $P(A)$. We get the action on maps $f : A \to B$ as the left Kan extension $(y_B.f)^\sharp$, which is an M-algebra map by point 2. The unit $y : A \to P(A)$ is an M-algebra map by point 1, and the naturality of that is immediate. The multiplication of the monad is given as the left Kan extension $(1_{PA})^\sharp$ of the identity, and again that is an M-algebra map by point 2.

4.4. *From monad to comonad*

In the previous section I explained why the monad M for $\wedge$-semilattices extends from **Pos** to **Lin**. Now **Lin** is compact closed: the tensor product is given by product of posets and the dual of a poset A is A^{op}, the opposite poset with the order reversed. Hence we can dualise the monad M on **Lin** to give a comonad $S = M^*$ which is defined as a functor by

$$S(A) = M^*(A) = (M(A^{op}))^{op}\,.$$

On objects $S(A)$ is the free $\vee$-semilattice on A.

Each $M(A)$ is a commutative monoid in **Pos** and so in **Lin**. Also maps $M(A) \to M(B)$ of free M-algebras in **Pos** preserve the monoid structure, and (there is something to check here) this extends to **Lin**. It follows that $S(A)$ is a commutative comonoid in **Lin** and that maps $S(A) \to S(B)$ of free S-algebras in **Lin** preserve the comonoid structure. Finally one can show that S is a monoidal comonoid in **Lin**. Thus S is a linear exponential comonad and **Lin** is a model for Linear Logic.

4.5. *The Kleisli category for the comonad*

One can identify the Kleisli category $\mathbf{Kl}(S)$ for the comonad S as a category of domains as follows. By definition the objects are posets and the maps $A \multimap\!\!\!\!\bullet\ B$ are maps of linear systems $SA \mathrel{-\!\!\!+\!\!\!\to} B$, that is maps of posets $SA \to PB$. Now consider that $PB \cong DSB$, so one has maps of posets $SA \to DSB$ and so directed sup preserving (that is Scott continuous) maps $DSA \to DSB$ or equivalently $PA \to PB$.

The whole discussion of this section is based on the category **Pos**: everything is at least enriched there. In this context it seems appropriate to say that an algebraic lattice is *free* just if it is of the form PA for a poset A, that is when it is a free $\bigvee$-lattice on a poset. Equivalently an algebraic lattice is free just if it is the result of taking a poset A, forming the free $\vee$-semilattice SA, and then taking the directed completion DSA. Thus one can identify $\mathbf{Kl}(S)$ with the category **FAlglatt** of free algebraic lattices.

Since one has a model for Linear Logic, the discussion in section 1 applies, and one derives the following.

Proposition 4.1. The Kleisli category $\mathbf{Kl}(S)$, equivalently the category **FAlglatt** of free algebraic lattices and Scott continuous maps, is cartesian closed.

This is all one gets by simple application of ideas coming from Linear Logic. To get a handle on Domain Theory one needs to go further and explain the cartesian closure of the category **Alglatt**, of all algebraic lattices and Scott continuous maps.

4.6. *Cpos*

To explain domains one has to go beyond the category of free algebraic lattices. But it is easy to go too far, and instructive to see why. Let **Dcpo** be the category of directed complete partial orders and Scott continuous maps. Then **Dcpo** is clearly the category D-Alg of algebras for the directed completion monad D. Now D is a commutative monad. (This holds either concretely because directed limits commute with directed limits, or else as D is lax idempotent and so commutative by an easy special case of section 6.7 of (Lopez Franco 2008).) But D is very special, indeed it is the main example in Kock (Kock 1971b) of a cartesian closed monad, that is, one in which the evident map

$$D(A \times B) \to DA \times DB$$

is inverse to the monoidal structure

$$DA \times DB \to D(A \times B)\,.$$

So applying the main result of (Kock 1971b) we get the following.

Proposition 4.2. The category **Dcpo**, of directed complete partial orders and Scott continuous maps, is cartesian closed.

This result is too easy. As a result it seems to me unhelpful to think of the cartesian closure of **Alglatt** as a matter of finding some cartesian closed subcategory of **Dcpo**.

4.7. *Algebraic Lattices*

It is now time to understand the category **Alglatt** in terms of the theory developed above. I first take another look at the category **FAlglatt** of free algebraic lattices. There are the following equivalent perspectives.

— The Kleisli category of the comonad S on **Lin**. The objects are posets A, B, C, say, and the maps $A \to B$ are maps $SA \twoheadrightarrow B$ in **Lin** that is, maps $SA \to PB$ in **Pos**.
— The category of free D-algebras on the underlying posets of free $\vee$-semilattices. Objects are free $\vee$-semilattices. SA, SB, SC, say, and maps $SA \to SB$ are maps $DSA \to DSB$ in **Dcpo** that is poset maps $SA \to DSB$.

Following the second view, the category **Alglatt** can be presented as follows. The objects are (arbitrary) $\vee$-semilattices $\mathcal{A} = (SA \to A)$, $\mathcal{B} = (SB \to B)$, $\mathcal{C} = (SC \to C)$, say, and maps $\mathcal{A}$ to $\mathcal{B}$ are D-algebra maps from the free D-algebra on the underlying poset A to that on B. I seek now to explain why **Alglatt** is cartesian closed.

Products in **Alglatt** are straightforward. The category S-Alg of $\vee$-semilattices is complete, indeed the forgetful functor creates limits. So in particular there is a product $\mathcal{A}\times\mathcal{B}$ given by the evident composite $S(A \times B) \to SA \times SB \to A \times B$. Then one checks

$$\begin{aligned}\mathbf{Alglatt}(\mathcal{C}, \mathcal{A}\times\mathcal{B}) &\cong D\text{-Alg}(DC, D(A\times B))\\ &\cong D\text{-Alg}(DC, DA\times DB)\\ &\cong D\text{-Alg}(DC, DA)\times D\text{-Alg}(DC, DB)\\ &\cong \mathbf{Alglatt}(\mathcal{C},\mathcal{A})\times\mathbf{Alglatt}(\mathcal{C},\mathcal{B})\,.\end{aligned}$$

Thus at the level of the generating $\vee$-semilattices products are given by products in S-Alg.

The question of function spaces in the category of algebraic lattices is very much more subtle. I start by explaining the issue. Suppose again that $\mathcal{A} = (a : SA \to A)$, $\mathcal{B} = (b : SB \to B)$ and $\mathcal{C} = (c : SC \to C)$ are $\vee$-semilattices. Then

$$\begin{aligned}\mathbf{Alglatt}(\mathcal{A}\times\mathcal{B}, \mathcal{C}) &\cong D\text{-Alg}(D(A\times B), DC)\\ &\cong \mathbf{Pos}(A\times B, DC)\\ &\cong \mathbf{Pos}(A, B\Rightarrow DC)\end{aligned}$$

To continue one clearly needs to show that $B \Rightarrow DC$ is of the form $D(E(B, C))$ for some $\vee$-semilattice $E(B, C)$. Now why is that? The first thing to note is that in case

$$\mathcal{C} = FX = (\mu_X : S^2X \to SX)$$

is a free $\vee$-semilattice FX on a poset X, then the matter is essentially no different from the construction in the category of all such **FAlglatt**. For

$$(B \Rightarrow DSX) \cong (B \Rightarrow PX) \cong P(B^{op}\times X) \cong DS(B^{op}\times X)\,.$$

So for $\mathcal{C} = FX$,

$$\begin{aligned}\mathbf{Alglatt}(\mathcal{A}\times\mathcal{B}, \mathcal{C}) &\cong \mathbf{Pos}(A, DS(B^{op}\times X))\\ &\cong D\text{-Alg}(DA, DS(B^{op}\times X))\\ &\cong \mathbf{Alglatt}(\mathcal{A}, \mathcal{B}\Rightarrow\mathcal{C})\end{aligned}$$

where $\mathcal{B} \Rightarrow \mathcal{C} = F(B^{op}\times X)$ is itself a free algebraic lattice. Note that the traditional approach to function spaces via step functions hides the fact that the space of functions into a free algebraic lattice is free. Though it is straightforward to confirm the observation by direct calculation, I do not recall it from the literature. So even at this point one has maybe gained some mathematical understanding.

To treat the space of functions $\mathcal{B} \Rightarrow \mathcal{C}$ for a general $\mathcal{C}$ requires more work. The argument just given depends essentially on the fact that $PX \cong (X^{op} \Rightarrow 2)$ in **Pos** so that

$$(B \Rightarrow PX) \cong (B \Rightarrow (X^{op} \Rightarrow 2)) \cong ((B \times X^{op}) \Rightarrow 2) \cong P(B^{op} \times X)\,.$$

So I need a substitute for that fact.

The substitute depends on the fact that the categories of $\wedge$-semilattices and of $\vee$-semilattices are symmetric monoidal closed. This is immediate by Proposition 1.2, but is also very easy to prove directly. Now consider a special case of the characterization of the Ind-completion. For C a $\vee$-semilattice the directed completion can be identified with $C^{op} \multimap 2$ the (complete) poset of $\wedge$-semilattice maps from C^{op} to 2. (Here $\multimap$ is the closed structure in $\wedge$-semilattices.) There are isomorphisms

$$\begin{aligned} B \Rightarrow (C^{op} \multimap 2) &\cong MB \multimap (C^{op} \multimap 2) \\ &\cong MB \otimes C^{op} \multimap 2\,. \end{aligned}$$

Now $(MB \otimes C^{op})^{op} \cong SB^{op} \otimes C$ where context determines that on the left hand side the tensor is in $\wedge$-semilattices while on the right hand side it is on $\vee$-semilattices. Thus for general algebraic lattices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ there is a natural isomorphism

$$\mathbf{Alglatt}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \quad\cong\quad \mathbf{Alglatt}(\mathcal{A}, \mathcal{B} \Rightarrow \mathcal{C})$$

where $(\mathcal{B} \Rightarrow \mathcal{C}) = D(S(B^{op}) \otimes C)$ is determined as an algebraic lattice by its finite elements $(S(B^{op}) \otimes C)$.

This explanation of the cartesian closure of **Alglatt** may appear to give an explanation of the function space $\mathcal{B} \Rightarrow \mathcal{C}$ which is less concrete than the usual one in terms of finite sups of step functions. But that is illusory. One has to work to get a concrete handle on the formal relation between finite sups of step functions. And the description of the finite elements of $\mathcal{B} \Rightarrow \mathcal{C}$ as $SB^{op} \otimes C$ readily gives a description by generators and relations, in the sense of order enriched algebra. I explain this in the following remarks

Remark 1 If X is a poset then SX is isomorphic to the $\vee$-semilattice of finitely generated downsets of X, that is, to the finitary elements of PX. A presentation can be given as follows. For a and b arbitrary finite subest of X define a preorder $a \leq b$ by

$$a \leq b \quad \text{if and only if} \quad \forall x \in a.\, \exists y \in b\,.x \leq y$$

In terms of generators and relations, take elements $x \in X$ and form terms in $\perp$ and $\vee$ subject to the laws for a commutative idempotent monoid together with $x \vee y = y$ whenever $x \leq y$.

Remark 2 If A and B are $\vee$-semilattices then $A \otimes B$ is generated as a $\vee$-semilattice from the product poset $A \times B$ with elements $a \otimes b$ subject to

$$(a \vee a') \otimes b = a \otimes b \vee a' \otimes b \qquad \perp \otimes b = \perp$$

$$a \otimes (b \vee b') = a \otimes b \vee a \otimes b' \qquad a \otimes \perp = \perp$$

as additional equations.

Remark 3 If X is a poset and B a $\vee$-semilattice, then the presentation of $S(X) \otimes B$ obtained by the previous remarks can be simplified. It is the generated as a $\vee$-semilattice from the poset $X \times B$ with elements $x \otimes b$ subject to

$$x \otimes (b \vee b') = x \otimes b \vee x \otimes b' \qquad x \otimes \bot = \bot$$

as additional equations.
Remark 4 If one applies Remark 3 to the case of $S(B^{op}) \otimes C$ one gets a presentation based on the poset $B^{op} \times C$ whose elements I now write as $[b, c]$ with

$$[b, (c \vee c')] = [b, c] \vee [b, c'] \qquad [b, \bot] = \bot$$

as additional equations. These are exactly the relations between the basic step functions

$$[b, c] = \begin{cases} c & \text{if } a \geq b, \\ \bot & \text{otherwise,} \end{cases}$$

which appear in the standard treatments.

5. Conclusion

In this paper I have concentrated on algebraic lattices, because the story for them is straightforward and immediately appealing. It is possible to treat Scott domains in a similar spirit, but I leave that for another occasion. What I am keen to stress here is a simple story. The starting point, the relational model of Linear Logic, is very natural. Generalizing from sets to categories gives a range of models based on the bicategory of profunctors. Specializing one of these from categories to posets gives a model leading to the cartesian closed category of free algebraic lattices. One uses absolutely basic abstract mathematics to move from that to the cartesian closed category of algebraic lattices.

Some proponents of Stable Domain Theory and its many relations castigate the theory of Scott domains as somehow trivial. I suppose that a number of thoughts lie behind that, most obviously the following.

— Scott domains are a rather boring kind of topological space.
— The properties of Scott domains, including the cartesian closure of the category are mathematically uninteresting.
— There is no useful connection between Scott domains and Linear Logic.

The abstract approach in this paper does I believe provide responses at various levels.

— Algebraic lattices are only accidentally spaces. They naturally arise by specializing ideas from the theory of Ind-completion to the poset case.
— That the category of algebraic lattices is cartesian closed depends on some elegant abstract mathematics.
— There is a connection with Linear Logic, but considerations beyond the standard Girard construction are needed to account for the category and its properties.

The reconstruction of Domain Theory given in this paper is inspired by categorical generalizations and the work of Winskel (Cattani and Winskel 2005) in particular. I hope that readers will feel that it does support Kreisel's idea that one justification for generalization is that it leads to an increase in mathematical understanding.

References

S. Abramsky. Semantic Foundations for Applicative Multiprogramming. In J. Diaz (editor), *Automata, Languages and Programming, Proceedings of ICALP'83*, Springer Lecture Notes in Computer Science **154**, 1983, 1-14.

J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories.* LMS Lecture Notes Series **189**, Cambridge University Press, 1994.

S. Awodey. *Category Theory.* Oxford Logic Guides **49**, Clarendon Press, Oxford, 2006.

H. Barendregt, M. Coppo and M. Dezani-Ciancaglini. A Filter Lambda Model and the Completeness of Type Assignment. *Journal of Symbolic Logic* **48**, 1983, 931-940.

M. Barr and C. Wells. *Toposes, Triples and Theories*. Grundlehren der mathematischen Wissenschaften **278**, Springer-Verlag, 1984.

A. Bauer and P. Taylor. The Dedekind Reals in Abstract Stone Duality. *Mathematical Structures in Computer Science* **19**, 2009, 757-838.

J. M. Beck. *Triples, Algebras and Cohomology.* Ph D Dissertation, Columbia University, 1967.

N. Benton, G. Bierman, V de Paiva and M. Hyland. A Term Calculus for Intuitionistic Linear Logic. In M. Bezem and J. F. Groote (editors), *Typed Lambda Calculi and Applications*, Springer Lecture Notes in Computer Science **664**, 1993, 75-90.

N. Benton, G. Bierman, V de Paiva and M. Hyland. Linear lambda-calculus and categorical models revisited. In E. Börger, G. Jäger, H. Kleine Büning, Simone Martini and M. M. Richter (editors), *Proceedings of Computer Science Logic Conference in San Miniato (September 1992)*, Springer Lecture Notes in Computer Science **702**, 1993, 61-84.

C. Berline. From Computation to Foundations via Functions and Application : the Lambda-calculus and its Webbed Models. *Theoretical Computer Science 249*, 2000, 81-161.

G. Berry. *Modèles completement adéquats et stables des lambda-calculs typés.* Thèse de Doctorat d'Etat, Université de Paris VII, 1979.

G. L. Cattani and G. Winskel. Profunctors, open maps and bisimulation. *Mathematical Structures in Computer Science* **15**, 2005, 553-614.

M. Coppo, M. Dezani-Ciancaglini, F. Honsell and G. Longo. Extended Type Structures and Filter Lambda Models. In G. Lolli, G. Longo and A. Marcja (editors), *Logic Colloquium 82.* North-Holland, 1984, 241–262.

T. Coquand, C. A. Gunter and G. Winskel. Domain Theoretic Models of Polymorphism. *Information and Computation* **81**, 1989, 123-167.

P.-L. Curien, G. D. Plotkin and G. Winskel. Bistructures, Bidomains and Linear Logic. In G. Plotkin, C. Stirling and M.Tofte (editors), *Proof, Language, and Interaction. Essays in Honour of Robin Milner*, MIT Press, 2000, 21-54.

Y. Diers. Multimonads and Multimonadic Categories. *Journal of Pure and Applied Algebra* **17**, 1980, 153-170.

A. Edalat. Domains for Computation in Mathematics, Physics and Exact Real Arithmetic. *Bulletin of Symbolic Logic* **3**, 1997, 401-452.

S. Eilenberg and G. M. Kelly. Closed Categories. In *Proceedings of the Conference on Categorical Algebra, La Jolla 1965*, 1966, 421-562.

S. Eilenberg and J. B. Wright. Automata in general algebras. *Information and Control* **11**, 1967, 452-470.

E. Engeler. Algebras and Combinators. *Algebra Universalis* **13**, 1981, 389-392.

M. P. Fiore, G. D. Plotkin and J. Power. Complete cuboidal sets in axiomatic domain theory. In *Proceedings of the 12th Annual IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press, 1997, 268-279.

M. P. Fiore and G. D. Plotkin. An extension of models of axiomatic domain theory to models of synthetic domain theory. In D. van Dalen and M. Bezem (editors), *Computer Science Logic, Proceedings of 10th International Workshop, CSL'96*, Springer Lecture Notes in Computer Science **1258**, 1997, 129-149.

M. Fiore, N. Gambino, M. Hyland and G. Winskel. The cartesian closed bicategory of generalised species of structures. *Journal of the London Mathematical Society* **77**, 2008, 203-220.

M. Fiore, N. Gambino, M. Hyland and G. Winskel. Kleisli bicategories. (In preparation.)

J.-Y. Girard. Linear Logic. *Theoretical Computer Science* **50**, (1987), 1-102.

A. Grothendieck M. Artin and J. L. Verdier. *Théorie des Topos et Cohomologie Étale des Schemas.* SGA 4, vol. 1, Springer Lecture Notes in Mathematics **269**, 1972.

R. Hasegawa. Two applications of analytic functors. *Theoretical Computer Science* **272**, 2002, 113-175.

R. Houston. Finite products are biproducts in a compact closed category. *Journal of Pure and Applied Algebra* **212**, 2008, 394-400.

J. M. E. Hyland. A syntactic characterization of the Equality in some Models for the Lambda Calculus. *Journal of the London Mathematical Society* **12**, 1976, 361-370.

J. M. E. Hyland. The effective topos. In A.S. Troelstra and D van Dalen (editors), *The L.E.J. Brouwer Centenary Symposium*, North-Holland, 1982, 165-216.

J. M. E. Hyland. First steps in synthetic domain theory. In A. Carboni, M.-C. Pedicchio and G. Rosolini (editors), *Category Theory*, Springer Lecture Notes in Mathematics **1488**, 1991, 280-301.

J. M. E. Hyland. Proof Theory in the Abstract. *Annals of Pure and Applied Logic* **114**, 2002, 43-78.

J. M. E. Hyland and A. M. Pitts. The Theory of Constructions: Categorical Semantics and Topos Theoretic Models. In J. W. Gray and A. Scedrov (editors) *Categories in Computer Science and Logic*, Contemporary Mathematics **92**, 1989, 137-199.

M. Hyland and J. Power. Pseudo-commutative monads and pseudo-closed 2-categories. *Journal of Pure and Applied Algebra* **175**, 2002, 141-185.

M. Hyland and A. Schalk. Glueing and Orthogonality for Models of Linear Logic. *Theoretical Computer Science* **294**, 2003 183-231.

M. Hyland, M. Nagayama, J. Power and G. Rosolini. A Category-Theoretic Formulation for Engeler-style Models of the Untyped lambda-Calculus. Proc MFCSIT 2004, *Electronic Notes in Theoretical Computer Science* **161**, 2006, 43-57.

G. B. Im and G. M. Kelly. A universal property of the convolution monoidal structure. *Journal of Pure and Applied Algebra*, **43**, 1986, 75-88.

P. T. Johnstone. *Stone Spaces.* Cambridge Studies in Advanced Mathematics **3**, Cambridge University Press, 1982.

A. Joyal. Une théorie combinatoire des séries formelles. *Advances in Mathematics* **42**, 1981, 1-82.

G. M. Kelly. *Basic Concepts of Enriched Category Theory.* LMS Lecture Note Series **64**, Cambridge University Press, 1982.

G. M. Kelly and M. Laplaza. Coherence for compact closed categories. *Journal of Pure and Applied Algebra* **19**, 1980, 193-213.

A. Kock. Monads on Symmetric Monoidal Closed Categories. *Archiv der Mathematik* **21**, 1970, 1-10.

A. Kock. Closed categories generated by Commutative Monads, *Journal of the Australian Mathematical Society* **12**, 1971, 405-424.

A. Kock. Bilinearity and Cartesian Closed Monads, *Mathematica Scandinavica* **29**, 1971, 161-174.

A. Kock. Strong Functors and Monoidal Monads. *Archiv der Mathematik* **23**, 1972, 113-120.

G. Kreisel. Some reasons for generalizing recusion theory. In R. O. Gandy and C. E. M. Yates (editors), *Logic Colloquium '69, Proceedings of the summer school and colloquium in Mathematical Logic*, North-Holland, 1971, 139–198.

J. Lambek and P. J. Scott. *Introduction to higher order categorical logic*. Cambridge Studies in Advanced Mathematics **7**, Cambridge University Press, 1986.

D. J. Lehmann. Categories for fixed point semantics. In *Proceedings of the 17th I.E.E.E. Annual Symposium on Foundations of Computer Science*, 1976, 122-126.

J. R. Longley. On the ubiquity of certain total type structures. *Mathematical Structures in Computer Science* **17**, 2007, 841-953.

I. Lopez Franco. *Autonomous pseudo-monoids*. PhD Dissertation, University of Cambridge, 2008.

S. Mac Lane. *Categories for the working mathematician*. Graduate Texts in Mathematics **5**, Springer, 1971.

R. A. Platek. *Foundations of Recursion Theory*. Ph.D. Dissertation, Stanford University, 1966.

G. D. Plotkin. A Power Domain Construction. *SIAM Journal on Computing* **5**, 1976, 452-487.

G. D. Plotkin and G. Winskel. Bistructures, Bidomains and Linear Logic. In *Proceedings of ICALP*, 1994, 352–363.

B. Reus and T. Streicher. General Synthetic Domain Theory - A Logical Approach. *Mathematical Structures in Computer Science* **9**, 1999, 177–223.

D. S. Scott. Continuous lattices. In F.W. Lawvere (editor), *Toposes, Algebraic Geometry and Logic*. Springer Lecture Notes in Mathematics **274**, 1971, 97–136.

P. Taylor. *Recursive Domains, Indexed category Theory and Polymorphism*. PhD Dissertation, University of Cambridge, 1986.

P. Taylor. An Algebraic Approach to Stable Domains. *Journal of Pure and Applied Algebra* **64**, 1990, 171-203.

P. Taylor. The fixed point property in synthetic domain theory. In *Proceedings of the Sixth Annual IEEE Symposium on Logic in Computer Science*, IEEE Computer Society Press, 1991, 152–160.

P. Taylor. Geometric and Higher Order Logic in terms of ASD. *Theory and Applications of Categories* **7**, 2000, 284-338.

P. Taylor. Sober Spaces and Continuations. *Theory and Applications of Categories* **10**, 2002, 248-299.

P. Taylor. Subspaces in ASD. *Theory and Applications of Categories* **10**, 2002, 300–366.

J. van Oosten. *Realizability: An Introduction to its Categorical Side*. Studies in Logic and The Foundations of Mathematics **152**, Elsevier, 2008.

G. Winskel. *Events in Computation*. PhD dissertation, University of Edinburgh, 1980.

G. Winskel. *The formal semantics of programming languages, an introduction*. MIT Press, 1993.

G. Winskel. Stable Bistructure Models of PCF. In *Proceedings of the 19th International Symposium on Mathematical Foundations of Computer Science*, Springer Lecture Notes in Computer Science **841**, 1994, 177–197.

G. Winskel and F. Zappa Nardelli. NEW-HOPLA a higher-order process language with name generation. In *Proceedings of TCS 2004, Third IFIP International Conference on Theoretical Computer Science*, 2004, 521-534.