

Discrete Lawvere Theories and Computational Effects

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Abstract

Countable Lawvere theories model computational effects such as exceptions, side-effects, interactive input/output, nondeterminism and probabilistic nondeterminism. The category of countable Lawvere theories has sums, tensors, and distributive tensors, modelling natural combinations of such effects. It is also closed under taking images. Enrichment in a category such as ωCpo allows one to extend this modelling of computational effects to account for partiality and recursion. Sum and tensor extend to enriched countable Lawvere theories, but distributive tensor and image do not. So here we introduce discrete countable enriched Lawvere theories in order to allow natural definitions and accounts of distributive tensor and image. A discrete countable enriched Lawvere theory is, in a sense we make precise, an enriched Lawvere theory with discrete arities. We show that they include all our leading examples of computational effects and are closed under sum and tensor. And we develop notions of enriched operad and enriched multicategory to support the definition.

Key words: Countable Lawvere V -theory, discrete countable Lawvere V -theory, model, sum, tensor, distributive tensor, image.

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1 Introduction

Lawvere theories are a category-theoretic formulation of universal algebra for which the notion of operation is primitive. Unlike universal algebra, the notion of Lawvere theory is presentation-independent, i.e., the category of models determines the theory uniquely up to coherent isomorphism. The concept has proved to be particularly fruitful, generalising to the study of finite limit theories and beyond [2,3]. It corresponds to the study of finitary monads on the category *Set*, but its definition is more in the spirit of universal algebra, the notion of Lawvere theory being essentially an axiomatisation of the notion of a clone of an equational theory.

Denotational semantics is less concerned with sets with structure than it is with ω -cpo's with structure, as the latter allow for an account of partiality and recursion. So one would like to extend the notion of, and results about, Lawvere theories to include ω -cpo's. Many results, several of them explained herein, can be extended axiomatically by reference to enriched Lawvere theories [32] with enrichment in the category $V = \omega\mathit{Cpo}$: to allow for recursion, one needs to replace finitariness assumptions by countability assumptions, but that amounts to a minor technical adjustment.

As we shall explain in Section 3, if V is locally countably presentable as a symmetric monoidal closed category, e.g., V is $\omega\mathit{Cpo}$ or *Set* or *Poset* or *Cat*, a countable Lawvere V -theory is a small V -category L with countable cotensors, which we shall define, together with an identity-on-objects strict countable cotensor preserving V -functor from $V_{\aleph_1}^{op}$ to L , where V_{\aleph_1} is a skeleton of the full sub- V -category of V determined by the countably presentable objects of V . A model of L in a V -category C with countable cotensors is a countable cotensor preserving V -functor from L to C . Extending the result for ordinary Lawvere theories, countable Lawvere V -theories correspond to V -monads on V with countable rank.

Taking V to be *Set* in the definition of countable Lawvere V -theory, we recover a countable version of the usual notion of Lawvere theory, together with its associated body of theory. Taking V to be *Cat*, enrichment yields a body of theory for categories with equational structure such as finite or countable product structure, finite or countable limit structure, finite or countable co-product structure, and various forms of monoidal structure. It is similarly fruitful where V is *Poset* or $\omega\mathit{Cpo}$ or any number of other naturally arising base categories. In this paper, which is oriented towards computer science, we focus on the example of V being $\omega\mathit{Cpo}$.

The category *Law* of Lawvere theories has sums, a tensor, and a distributive tensor. These constructs appear widely in both mathematics and computer

science. The sum and the tensor of Lawvere theories enrich without fuss, yielding an account of the two most common ways of combining computational effects [10,11]. In contrast, the notion of distributive tensor does not routinely enrich, but it too plays a prominent role in combining effects. For instance, in combining probabilistic nondeterminism with ordinary nondeterminism, the operator for probabilistic nondeterminism distributes over the operator for nondeterminism, as described implicitly in [23], see also [35,36]. And in combining internal and external nondeterminism, each operator distributes over the other [7]. So in this paper, we modify the notion of enriched Lawvere theory so that, while continuing to include our leading examples of computational effects, and while continuing to admit the constructions and characterisations of the sum and tensor product of enriched Lawvere theories, we can incorporate an enriched account of the distributive tensor, extending our two leading examples of it. In doing so, fortuitously, we also include an account of the image, which exists for ordinary Lawvere theories but not for enriched theories in general: the image allows us to take operations and observations, rather than operations and equations, as primitive notions in analysing computational effects [33]. In general, our analysis builds upon and extends the ideas of [10,11,28–30], which began to develop a unified semantics for modelling the operations associated with computational effects. Note that our analysis here does not, a priori, include continuations, but some of it can be extended to do so [9].

There is a very general setting for our analysis that is yet to be investigated and that we do not study here. The situation might best be explained by analogy. One has long had a notion of monoid, given by a set M together with an associative binary multiplication \cdot with left and right unit e . But the theory of monoids is perhaps best understood by generalising from Set to an axiomatically given monoidal category \mathcal{C} . The structure of a monoidal category mirrors that of a monoid but at one higher dimension; it provides a particularly natural setting in which to define and analyse the notion of monoid; and it allows a treatment of constructs such as that of a ring, which is exactly a monoid in the monoidal category Ab of abelian groups. Such a construct exists here too: there is a higher-dimensional version of the notion of distributive tensor that mirrors the ordinary definition but at one higher dimension; and it provides a natural setting in which to define and analyse the notion of distributive tensor. But, in contrast to the situation for monoids, we do not have any compelling example of that higher-dimensional structure beyond a canonical one determined by standard operations on countable numbers. So we do not develop the higher dimensional structure in this paper, but rather we restrict ourselves to studying its leading instance: and that yields what we call a *discrete* countable Lawvere V -theory.

For a discrete countable Lawvere V -theory, L is still a small V -category, but it comes equipped with an identity-on-objects functor from a skeleton of \aleph_1^{op}

rather than from $V_{\aleph_1}^{op}$, so an object of L is either a natural number or is \aleph_0 rather than being an arbitrary countably presentable object of V such as Sierpinski space when V is *Poset*. Taking V to be *Set*, we recover the notion of countable Lawvere V -theory. In contrast, when V is *Cat*, the restriction from V -theories to discrete countable Lawvere V -theories is substantial, cutting out examples such as those involving limits and colimits. When V is *Poset* or ωCpo , one loses some examples but taps into an already existing body of logical theory, primarily developed by Jose Meseguer over several decades [22]. In particular, when V is ωCpo , discrete countable Lawvere V -theories still include the bulk of the structures that appear in practice, including all those of primary interest in analysing computational effects [10,11], except for continuations, which are not included in [10,11] anyway. There is a sense in which the correspondence between Lawvere theories and monads extends to one between discrete Lawvere theories and a kind of monads, but that occurs more by fiat than by a natural condition on the definition of monad [18]. In Section 3, we develop the definition of discrete countable Lawvere V -theory, and in Sections 4 and 5, we analyse sum and tensor in this context, checking that they agree with those for arbitrary Lawvere V -theories.

The central reason for introducing discrete Lawvere V -theories is to define and analyse a distributive tensor of V -theories extending that for ordinary Lawvere theories. So we define the notion in Section 6. We want not only a definition of distributive tensor but also a universal characterisation of it. The natural universal characterisation is subtle, involving an enriched definition of operad. If L is finitary and L' is commutative, as is the case for our leading examples of distributivity, subject to cartesianness restrictions on the base and enriching categories, we can give stronger results than are possible in full generality, developing a relationship with the theory of monads. So we characterise the distributive tensor under these restricted assumptions in Section 7 and we give a somewhat more complicated characterisation in full generality in Section 8.

Finally, in Section 9, we analyse the construction of an image: the image exists for ordinary Lawvere theories but not for enriched ones. But it does exist for finitary discrete Lawvere V -theories. Its significance is in allowing one to take operations and observations, rather than operations and equations, as primitive in defining computational effects.

We use Kelly's book [16] as the source book for all definitions and notation for enriched categories, with [17] being the basic text for locally presentable V -categories. This paper is an extended version of the conference paper [34], the main new work appearing in Sections 7 and 8.

2 Enriched Lawvere Theories

In this section, we recall the notions of Lawvere theory and enriched Lawvere theory and one strand of thought that motivates their use in computer science. The work in this section is adapted from [10,11], which in turn was motivated by the desire for a more profound formulation of Moggi's unification of computational effects as monads in [24,25].

Definition 1 *A Lawvere theory consists of a small category L with finite products together with a strict finite product preserving identity-on-objects functor $I : \text{Nat}^{\text{op}} \rightarrow L$, where Nat is the category of all natural numbers and maps between them [2,3]. A model of a Lawvere theory L in a category C with finite products is a finite-product preserving functor from L to C .*

Implicit in the definition is the fact that the objects of L are exactly the natural numbers. The definition provides a category theoretic formulation of universal algebra, with the notion of operation taken as primitive: a map in L from n to m is understood as being given by m operations of arity n . Unlike the notion of equational theory, the concept of Lawvere theory is presentation-independent, i.e., if a pair of Lawvere theories have equivalent categories of models, the two theories are isomorphic.

The definition of model extends to the definition of the category $\text{Mod}(L, C)$ of models of L in any category C with finite products: maps of models are defined to be natural transformations. Note that naturality forces maps of models to respect the product structure in the definition of model. Observe also that we do not demand strict preservation in the definition of model: to do so would eliminate many of the leading examples [31]!

For any Lawvere theory L and any locally finitely presentable category C , the functor $ev_1 : \text{Mod}(L, C) \rightarrow C$ has a left adjoint, inducing a monad T_L on C : we shall return to this later in the section.

The usual way in which one obtains Lawvere theories is by means of sketches, with the Lawvere theory given freely on the sketch: Barr and Wells' book [3] treats sketches in loving detail and gives a range of examples of both sketches and Lawvere theories. To give a sketch for a Lawvere theory amounts to giving operations and universally defined equations, i.e., an equational theory. The Lawvere theory is an axiomatisation of the notion of the clone of an equational theory, equivalently of a sketch.

Example 2 *The Lawvere theory L_E for exceptions is the free Lawvere theory generated by an E -indexed family of nullary operations with no equations. The monad on Set induced by L_E is $T_E = - + E$. More generally, if C is any category with finite powers and sums then $\text{Mod}(L_E, C)$ is equivalent to*

the category of algebras for the monad $- + \underline{E}$ where \underline{E} is the E -fold copower of 1 , i.e., $\coprod_E 1$.

Interactive input/output works similarly to exceptions [11], so we omit details. For the next example, we use the evident generalisation of the notion of Lawvere theory to countable Lawvere theory as used in [10,11] and as we shall make precise shortly: it allows us to use countable arities.

Example 3 *The countable Lawvere theory L_S for side-effects, where $S = V^L$, is the free countable Lawvere theory generated by the operations $\text{lookup} : V \rightarrow L$ and $\text{update} : 1 \rightarrow L \times V$ subject to seven natural equations as follows [30]: with lookup corresponding to the logical symbol l and with update corresponding to u , the equations can be expressed syntactically as*

- (1) $l_{loc}(u_{loc,v}(x))_v = x$
- (2) $l_{loc}(l_{loc}(t_{vv'})_v)_{v'} = l_{loc}(t_{vv})_v$
- (3) $u_{loc,v}(u_{loc,v'}(x)) = u_{loc,v'}(x)$
- (4) $u_{loc,v}(l_{loc}(t_{v'})_{v'}) = u_{loc,v}(t_v)$
- (5) $l_{loc}(l_{loc'}(t_{vv'})_v)_{v'} = l_{loc'}(l_{loc}(t_{vv'})_v)_{v'}$ where $loc \neq loc'$
- (6) $u_{loc,v}(u_{loc',v'}(x)) = u_{loc',v'}(u_{loc,v}(x))$ where $loc \neq loc'$
- (7) $u_{loc,v}(l_{loc'}(t_{v'})_{v'}) = l_{loc'}(u_{loc,v}(t_{v'}))_{v'}$ where $loc \neq loc'$.

These are expressed both as equations and diagrammatically in [30], where it is shown that L_S induces the side-effects monad. More generally, if C is any category with countable powers and copowers then, slightly generalising the result in [30], $\text{Mod}(L_S, C)$ is equivalent to the category of algebras for the monad $(S \times -)^S$ where we write $(S \times -)$ for the S -fold copower $\coprod_S -$, and $(-)^S$ for the S -fold power $\prod_S -$.

Example 4 *The Lawvere theory L_N for (binary) nondeterminism is the Lawvere theory freely generated by a binary operation $\vee : 2 \rightarrow 1$ subject to equations for associativity, commutativity and idempotence, i.e., the Lawvere theory for a semilattice. The induced monad on Set is the finite non-empty subset monad, F^+ , cf [8,26,27].*

Example 5 *The Lawvere theory L_P for probabilistic nondeterminism is that freely generated by $[0, 1]$ -many binary operations $+_\tau : 2 \rightarrow 1$ subject to the equations for associativity, commutativity and idempotence in [6]. The induced monad on Set is the distributions with finite support monad, D_f , cf [13,14].*

In denotational semantics, one is not primarily interested in sets but rather in ω -cpo's, as the latter may be used to account for recursion. We therefore seek to generalise the study of Lawvere theories from sets to ω -cpo's. This may be done axiomatically by recourse to the notion of a countable *enriched* Lawvere theory and considering the example of enrichment in the category of ω -cpo's [10,11,32]. Countability is essential to account for recursion, as seen

for instance in studying side-effects.

Axiomatically, the details are as follows. We first assume our base category V is locally countably presentable as a symmetric monoidal closed category [1,16,17]. For the purposes of this paper, we do not require a definition of that: we simply need to know that the categories Set , $Poset$, ωCpo and Cat are all examples. In all those examples, the relevant symmetric monoidal closed structure is, in fact, cartesian closed structure. So we shall assume that when convenient.

The least obvious point to note when enriching Lawvere theories is that the notion of countable product of a single generator does not generalise most naturally to a notion of countable product but rather to a notion of countable *cotensor* [16]. The notion of cotensor is the natural enrichment of the notion of a power-object. Given an object x of a category C and a set A , the A -fold power x^A satisfies the defining condition that there is a bijection of sets

$$C(y, x^A) \cong C(y, x)^A$$

natural in y . This enriches to the notion of cotensor as follows.

Definition 6 *Given an object x of a V -category C and given an object a of V , the cotensor x^a satisfies the defining condition that there is an isomorphism in V*

$$C(y, x^a) \cong C(y, x)^a$$

V -natural in y . The cotensor x^a is called countable if a is a countably presentable object of V .

When $C = V$, x^a is the exponential in V .

Example 7 *Taking V to be $Poset$, the notion of cotensor allows us not only to consider objects such as $x \times x$ ($= x^2$) in a locally ordered category, but also to consider objects such as x^\leq , where \leq is Sierpinski space, the two point partial order $\perp \leq \top$. This possibility allows us, in describing $Poset$ -theories, not only to retain countable products but also to consider a greater range of arities and, in particular, to incorporate inequations. For the latter, suppose one wishes to say that $f \leq g$ for two morphisms $f, g : x \rightarrow y$; this is accomplished by introducing a third morphism $h : x \rightarrow y^\leq$ and asserting the equations $f = y^\perp \circ h$ and $g = y^\top \circ h$, where \perp and \top are the two evident maps from 1 to \leq . Observe that y^\leq only appears here as a codomain of an operation, not as a domain of one. $Poset$ -enriched Lawvere theories are at the heart of Ghani and L uth's work on term rewriting systems in [5].*

In order to enrich the definition of Lawvere theory, we need not only the notion of a cotensor, but also its dual, i.e., tensor. Given an object x of a C -category C and given an object a of V , the tensor $a \otimes x$ is precisely given by taking the cotensor in the V -category C^{op} . The tensor is called *countable* if a is a countably presentable object of V . When $C = V$, the tensor $a \otimes x$ agrees with the monoidal structure of V .

Define V_{\aleph_1} to be a skeleton of the full sub- V -category of V determined by the countably presentable objects of V . It is equivalent to the free V -category with countable tensors on 1 [16,32].

Definition 8 *A countable Lawvere V -theory is a small V -category L with countable cotensors together with a strict countable-cotensor preserving identity-on-objects V -functor $I : V_{\aleph_1}^{op} \rightarrow L$. A map of countable Lawvere V -theories from L to L' is a strict countable-cotensor preserving V -functor from L to L' that commutes with I and I' . A model of L in a V -category C with countable cotensors is a countable-cotensor preserving V -functor $M : L \rightarrow C$.*

Routinely generalising the unenriched case, for any countable Lawvere V -theory L and any V -category with countable cotensors C , we have a V -category of models of L in C , $Mod(L, C)$; the homobjects are given by homobjects of all V -natural transformations [16], and the V -naturality condition implies they respect countable cotensors. There is a canonical forgetful V -functor U_L from $Mod(L, C)$ to C given by evaluation at the unit of V , which necessarily lies in V_{\aleph_1} and hence in L . If the forgetful V -functor has a left V -adjoint, as it does whenever C is a locally presentable V -category, this forgetful V -functor exhibits $Mod(L, C)$ as equivalent to the V -category $T_L\text{-Alg}$ for the induced V -monad T_L on C . We denote the category of countable Lawvere V -theories by Law_V .

To give a V -enriched V -monad is equivalent to giving a strong monad on V [21]. So, in order to make the comparison with Moggi's unified account of computational effects as modelled by strong monads a little more direct, we express the main abstract result of [32] in terms of strong monads.

Theorem 9 *If V is locally countably presentable as a symmetric monoidal closed category, the construction of T_L from L induces an equivalence of categories between the category of countable Lawvere V -theories on V and the category of strong monads on V with countable rank. Moreover, the comparison V -functor exhibits an equivalence between the V -categories $Mod(L, V)$ and $T_L\text{-Alg}$.*

A common and important way to generate countable Lawvere V -theories is by taking the free countable Lawvere V -theory on an unenriched countable Lawvere theory. For instance, let V be ωCpo . Given an unenriched countable Lawvere theory L , the free countable Lawvere ωCpo -theory L_ω on L is gen-

erated by the operations and equations of L , but it has more objects as there are countably presentable ω -cpo's other than flat ones, and these additional objects generate additional maps. See [11] for details, but suffice it for here to note that Examples 2, 3, 4 and 5 all thereby freely yield countable Lawvere V -theories where $V = \omega Cpo$.

For the leading example of a countable Lawvere V -theory that does not arise freely from an unenriched countable Lawvere theory, let V be ωCpo and consider nontermination.

Example 10 *The countable Lawvere ωCpo -theory L_Ω for nontermination is the theory freely generated by a nullary operation $\Omega : 0 \rightarrow 1$ subject to the condition that there is an inequality*

$$\begin{array}{ccc}
 1 & \xrightarrow{\quad} & 0 \\
 & \searrow \text{id} & \downarrow \Omega \\
 & & 1
 \end{array}
 \quad \geq$$

where the unlabelled map is the unique map determined because 0 is the initial object of V_{\aleph_1} and therefore the terminal object of $V_{\aleph_1}^{op}$. The models of L_Ω in ωCpo are the ω -cpo's with a least element. The corresponding strong monad T_Ω is the lifting construction $(-)_\perp$ which adds a new least element. Observe that there is at most one morphism from L_Ω to any other countable ωCpo -theory L , reflecting the fact that a least element is unique if it exists.

Adding a nontermination effect allows us to model recursion in the context of ωCpo . If we want to model other effects in addition to recursion, we have to combine them with nontermination: switching to L_ω from L does not suffice (see [11] for details). This generates for us many examples of countable Lawvere V -theories that are not freely generated by unenriched ones: combining any of Examples 2, 3, 4 and 5 with Example 10 as is invariably the case in practice yields examples; the combinations can be given by fiat or may more generally be seen as examples of the various ways in which one can combine Lawvere theories that we investigate later in the paper.

3 Discrete Lawvere Theories

As we have seen, countable Lawvere V -theories, for $V = \omega Cpo$, allow us to account for all the leading examples of computational effects other than continuations. Moreover, they are closed under two constructions that model natural ways in which computational effects are combined: sum and tensor [10,11].

But there are other constructions that may be made of countable Lawvere theories in the original *Set*-based case, that do not exist for arbitrary countable Lawvere V -theories where $V = \omega\mathit{Cpo}$, but that do appear in combining computational effects in practice. So we seek a refinement of the notion of countable Lawvere V -theory to account for them.

In particular, as we shall discuss in Section 6, consider the operations of one theory L distributing over those of another theory L' , as appear in concurrency [7] and in the combination of nondeterminism and probabilistic nondeterminism [23]. The construction does not readily enrich. The problem lies with the arities: an arbitrary countable Lawvere V -theory has arities that may be any countably presentable objects of V , but the notion of distributivity a priori only makes sense for arities that are sets (see Section 6 for details).

Lawvere V -theories also are not closed under taking the image of a model, as one wants when taking observations as a primitive notion. We give an outline of this in Section 9, cf [33]. Again, the problem lies with the arbitrariness of the arities.

So in this section, we refine the notion of enriched Lawvere theory by restricting the arities to be sets, but still including all our examples, still allowing us to take sum and tensor, but also allowing us to take a distributive tensor and an image. This yields our notion of discrete Lawvere theory. Motivated by recursion, we describe a countable rather than a finite version here.

The notion of discrete countable Lawvere theory lies between that of ordinary countable Lawvere theory and that of countable Lawvere V -theory: in the former, the arities are objects of \aleph_1 and the homs are sets; in the latter, arities are countably presentable objects of V and the homs are objects of V ; but for discrete countable Lawvere V -theories, we demand that the arities lie in \aleph_1 but allow the homs to be objects of V . The definition, using the notation we have already established, is as follows. The underlying ordinary category of a V -category C is denoted by C_0 [16].

Definition 11 *A discrete countable Lawvere V -theory is a small V -category L with countable products and a strict countable-product preserving identity-on-objects functor $I : \aleph_1^{op} \rightarrow L_0$. A map of discrete countable Lawvere V -theories from L to L' is a (necessarily strict countable-product preserving) V -functor from L to L' that commutes with I and I' . A model of L in a V -category C with countable products is a countable product preserving V -functor $M : L \rightarrow C$. (Slightly more generally it suffices to have products of countably many copies of the object $X = M(1)$.)*

So an object of a discrete countable Lawvere theory L is a countable ordinal, the countable products of L are given by a choice of countable sum of the countable ordinals, with projections given by the images of the corresponding

injections of the category \aleph_1 . Just as for ordinary Lawvere theories, discrete countable Lawvere theories are typically generated by sketches, as we shall illustrate. We denote the category of discrete countable Lawvere V -theories by $DLaw_V$.

The definition of model extends, as before, to yield a natural V -category $Mod(L, C)$ of models of L in any V -category C with countable products: the homobjects are given by the usual internalisation to V of the set of all V -natural transformations between models [16].

Let us consider, in concrete terms, what a model M of a discrete countable Lawvere V -theory L in a V -category C with countable products is. First, one must send the object 1 to an object X of C . Since M preserves countable products and since all objects of L are countable products of copies of 1, this completely determines the behaviour of M on objects, indeed also on all maps in \aleph_1^{op} , up to coherent isomorphism. It remains to give the behaviour of M on homs. That amounts to giving, for each a and b in \aleph_1 , a map in V of the form

$$M_{a,b} : L(a, b) \longrightarrow C(X^a, X^b)$$

subject to preservation of identities and composition. So, in particular, if L is freely generated by some sort of signature and equations, equivalently by some generalised notion of sketch (see [20] for a general definition and treatment of the sketch idea), it simply amounts to giving a model of each constructor of the generalised signature subject to the generalised equations, exactly as for Lawvere V -theories as in Section 2 but with arities restricted to be objects of \aleph_1 rather than arbitrary countably presentable objects of V .

One can construct a forgetful functor from the category Law_V of countable Lawvere V -theories to $DLaw_V$ as follows. Observe that \aleph_1^{op} being the free category with countable products on 1 and $V_{\aleph_1}^{op}$ including the object $1 \otimes I$ (where I is the unit of V , and so $1 \otimes I$ is I) and having countable products determine a canonical functor $J : \aleph_1^{op} \longrightarrow V_{\aleph_1}^{op}$.

Definition 12 *The forgetful functor $U : Law_V \longrightarrow DLaw_V$ sends a countable Lawvere V -theory $I : V_{\aleph_1}^{op} \longrightarrow L$ to the V -category and functor determined by factorising the composite*

$$\aleph_1^{op} \xrightarrow{J} V_{\aleph_1}^{op} \xrightarrow{I} L$$

as an identity-on-objects functor followed by full faithful one, and using the latter functor to induce V -structure on the factorisation from that on L .

So the objects of the factorisation are exactly the objects of \aleph_1^{op} , with the hom from a to b given by $L(IJa, IJb)$, and with composition determined by that

of L . This tells us that, given a countable Lawvere V -theory L , the discrete countable Lawvere V -theory $U(L)$ is given by taking the full sub- V -category of L determined by the countable products of the generating object 1 .

Proposition 13 *If V is locally countably presentable as a cartesian closed category, the functor $V(1, -) : V \rightarrow \text{Set}$ has a left adjoint, and the left adjoint sends countable sets to countably presentable objects of V .*

By the proposition, if a is an object of \aleph_1 , it follows that the free object of V on it, which, with mild overloading of notation, we also denote by a , is an object of V_{\aleph_1} .

Theorem 14 *The forgetful functor $U : \text{Law}_V \rightarrow D\text{Law}_V$ has a left adjoint F . Moreover, given any V -category C with countable cotensors, the unit of the adjunction determines a canonical equivalence*

$$\text{Mod}(L, C) \simeq \text{Mod}(F(L), C)$$

where the first occurrence of Mod refers to models of a discrete theory, thus V -functors that preserve countable products, while the second refers to models of a V -theory, thus V -functors preserving countable V -cotensors [4].

PROOF. The first statement may be seen as an instance of the theory of (countable) essentially algebraic theories [1,3] or alternatively via the study of V -categories with equational structure [4,19]. The second follows from the work relating strict maps and pseudo-maps of categories with structure in [4]. Alternatively, one can check it by direct calculation.

So, for every countable discrete Lawvere theory L , there is a countable Lawvere V -theory, namely $F(L)$, whose V -category of models in any V -category C with countable cotensors agrees with the V -category of models of L in C *qua* V -category with countable products of copies of any object.

There is also a forgetful functor U' from $D\text{Law}_V$ to the category of ordinary countable Lawvere theories Law_c : it simply forgets the V -structure of L by taking its underlying ordinary category L_0 .

Theorem 15 *The forgetful functor $U' : D\text{Law}_V \rightarrow \text{Law}_c$ has a left adjoint F' with the additional property that for any V -category C with countable products, the unit of the adjunction determines a canonical equivalence*

$$\text{Mod}(L, C_0) \simeq \text{Mod}(F'(L), C)$$

where here, the first occurrence of Mod refers to models of an ordinary theory, thus functors that preserve countable products, while the second refers to models of a discrete V -theory, thus V -functors preserving countable V -products [4].

PROOF. The free V -category (without insisting upon a countable product condition) on L in fact has countable products and acts as $F'(L)$: one must observe that countable products of the former freely yield countable products of the latter.

Thus, for any ordinary countable Lawvere theory L , there is a discrete countable Lawvere V -theory, namely $F'(L)$, whose category of models in any V -category C with countable products agrees with the category of models of L in the underlying category C_0 of C , which necessarily has countable products in the ordinary sense.

By Theorem 15, Examples 2, 3, 4 and 5 may all be seen as discrete countable Lawvere V -theories: it is safe for us to identify these ordinary countable Lawvere theories with the discrete countable Lawvere V -theories they freely generate. A non-free example is as follows.

Example 16 *Consider Example 10, the Lawvere ωCpo -theory for nontermination. As presented, it gives a sketch from which the Lawvere ωCpo -theory is given freely. But one can see by inspection that the sketch may equally be seen as a sketch for a discrete ωCpo -theory, in that one could equally consider the free discrete ωCpo -theory generated by it and consider its models: the arities appearing in Example 10 are all discrete sets, in this case finite sets, and the homs of a discrete Lawvere V -theory may be arbitrary objects of V , in this case allowing us to express the inequality in the example, i.e., we can replace the use of $y \leq$ as the codomain of an operation in Example 10 by two operations into y with an inequality between them. But the example does not, in any reasonable sense, freely generate an ordinary Lawvere theory as ordinary Lawvere theories do not allow us to treat inequality nontrivially.*

Extending the remarks after Example 10, an assortment of further non-free examples of discrete countable Lawvere V -theories is given by combining any of Examples 2, 3, 4 and 5 with Example 16. One can check that directly, or one can see it as a consequence of the following axiomatic body of theory.

There are four constructions on Lawvere theories that are of primary interest to us: the sum, the tensor, the distributive tensor, and the image. These all arise in the study of computational effects. One considers the sum in combining exceptions with any other computational effect [10,11]; one considers the tensor in combining side-effects with all other effects other than exceptions [10,11]; the distributive tensor or a two-sided version of it appears in

combining two sorts of nondeterminism, for instance internal and external nondeterminism as used by Hennessy in modelling concurrency [7], or in combining ordinary nondeterminism with probabilistic nondeterminism [23]; and one considers the image in deriving equations from a theory of observations. So, in forthcoming sections we consider these constructions in turn.

4 Sum

In this section, we consider the sum of discrete Lawvere V -theories. We only do so briefly as we already investigated sums of arbitrary Lawvere V -theories [10,11], so we just need to check that the construction of a sum restricts from arbitrary Lawvere V -theories to discrete ones. A fortiori, our various examples of sums restrict from arbitrary V -theories to discrete ones.

Theorem 17 *The category of discrete countable Lawvere V -theories is co-complete.*

One way to prove this is by observing that the category of discrete countable Lawvere V -theories is locally countably presentable [1]. An explicit construction of the sum is complicated as a general construction involves a transfinite induction, with inductive steps being given by a complicated coequaliser, cf [15]. But, just as in [11], all our examples of discrete countable Lawvere theories are given freely by sketches. And in those terms, the sum is easy to describe: one takes all operations of both equational theories, including information such as partial order information, subject to all axioms of both. The complication arises in passing from the induced sketch to the Lawvere theory freely generated by it, as, in doing so, one may apply the operations of one theory to the operations of the other, hence the transfinite induction.

Even in terms of sketches, care is required. For instance, given Lawvere theories L and L' , there are always maps of Lawvere theories given by coprojections $L \rightarrow L + L'$ and $L' \rightarrow L + L'$. But these coprojection functors need not be faithful. For instance, L might be the trivially collapsing theory, i.e., its equations may force L to be equivalent to 1. In that case, $L + L'$ is also equivalent to 1, so the coprojection from L' is trivial.

It is a simple observation, comparing the above with [11], that the sum of discrete countable Lawvere V -theories qua countable Lawvere V -theories is discrete. One would certainly hope so as left adjoints, in particular the left adjoint F to the forgetful functor $U : Law_V \rightarrow DLaw_V$, preserve sums. For the same reason, a sum of countable Lawvere theories qua discrete countable Lawvere V -theories is free on the sum of ordinary countable Lawvere theories. For computational effects, the leading examples are given by the combination

of exceptions with all other effects and the combination of interactive I/O with most other effects [11].

5 Tensor

The tensor of theories arises when one wants to combine side-effects with most other computational effects. Along with Gordon Plotkin, we investigated the tensor product of countable Lawvere V -theories in detail in [11]. So we shall not repeat that analysis here.

It was shown in [11] that a tensor product of countable Lawvere V -theories always exists and may be characterised by a universal property or equivalently, more naturally, as a representing object. Here, we make the easy observations that we can adapt the construction and characterising theorem of the tensor given in [11] from countable Lawvere V -theories to discrete countable Lawvere V -theories and that the left adjoints F and F' to $U : Law_V \rightarrow DLaw_V$ and $U' : DLaw_V \rightarrow Law_c$ preserve tensor products. It follows that our examples of the tensor studied in [11] are immediately examples of the tensor described here: we shall not repeat the details.

Definition 18 *Given discrete countable Lawvere V -theories L and L' , a commutative cocone over L and L' consists of a discrete countable Lawvere V -theory L'' together with maps of discrete countable Lawvere V -theories from each of L and L' to L'' subject to commutativity of all diagrams of the form*

$$\begin{array}{ccc}
 L(a, b) \otimes L'(a', b') & \longrightarrow & L(a \times b', b \times b') \otimes L'(a \times a', a \times b') \\
 \downarrow & & \downarrow \\
 L(a \times a', b \times a') \otimes L'(b \times a', b \times b') & \longrightarrow & L''(a \times a', b \times b')
 \end{array}$$

where the right-hand and lower maps are canonically determined by the maps from each of L and L' to L'' , together with composition in L'' , and where the other maps are given axiomatically by the product structure.

Definition 19 [11] *Given discrete countable Lawvere V -theories L and L' , the discrete countable Lawvere V -theory $L \otimes L'$, which we call the tensor product of L and L' , together with canonical maps to it from each of L and L' , is defined to be the universal commutative cocone over L and L' , i.e., for any commutative cocone L'' over L and L' , there is a unique map of discrete countable Lawvere V -theories from $L \otimes L'$ to L'' making the two canonically determined triangles commute.*

The tensor product always exists and may be described as the free theory on an enriched sketch [20], cf [11]. It may equally, indeed more elegantly, be proved to exist by appeal to an enrichment of the delicate 2-categorical analysis of the form appearing in [12], from which the following result, adapting the central result about arbitrary V -theories in [11], also follows:

Theorem 20 • *The construction $L \otimes L'$ on discrete countable Lawvere V -theories extends to a symmetric monoidal structure on $D\text{Law}_V$, and*

- *for any small V -category C with countable products, there is a coherent equivalence of V -categories between $\text{Mod}(L \otimes L', C)$ and $\text{Mod}(L, \text{Mod}(L', C))$.*

Just as in [11], the unit for the tensor product is the initial discrete countable Lawvere V -theory, i.e, the theory generated by no operations and no equations. This is the initial object of the category of discrete countable Lawvere V -theories, so is also the unit for the sum; it corresponds to the identity monad.

As we have already observed, Examples 2, 3, 4, 5 and 10 may all be seen as discrete countable Lawvere V -theories, and so we may simply translate our study of tensor of countable Lawvere V -theories in [11] to that of discrete countable Lawvere V -theories, while retaining all our examples. Its importance, as studied in detail in [11], lies in combining side-effects with almost all other effects, the main counter-example to that being in the combination with exceptions, which is covered by the previous section. As was the case for sum, the left adjoints F and F' preserve the tensor product, making the adjunctions $U : \text{Law}_V \rightarrow D\text{Law}_V$ and $U' : D\text{Law}_V \rightarrow \text{Law}_c$ into symmetric monoidal adjunctions.

6 Distributive Tensor

We now turn to the distributive tensor. This is where the concept of discrete countable Lawvere V -theory starts to yield its value relative to arbitrary countable Lawvere V -theories: one can speak of a distributive tensor product of ordinary Lawvere theories without difficulty, but there is not a natural way to speak of a distributive tensor of arbitrary countable Lawvere V -theories in the absence of further higher dimensional data or conditions, although it appears in computational practice.

For ordinary Lawvere theories, the distributive tensor is defined similarly to the tensor except that the two sets of operations are not required to commute, but rather the first are required to distribute over the second.

Consider Lawvere theories L and L' and what it means to say that the operations of L distribute over the operations of L' . The idea of distributivity is that

one has an operation $f : X^{n+1} \rightarrow X$ of L , and an operation $g : X^{n'} \rightarrow X$ of L' , and one wants the diagram

$$\begin{array}{ccc}
 X^n \times X^{n'} & \xrightarrow{(f \cdot (X^n \times \pi_{i'}))_{i' \in n'}} & X^{n'} \\
 \downarrow X^n \times g & & \downarrow g \\
 X^{n+1} & \xrightarrow{f} & X
 \end{array}$$

where the top horizontal map is given by the n' -tuple of maps into 1 with i' -th component $f \cdot (X^n \times \pi_{i'})$, to commute, and similarly for all other n variants of the diagram given by varying the choice of an element of $n+1$ in the exponent in the bottom left-hand corner from being the last element to being any of its n predecessors. This motivates the following:

Definition 21 *Given Lawvere theories L and L' , the Lawvere theory $L \triangleright L'$, called the distributive tensor of L over L' , is defined by the universal property of having maps of Lawvere theories from L and L' to $L \triangleright L'$, with all operations of L distributing over all operations of L' , i.e., given $f : (n+1) \rightarrow 1$ in L and $f' : n' \rightarrow 1$ in L' , we demand commutativity of the diagram*

$$\begin{array}{ccc}
 n+n' & \xrightarrow{(f \cdot (n \times \pi_{i'}))_{i' \in n'}} & n' \\
 \downarrow id_n \times f' & & \downarrow f' \\
 n+1 & \xrightarrow{f} & 1
 \end{array}$$

together with commutativity of all other n variants of the diagram given by varying the choice of an element of $n+1$ in the bottom left-hand corner from being the last element to being any of its n predecessors.

In succeeding sections, we shall characterise the distributive tensor in terms of models of L in $Mod(L', C)$ along the lines of Theorem 20: it involves the use of operads and symmetric monoidal structure or more generally multicategory structure on $Mod(L', C)$. But for the moment, we just remark that the distributive tensor does arise naturally in computation, sometimes in a two-sided framework such as in Hennessy's modelling of internal and external choice in [7] for modelling concurrency, and sometimes for modelling the combination of probabilistic nondeterminism and ordinary nondeterminism [14,23].

Now try extending this to arbitrary countable Lawvere V -theories: one would replace the $n+1$ in the bottom left-hand corner of the diagram by an arbitrary finitely presentable object v of V that, in some coherent sense, can be

subdivided into an element together with the rest of v . But it is not clear how to do that. For instance, when V is *Poset*, it is not clear how to handle an exponent such as that given by Sierpinski space, i.e., the two element poset $\perp \leq \top$.

The most obvious simple solution that includes all our examples seems to be to restrict to discrete countable Lawvere V -theories: so the arities are all discrete sets, from which we can choose an element as a possible codomain, while leaving the rest of the set alone. We are still able to consider inequalities as in Example 10 seen as a discrete theory in Example 16. So that is how we proceed here, while remarking that this is an instance of a more general construct at one higher dimension, for which we do not yet have other compelling examples.

Definition 22 *Given discrete countable Lawvere V -theories L and L' , the discrete countable Lawvere V -theory $L \triangleright L'$, called the distributive tensor of L over L' , is defined by the universal property of having maps of discrete countable Lawvere V -theories from L and L' to $L \triangleright L'$, with all operations of L distributing over all operations of L' . I.e., for all objects a and a' of \aleph_1 and for all elements x of a (so $a \geq 1$), suppressing canonical isomorphisms, we demand commutativity of the diagram*

$$\begin{array}{ccc}
 L(a, 1) \otimes L'(a', 1) & \xrightarrow{L(\pi_{i'}, 1)_{i' \in a'} \otimes id} & L(a - x + a', a') \otimes L'(a', 1) \\
 \downarrow id \otimes (a - x + (-)) & & \downarrow comp \\
 L(a, 1) \otimes L'(a - x + a', a) & \xrightarrow{comp} & (L \triangleright L')(a - x + a', 1)
 \end{array}$$

This definition internalises a countable version of the notion of distributivity, in particular allowing us to include Example 16, the example of nontermination, without running into an arity problem. Thus we can extend our list of constructions on discrete countable Lawvere V -theories to include combinations involving various forms of nondeterminism.

Note that we really only need discreteness of L in order to make this definition: one can readily extend the definition to allow for non-discrete L' , but the additional generality of allowing L' not to be discrete has not arisen in practice yet, so we shall not give it much attention.

7 Characterising distributivity: the commutative case

The notion of tensor $L \otimes L'$ of Lawvere theories is validated by Theorem 20, which characterises the category of models of $L \otimes L'$ in terms of categories of models of L and L' . We should like to do similarly for the distributive tensor, and we shall do so in full generality in Section 8. But all our leading examples of distributivity, all of which arise in regard to various forms of nondeterminism, satisfy the assumptions that L is finitary, L' is *commutative*, V is cartesian closed, and the V -category C in which we take models is also cartesian closed. And under those circumstances, we can give a smoother account that can also naturally be formulated in terms of monads.

The notion of commutativity of a theory can be defined for ordinary Lawvere theories, for countable Lawvere V -theories, or for discrete countable Lawvere V -theories. The various assertions are respected by both forgetful functors and their adjoints. For generality, we shall define it for an arbitrary countable Lawvere V -theory.

Emulating Definition 19, commutativity of a countable Lawvere V -theory L is the assertion that, suppressing canonical isomorphisms (meaning those induced by the structural symmetries, associativities, and units),

$$\begin{array}{ccc}
 L(a, b) \otimes L(a', b') & \longrightarrow & L(a \otimes b', b \otimes b') \otimes L(a \otimes a', a \otimes b') \\
 \downarrow & & \downarrow \text{comp} \\
 L(a \otimes a', b \otimes a') \otimes L'(b \otimes a', b \otimes b') & \xrightarrow{\text{comp}} & L(a \otimes a', b \otimes b')
 \end{array}$$

always commutes. The definition relates closely with that of commutativity for a monad:

Proposition 23 *If L is a commutative countable Lawvere V -theory and C is locally countably presentable as a symmetric monoidal closed V -category, the V -monad T_L induced by $U_L : \text{Mod}(L, C) \rightarrow C$ acquires a canonical commutativity.*

PROOF. As in Section 2, the V -category $\text{Mod}(L, C)$ enriches canonically over C , and $U_L : \text{Mod}(L, C) \rightarrow C$ has a left C -enriched adjoint. Thus the V -monad T_L enriches over C , and thus its underlying ordinary monad has a canonical strength relative to C . The commutativity condition of L yields that for T_L by the (easy direction of the) enriched Yoneda embedding.

This result has a converse when $C = V$: in that case, if T_L is commutative as a V -monad, then L is commutative as a countable Lawvere V -theory. Commutativity for a V -monad was characterised by Kock in [21] essentially as follows:

Theorem 24 *Let V be locally countably presentable as a symmetric monoidal closed category, and let T be a commutative V -monad with countable rank on V . Then the V -category $T\text{-Alg}$ is symmetric monoidal closed, and the canonical V -adjunction $F \dashv U : T\text{-Alg} \rightarrow V$ is a symmetric monoidal closed adjunction.*

The last line of the theorem requires some explanation. It implies that the left adjoint F preserves symmetric monoidal structure up to coherent isomorphism, but it does not imply that the forgetful functor $U : T\text{-Alg} \rightarrow V$ does so. It implies that the latter is a *symmetric monoidal functor*, i.e., it comes equipped with maps $UA \otimes UB \rightarrow U(A \otimes B)$ and $1 \rightarrow UI$ satisfying coherence conditions.

A map of the form $(X, x) \otimes (Y, y) \rightarrow (Z, z)$ in $T\text{-Alg}$ may be characterised as a map $f : X \otimes Y \rightarrow Z$ in V that acts as a map of T -algebras with respect to each of (X, x) and (Y, y) separately. We can duly express our development so far directly and a little more generally in terms of models of L as follows:

Corollary 25 *Let L be a commutative countable Lawvere V -theory and let C be a cocomplete symmetric monoidal closed V -category. Then the V -category $\text{Mod}(L, C)$ is canonically symmetric monoidal closed, and the forgetful V -functor $U : \text{Mod}(L, C) \rightarrow C$ is part of a symmetric monoidal closed adjunction. The maps $M \otimes N \rightarrow P$ amount to maps in C of the form $UM \otimes UN \rightarrow UP$ that respect the L -structure of each of M and N separately.*

So, for the unenriched setting, given a finitary Lawvere theory L and a commutative Lawvere theory L' , a model of $L \triangleright L'$ in Set amounts essentially to a model of L in $\text{Mod}(L', \text{Set})$ but using the canonical symmetric monoidal closed structure of $\text{Mod}(L', \text{Set})$ rather than its finite product structure. The details of such a characterisation require a little care at two points: because we are not dealing with the finite product structure of $\text{Mod}(L', \text{Set})$, we cannot quite speak of a model of L in it as we need some way to discard the diagonals of L ; and, given such a partial model $M : L \rightarrow \text{Mod}(L', \text{Set})$, we need to add coherence detail to relate the composite $UM : L \rightarrow \text{Set}$, a symmetric monoidal functor from L to Set , with a model in the usual sense of L in Set .

Our assumption of finitariness of L here is merely a convenience: the tensor product on $\text{Mod}(L, C)$ extends readily to a countable tensor determined by a colimit similar to that in the usual construction of the canonical symmetric monoidal structure on $\text{Mod}(L, C)$; we shall explore it more in the next section.

But assuming finiteness, we can characterise $Mod(L \triangleright L, C)$ as follows.

Definition 26 *An operad consists of, for each natural number n , a set O_n , and for each $(n, (m_i)_{i \in n})$, a composition operation $\cdot : O_n \times \prod_{i \in n} O_{m_i} \longrightarrow O_{\sum_{i \in n} m_i}$, and an identity operation $1 \in O_1$, such that composition is associative and 1 yields both left and right identity for composition.*

The definition of operad extends routinely to give a category $Oper$, the maps of which are indexed families of functions respecting composition and identities.

Every Lawvere theory L has an underlying operad $Op(L)$ defined by putting $Op(L)_n = L(n, 1)$, with the composition of $Op(L)$ determined by that of L together with the bifactoriality of product in L , with the identity given by the identity of L at 1 . This yields a functor $Op : Law \longrightarrow Oper$.

Proposition 27 *The functor $Op : Law \longrightarrow Oper$ has a left adjoint F .*

We shall use the left adjoint, together with cocompleteness of Law , to characterise $L \triangleright L'$, which is the free theory generated by all operations of L and L' such that the operations of $Op(L)$ distribute over all operations of L' .

One can define the notion of model of an operad in any symmetric monoidal category. For an object X of a symmetric monoidal category C and a natural number n , let $X^{(n)}$ denote a choice of n -fold tensor product of X .

Definition 28 *Let O be an operad and let C be a symmetric monoidal category. A model of O in C consists of an object X of C together with, for each $f \in O_n$, a map $M(f) : X^{(n)} \longrightarrow X$, respecting the composition and identity. (Equivalently, we have a map of operads from O to the endomorphism operad of X .)*

Any symmetric monoidal functor $U : C \longrightarrow D$ induces, by composition, a functor $Mod(O, U) : Mod(O, C) \longrightarrow Mod(O, D)$. So for any operad O , any commutative Lawvere theory L' , and any cocomplete cartesian closed category C (for instance $C = Set$), the forgetful functor $U' : Mod(L', C) \longrightarrow C$ induces a functor $Mod(O, U') : Mod(O, Mod(L', C)) \longrightarrow Mod(O, C)$. It follows for general reasons [3,19] that the category $Mod(O, Mod(L', C))$ is of the form $Mod(O * L', C)$ for some Lawvere theory $O * L'$, and the functor $Mod(O, U')$ is of the form $Mod(h, C)$ for some map of Lawvere theories $FO \longrightarrow O * L'$.

Theorem 29 *For any Lawvere theory L and commutative Lawvere theory L' ,*

and for any cocomplete cartesian closed category C , the pullback

$$\begin{array}{ccc}
 P & \longrightarrow & \text{Mod}(Op(L), \text{Mod}(L', C)) \\
 \downarrow & & \downarrow \text{Mod}(Op(L), U') \\
 \text{Mod}(L, C) & \xrightarrow{U} & \text{Mod}(Op(L), C)
 \end{array}$$

is given by $\text{Mod}(L \triangleright L', C)$.

PROOF. An object of P consists of an object X of C together with structures for each of a model of L and a model of L' in C , such that the operations of $Op(L)$ distribute over all operations of L' , i.e., a model of $L \triangleright L'$.

Corollary 30 *The Lawvere theory $L \triangleright L'$ is characterised by the pushout in Law*

$$\begin{array}{ccc}
 FOp(L) & \longrightarrow & Op(L) * L' \\
 \downarrow & & \downarrow \\
 L & \longrightarrow & L \triangleright L'
 \end{array}$$

A countable enriched version of the above is as follows:

Definition 31 *If V is locally countably presentable as a symmetric monoidal closed category, a countable V -operad consists of, for each object a of \aleph_1 , an object O_a of V , and for each $(a, (b_i)_{i \in a})$, a composition operation $\cdot : O_a \otimes \prod_{i \in a} O_{b_i} \longrightarrow O_{\sum_{i \in a} b_i}$, and an identity operation $1 : 1 \longrightarrow O_1$, such that composition is associative and 1 yields both left and right identity for composition.*

The definition of countable V -operad extends routinely to give a category $V\text{-Oper}$, the maps of which are indexed families of maps in V respecting composition and identities.

Every discrete countable Lawvere V -theory L has an underlying countable V -operad $Op(L)$ defined by putting $Op(L)_a = L(a, 1)$, with composition determined by the composition of L and with the identity given by the identity of L at 1 . This yields a functor $Op : DLaw_V \longrightarrow V\text{-Oper}$.

Proposition 32 *The functor $Op : DLaw_V \longrightarrow V\text{-Oper}$ has a left adjoint F .*

It is routine to generalise Definition 28 to define the notion of a model of a countable V -operad in a category C with a countable generalisation of symmetric monoidal V -structure. But the axioms for the countable version are a little complex, so we shall not develop it here. For a model of a finitary V -theory, one routinely generalises the construction, for each $f \in O_n$ of $M(f)$, in Definition 28 to the request for a map in V from O_n to $C(X^{(n)}, X)$, subject to the evident coherence conditions.

Using that definition, and assuming that the symmetric monoidal structures of V and C are cartesian, we obtain exactly the same characterisation in the enriched setting as we have above for the unenriched setting as follows:

Theorem 33 *Let V be locally countably presentable as a cartesian closed category, and let C be a cocomplete cartesian closed V -category. For any finitary discrete Lawvere V -theory L and any commutative discrete countable V -theory L' , the pullback in $V\text{-Cat}$*

$$\begin{array}{ccc}
 \text{Mod}(L \triangleright L', C) & \longrightarrow & \text{Mod}(\text{Op}(L), \text{Mod}(L', C)) \\
 \downarrow & & \downarrow \text{Mod}(\text{Op}(L), U') \\
 \text{Mod}(L, C) & \xrightarrow{U} & \text{Mod}(\text{Op}(L), C)
 \end{array}$$

is given by $\text{Mod}(L \triangleright L', C)$.

PROOF. A proof is given by the same argument as that for Theorem 29, with the additional routine checking that the argument enriches and is not dependent upon finitariness of Set .

As we have mentioned before, our results do not make any substantial use of the discreteness of L' . They can also routinely be extended to dropping finitariness of L , and, at expense of the relationship with commutative monads, does not require V to be cartesian or C to be closed.

Corollary 34 *For a finitary discrete Lawvere V -theory L and a commutative discrete countable Lawvere V -theory L' , the discrete Lawvere V -theory $L \triangleright L'$*

is characterised by the pushout in $D\text{Law}_V$

$$\begin{array}{ccc}
F\text{Op}(L) & \longrightarrow & \text{Op}(L) * L' \\
\downarrow & & \downarrow \\
L & \longrightarrow & L \triangleright L'
\end{array}$$

8 Characterising distributivity in general

In this section, we drop the assumptions of finitariness of L , commutativity of L' , cartesianness of V and closedness of C , yet still characterise the distributive tensor $L \triangleright L'$, albeit in somewhat less agreeable terms.

Definition 35 An \aleph_1 -multi- V -category C consists of

- a set $\text{Ob}C$ of objects
- for each a in \aleph_1 , each a -indexed family $(X_i)_{i \in a}$ of objects of C and each object X of C , an object $C((X_i)_{i \in a}, X)$ of V
- composition and identity data

such that composition is associative and the identities act as identities.

For simplicity of exposition, we shall express the following proposition in un-enriched terms: the enrichment is routine but looks a little less clear.

Proposition 36 Let L be a countable Lawvere theory and let C be a category with countable products. The following data forms an \aleph_1 -multicategory we call $\text{Mod}(L, C)$:

- an object is a model $M : L \rightarrow C$
- $\text{Mod}(L, C)((M_i)_{i \in a}, M)$ is given by the subset of $C(\prod_{i \in a} M_i 1, M 1)$ determined by those maps $f : \prod_{i \in a} M_i 1 \rightarrow M 1$ such that for each $i \in a$ and each map $f' : n' \rightarrow 1$ in L , the diagram, suppressing commutativities,

$$\begin{array}{ccc}
(M_i 1)^{n'} \times \prod_{i' \in (a - \{i\})} M_{i'} 1 & \longrightarrow & (\prod_{i \in a} M_i 1)^{n'} \xrightarrow{f^{n'}} (M 1)^{n'} \\
\downarrow M_i f' \times 1 & & \downarrow M f' \\
\prod_{i \in a} M_i 1 & \xrightarrow{f} & M 1
\end{array}$$

commutes

- *composition and identities lifted from C*

The principal difference between this and the work of Section 7 is that here we do not demand the existence and functoriality of a representing object in $Mod(L, C)$, i.e., a tensor product in $Mod(L, C)$ with universal property determined by Proposition 36. Our only use of commutativity of L in Section 7 was to force the existence and functoriality of such an object. The second main difference is that we use product structure of C rather than symmetric monoidal structure: the only reason for using symmetric monoidal structure in Section 7 was to characterise the work in terms of commutative monads.

The notion of a *model* of a countable operad in any \aleph_1 -multicategory C is evident, upon which we can consider the category $Mod(O, C)$: note that this is naturally a category rather than a multicategory. We can now routinely emulate the latter part of Section 7 to characterise $Mod(L \triangleright L', C)$ in terms of the V -category of models of $Op(L)$ in the \aleph_1 -multi- V -category $Mod(L', C)$. Note that a V -category C with countable products can trivially be regarded as an \aleph_1 -multi- V -category, for instance by a trivial application of Proposition 36.

Theorem 37 *For discrete countable Lawvere V -theories L and L' and V -category C with countable products, the following diagram forms a pullback in V -Cat:*

$$\begin{array}{ccc}
 Mod(L \triangleright L', C) & \longrightarrow & Mod(Op(L), Mod(L', C)) \\
 \downarrow & & \downarrow Mod(Op(L), U') \\
 Mod(L, C) & \xrightarrow{U} & Mod(Op(L), C)
 \end{array}$$

PROOF. A proof of this is given by the proof of Theorem 29, subject to the observations that the proof generalises to allow for enrichment and to the situation of $Mod(L', C)$ being an \aleph_1 -multi- V -category without the necessity of that structure inducing symmetric monoidal structure.

Corollary 38 *For discrete countable Lawvere V -theories L and L' , the dis-*

crete Lawvere V -theory $L \triangleright L'$ is characterised by the pushout in $DLaw_V$

$$\begin{array}{ccc}
 FOp(L) & \longrightarrow & Op(L) * L' \\
 \downarrow & & \downarrow \\
 L & \longrightarrow & L \triangleright L'
 \end{array}$$

We again remark that the discreteness of L' is only used incidentally here.

9 Image

The image construction is of a somewhat different nature to those we have studied so far, and it is used for a somewhat different purpose. Here, rather than starting with a pair of Lawvere theories, one starts with a single Lawvere theory and a model of it. We first consider a construction that is not quite what we want but which does exist for arbitrary countable Lawvere V -theories.

Definition 39 *Given a countable Lawvere V -theory L and a model $M : L \rightarrow C$, the full image $L_f M$ of M is the (bijective-on-objects, fully faithful) factorisation of M , i.e., up to isomorphism of V -categories, it is determined by putting*

$$(L_f M)(m, n) = C(Mm, Mn)$$

A variant of this, where one removes the size limitation at the cost of considerably less elegance, appears in the study of continuations, where M is typically taken to be the free model on a test set R [9]. But what we want to do here is a little more subtle. First we consider the situation for ordinary Lawvere theories.

The image, as opposed to the full image, appears when one wants to take observations rather than equations as primitive [33]. The idea is that one only considers the signature with which one starts, and then puts equations or inequations between derived terms depending upon what an observational model demands. One cannot just take the full image, i.e., the (bijective-on-objects, fully faithful) factorisation, as that would include maps that are not generated by a signature. So we need a more subtle factorisation, one which, in a precise sense, moves the fullness from the right to the left of the factorisation system. The factorisation system we need is defined as follows:

Definition 40 *Given a Lawvere theory L and a model $M : L \rightarrow C$, the*

image L_M of M is determined by the (bijective-on-objects and full, faithful) factorisation of M , i.e., up to isomorphism of categories, it is determined by putting $L_M(m, n)$ equal to the image of the function

$$M_{m,n} : L(m, n) \longrightarrow C(Mm, Mn)$$

Example 41 Following Example 3, let $S = V^L$. The standard semantics of a command is generally understood to be a state-changing function, i.e., a function of the form $S \longrightarrow S$. So the operations lookup and update should act on powers of the set S . They are generally deemed to act as follows: the operation lookup is modelled by the function

$$(S \rightarrow S)^V \longrightarrow (S \rightarrow S)^L$$

determined by composition with the function from $L \times S$ to $V \times S$ that, given (loc, σ) , “looks up” loc in $\sigma : L \rightarrow V$ to determine its value, and is given by the projection to S ; and the operation update is modelled by the function

$$(S \rightarrow S) \longrightarrow (S \rightarrow S)^{L \times V}$$

determined by composition with the function from $L \times V \times S$ to S that, given (loc, v, σ) , “updates” $\sigma : L \rightarrow V$ by replacing the value at loc by v . We wish to set a pair of operations generated by lookup and update equal precisely when they yield the same functions on powers of $S \rightarrow S$. And that is given as follows: first take the free countable Lawvere theory $L_{u,l}$ generated by operations lookup and update as in Example 3 but without imposing the axioms of Example 3. Then take the model $M : L_{l,u} \longrightarrow Set$ of $L_{l,u}$ determined by the set $S \rightarrow S$ together with the functions defined above. Now take the image determined by M . The result is exactly L_S as defined in Example 3.

The above example appears, along with several others, in [33]. The difficulty with enriching the idea arises even when V is *Poset*. If one replaces categories by V -categories with finite cotensors in the definition of image, the factorisation need not have finite cotensors: the problem arises when one considers cotensors with non-discrete posets such as Sierpinski space. Looking harder at why that is not a problem for ordinary Lawvere theories, one notes that the homs are sets, and in *Set*, all epimorphisms are retracts, so are preserved by all functors, whereas that is not the case in *Poset*. In contrast, products often do preserve epimorphisms or at least strong epimorphisms. For instance, in *Poset*, an epimorphism is pointwise a surjective function, and the product of surjective functions is again surjective.

For simplicity of exposition and because it includes all our examples, we shall assume that V is cartesian closed: one needs to add some routine additional assumptions if it is not. The central result we need is as follows:

Theorem 42 *Given a factorisation system $(\mathcal{E}, \mathcal{M})$ on V for which \mathcal{E} is closed under countable products, and given a discrete countable Lawvere theory L and a model $M : L \rightarrow C$, taking the $(\mathcal{E}, \mathcal{M})$ -factorisation of the map in V*

$$M_{a,b} : L(a, b) \rightarrow C(Ma, Mb)$$

yields a factorisation of M as a V -functor that preserves countable products.

PROOF. The statement of the theorem determines the objects and homs of the putative image of M : the objects are those of L , with $L_M(a, b)$ given by the factorisation. We must now determine the composition of L_M . Consider the diagram

$$\begin{array}{ccccc} L(b, c) \times L(a, b) & \longrightarrow & L_M(b, c) \times L_M(a, b) & \longrightarrow & C(Mb, Mc) \times C(Ma, Mb) \\ \text{comp}_L \downarrow & & & & \downarrow \text{comp}_C \\ L(a, b) & \longrightarrow & L_M(a, b) & \longrightarrow & C(Ma, Mc) \end{array}$$

with the upper and lower arrows determined by factorising. Because \mathcal{E} is closed under countable products, it is necessarily closed under finite products, and so the upper left map lies in \mathcal{E} . The lower right arrow lies in \mathcal{M} . So the factorisation condition yields a unique composition for L_M that allows M to factorise as a V -functor. It is routine to verify associativity and unit conditions for L_M . The unicity up to isomorphism of a factorisation system, together with the closedness of \mathcal{E} under products, allows one, by a similar argument, to deduce that L_M has countable products and that the induced V -functors from L and to C preserve countable products.

It follows from the statement of Theorem 42 that the factorisation L_M is a discrete countable Lawvere V -theory. That allows us to make the following definition:

Definition 43 *Given a factorisation system $(\mathcal{E}, \mathcal{M})$ on V for which \mathcal{E} is closed under countable products, and given a discrete countable Lawvere theory L and a model $M : L \rightarrow C$, the $(\mathcal{E}, \mathcal{M})$ -image L_M of M is determined by the (bijective-on-objects and locally $(\mathcal{E}, \mathcal{M})$) factorisation of M , i.e., determined by putting $L_M(a, b)$ equal to the $(\mathcal{E}, \mathcal{M})$ -factorisation of the map in V*

$$M_{a,b} : L(a, b) \rightarrow C(Ma, Mb)$$

Example 44 *Let V be Poset and let \mathcal{E} be the class of epimorphisms, which are precisely the pointwise epimorphisms. So \mathcal{M} is necessarily the class of*

strong monomorphisms. The pointwise surjections of *Poset* are closed under countable product, so satisfy the conditions of Theorem 42. Thus one can, for instance, give a *Poset*-enriched version of Example 41. As remarked before, the pointwise surjections in *Poset* are not closed under cotensor, so the restriction to discrete theories is essential here.

Enrichment in the category ωCpo requires more care, as countable products there involve more subtlety than finite products do: a finite product of epimorphisms is always an epimorphism by cartesian closedness. The solution to that difficulty seems likely to involve making more subtle use of the relationships developed in this paper between discrete countable Lawvere V -theories and arbitrary countable Lawvere V -theories, and their relationships with finitary versions.

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