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A Category Theoretic Formulation for Engeler-style Models of the Untyped λ-Calculus

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Abstract

We give a category-theoretic formulation of Engeler-style models for the untyped λ-calculus. In order to do so, we exhibit an equivalence between distributive laws and extensions of one monad to the Kleisli category of another and explore the example of an arbitrary commutative monad together with the monad for commutative monoids. On *Set* as base category, the latter is the finite multiset monad. We exploit the self-duality of the category *Rel*, i.e., the Kleisli category for the powerset monad, and the category theoretic structures on it that allow us to build models of the untyped λ-calculus, yielding a variant of the Engeler model. We replace the monad for commutative monoids by that for idempotent commutative monoids, which, on *Set*, is the finite powerset monad. This does not quite yield a distributive law, so requires a little more subtlety, but, subject to that subtlety, it yields exactly the original Engeler construction.

Keywords: Engeler model, untyped λ-calculus, cartesian closed category, distributive law, commutative monoid, finite multiset, finite powerset, Kleisli construction

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1 Introduction

Dana Scott proved that any model of the untyped λ-calculus, or more directly any λ-algebra, generates a cartesian closed category in which the model can be seen as a reflexive object, i.e., an object D together with data exhibiting the exponential D^D as a retract of D [20]. One obtains such a category by taking the Cauchy completion, equivalently the category of retracts, generated by the model. Although a fine completeness result from a category-theoretic perspective, this result does not imply that any λ-algebra appears as an object of a *natural* cartesian closed category such as the category ωCpo or some category built from Rel. So, it is an ongoing natural question, from a category theoretic perspective, to see the various models of the untyped λ-calculus as reflexive objects in natural cartesian closed categories. In this paper, we investigate the particular situation of models in the spirit of those proposed by Engeler in [8], also investigated in [17].

Engeler's original models for the untyped λ-calculus are given by taking a set D for which $P_f D \times D$ is a subset of D, where $P_f X$ is the set of finite subsets of X. The set $P_f D \times D$ acts as a kind of exponential of D to itself. Every subset is the splitting of a retraction, thus $P_f D \times D$ is a retract of D. If one could make the sense in which $P_f D \times D$ is a kind of exponential precise via a natural construction, we would have answered our question.

Consider the following heuristic argument: if there was a distributive law of the monad P_f over the powerset monad P, we would have a lifting of the monad P_f to Rel. The latter category is self-dual, so the lifting can be seen as a comonad on Rel. The category Rel is symmetric monoidal closed with products and coproducts, and the lifting sends products to the symmetric monoidal structure of Rel, thereby making the Kleisli category for the comonad cartesian closed. It follows from the structure of Rel that the closed structure in the Kleisli category would be given by $P_f X \times Y$. With a little calculation, one could thus see the Engeler construction as a reflexive object of that Kleisli category. However, the argument fails: the natural construction does not yield a distributive law of P_f qua monad over P.

But we can step back. Suppose we replace P_f by M_f, the finite multiset monad. This is the monad for commutative monoids. We can prove that there is always a distributive law of the monad for commutative monoids over any commutative monad T on any base cocomplete symmetric monoidal closed category C: see Section 2. It follows axiomatically that M_f extends to a monad on the Kleisli category $Kl(T)$ of T: see Section 3. The Kleisli category $Kl(T)$ has a canonical symmetric monoidal structure, and the extension $\tilde{M}_f$ of M_f yields the monad for the category of commutative monoids in $Kl(T)$: see Section 4. If we then restrict to Set as base category and take T to be the powerset monad P, it follows that $Kl(T)$ is self-dual and is routinely seen to have enough extra structure to make $Kl(\tilde{M}_f)^{op}$ cartesian closed: we can give a more general analysis of parts of that. With a little calculation, it follows that this provides a mild variant of Engeler's models: see Section 5. For a range of other variants of Engeler's models, see [17].

The difference between M_f and P_f is instructive: M_f is the monad for commutative monoids, while P_f is the monad for idempotent commutative monoids. The latter involves a cartesian equation, namely $x + x = x$. It is this repetition of x on the left-hand side of the equation that makes the crucial difference here. (How the problem presents itself will be described in Example 2.4.) The argument we gave above for extension of a monad goes through without fuss in more general situations providing one only considers symmetric operadic structure rather than general algebraic structure: we give background in Section 2. But it does not extend to P_f without further effort.

However, returning to Engeler's original construction, suppose we try to find a distributive law of the monad P_f over P. We can readily find a distributive law of P_f qua endofunctor over P: see Section 3. And we can lift its multiplication natural transformation. The only difficulty is that the unit of the monad P_f does not lift from Set to Rel, i.e., it is natural in Set but not in Rel. But that is not so bad: the endofunctor and the multiplication allow us to build a Kleisli construction, but it is a semicategory rather than a category. And semifunctors, which are closely related to semicategories, were investigated precisely in regard to modelling untyped λ-calculus by Hayashi in [9]. Semicategories have long been investigated in category theory, see for instance [11]. The data for the unit still exists, and it provides pointwise idempotents. And that is enough for us to mimic the above argument, modulo the mild additional subtlety involved with taking and splitting idempotents, returning Engeler's original model as we explain in Section 5. A fully axiomatic treatment of this more refined argument will appear in a subsequent paper.

This work leaves one striking open question that we are keen to pursue: Engeler models are a simple case of filter models for the untyped λ-calculus. So, with a category theoretic formulation for Engeler models in hand, as future work, we intend to extend our category theoretic analysis to account for filter models. Since at least some filter models are naturally domains, this may require a reconstruction of domain theory. That is potentially a large job, hence our deferring it.

2 Distributive Laws and Liftings

Given a pair of monads S and T on a category C, the following result appears widely in the literature, e.g., in [1].

Theorem 2.1 *To give a distributive law of monads*

$$\lambda : ST \Rightarrow TS$$

of S over T is equivalent to giving a lifting of the monad T to the category S-Alg.

In this paper, we shall focus on a class of examples of such distributive laws, which we shall describe, where S is the monad for commutative monoids in a symmetric monoidal category C, when such a monad exists, and T is an arbitrary commutative monad on C. We briefly recall the relevant definitions.

Given a symmetric monoidal category C, a *strength* for a monad (T, η, μ) on C is a natural transformation with components of the form $t_{X,Y} : X \otimes TY \longrightarrow T(X \otimes Y)$ satisfying four axioms expressing coherence with respect to the monad structure of

T and the symmetric monoidal structure of C [15]. A monad with a strength is called *commutative* if the diagram

$$\begin{array}{ccccc} TX \otimes TY & \xrightarrow{t_{TX,Y}} & T(TX \otimes Y) & \xrightarrow{Tt^*_{X,Y}} & T^2(X \otimes Y) \\ {\scriptstyle t^*_{X,TY}} \downarrow & & & & \downarrow {\scriptstyle \mu_{X \otimes Y}} \\ T(X \otimes TY) & \xrightarrow[Tt_{X,Y}]{} & T^2(X \otimes Y) & \xrightarrow[\mu_{X \otimes Y}]{} & T(X \otimes Y) \end{array}$$

commutes for all X and Y, where t^* is defined from t using the symmetry of C [15].

Given an arbitrary symmetric monoidal category C, one can readily define the category $CMon(C)$ of commutative monoids in C. It inherently comes equipped with a forgetful functor $U : CMon(C) \longrightarrow C$. In full generality, the forgetful functor need not have a left adjoint. But it does have a left adjoint in a very wide class of cases, including all that are of primary interest to us: see [14] for some such general conditions. For a far more restricted class than necessary but one that includes our leading examples, if C is closed and cocomplete, the left adjoint exists. In the case that C is Set, the monad for commutative monoids in C is M_f, the finite multiset monad. Extending that notation, we shall denote by M_f the monad for commutative monoids in any symmetric monoidal category for which the left adjoint and hence the monad exists.

It is routine to verify that $U : CMon(C) \longrightarrow C$ always satisfies the other conditions of Beck's monadicity theorem, so the existence of the left adjoint is sufficient to prove monadicity of $CMon(C)$ over C [14]. Putting this together, we have the following.

Theorem 2.2 *If C is a cocomplete symmetric monoidal closed category, the category $CMon(C)$ of commutative monoids in C is monadic over C.*

The theorem allows us to express our leading class of examples of distributive laws as follows, cf [15].

Example 2.3 Let C be a cocomplete symmetric monoidal closed category, and let T be a commutative monad on C. It is routine to verify that T lifts to the category $CMon(C)$ of commutative monoids in C. So, Theorem 2.1 yields a distributive law of M_f over T. In particular, taking C to be Set, this induces a canonical distributive law of the monad M_f for finite multisets over any commutative monad T.

This example evidently extends, generalising from the category $CMon(C)$ of commutative monoids in C to the category of models in C of any symmetric operad. We shall not develop that point further here but we intend to do so in future work. For a class of non-examples of a distributive law of monads, but in the same spirit as Example 2.3 and whose structure we shall consider in detail later, consider the following.

Example 2.4 Let C be a category with finite products and let T be a commutative monad on it. Attempting to restrict Example 2.3 to the category $ICMon(C)$ of idempotent commutative monoids (semi-lattices) in C, one fails: the lifting described in Example 2.3 sends an idempotent commutative monoid to a commutative monoid that need not be idempotent. For the specific example relevant to this paper, let C be Set and let $T = P$ be the usual power-set monad. Now the lifting from Example 2.3 amounts to the following. If $(A, \cdot)$ is a commutative monoid, then $P(A)$ inherits the structure of a commutative monoid with multiplication given by $X \cdot Y = \{a \cdot b \,|\, a \in X,\ b \in Y\}$. But for A idempotent, $P(A)$ is generally not idempotent (with that multiplication). Now the monad for idempotent commutative monoids is P_f, the finite powerset monad, and we see that the distributive law of Example 2.3 does not quotient to give a distributive law of the monad P_f over the monad P. (It is routine to check the failure of lifting directly.)

It follows from the ideas in [21] that one can generalise Theorem 2.1 and the definitions in it to happen inside a 2-category subject to mild axiomatic conditions. Rather than have a category C, one has an object of a 2-category; similarly for functors and natural transformations; one can readily generalise the construction of the category S-Alg to a construction within a 2-category with some finite 2-categorical limits. The work of the next section, where we investigate the Kleisli construction $Kl(S)$ rather than S-Alg, can also be done in Street's setting and can be seen as a kind of dual.

3 Distributive Laws and Kleisli Extensions

Given a monad (T, η, μ) on a category C, we denote the Kleisli category for (T, η, μ) by $Kl(T)$, and we denote the canonical identity-on-objects functor from C to $Kl(T)$ by $J : C \longrightarrow Kl(T)$. Note that the functor J need not be faithful, so need not be an inclusion. Nevertheless, it usually is an inclusion, and it is usually harmless to think of it as such. In fact, the functor J is faithful if and only if η is a pointwise monomorphism.

Definition 3.1 Given a monad (T, η, μ) on a category C, and an endofunctor H on C, an *extension* of H to $Kl(T)$ is an endofunctor $\tilde{H}$ on $Kl(T)$ such that $\tilde{H}J = JH$.

Observe that, in the definition, we demand an equality of functors, not merely an isomorphism. That is not only convenient in avoiding coherence conditions, but it is also essential to providing a precise and reasonable result.

Definition 3.2 A *distributive law* of an endofunctor H over a monad T on a category C is a natural transformation

$$\lambda : HT \Rightarrow TH$$

subject to commutativity of the evident two diagrams expressing coherence with respect to the unit and multiplication of T.

Proposition 3.3 *To give a distributive law of an endofunctor H over a monad T is equivalent to giving an extension of H to $Kl(T)$.*

Proof. To go from a distributive law to an extension is routine. And, given an extension $\tilde{H}$, applying $\tilde{H}$ to the map in $Kl(T)$ from TX to X given by id_{TX} in C, i.e., to the counit of the canonical adjunction, yields a distributive law. It is routine to verify that the two constructions are mutually inverse. □

Given a distributive law $\lambda : HT \Rightarrow TH$, we denote the induced extension by $\tilde{H}_\lambda$. The combination of Theorem 2.1, Example 2.3 and Proposition 3.3 immediately yields a class of examples of extensions for us.

Example 3.4 Let C be a cocomplete symmetric monoidal closed category and let T be a commutative monad on C. As explained in Example 2.3, T lifts to a monad on the category $CMon(C)$ of commutative monoids in C. By Theorem 2.1, this lifting yields a canonical distributive law of the monad M_f for commutative monoids over T. A fortiori, this is a distributive law of M_f qua endofunctor over T. So, applying Proposition 3.3 yields a canonical extension of the functor M_f to $Kl(T)$.

Example 3.5 Taking C to be Set and P to be the powerset monad, the distributive law of Example 3.4 quotients to give a distributive law of the finite powerset functor P_f over P and hence an extension of P_f to Rel, i.e., to $Kl(P)$. This result does not seem to hold for an arbitrary commutative monad T in place of P. The distributive law sends a finite subset A of $P(X)$ to the subset Y of $P_f X$ determined by those finite subsets B of X for which $\forall b \in B \exists a \in A. b \in a$ and $\forall a \in A \exists x \in a. x \in B$.

Trivially, an extension of an endofunctor H to $Kl(T)$ induces an extension of the composite HH to $Kl(T)$: one requires a little care as there may be more than one extension of H to $Kl(T)$. It is routine to verify that a distributive law $\lambda : HT \Rightarrow TH$ induces a distributive law of HH over T given by

$$HHT \xrightarrow{H\lambda} HTH \xrightarrow{\lambda H} THH$$

which we denote by λ_2. It is straightforward to prove the following result.

Proposition 3.6 *Given a distributive law λ of an endofunctor H over a monad T, the extension $\tilde{H}_{\lambda_2}$ is exactly the composite $\tilde{H}_\lambda \tilde{H}_\lambda$.*

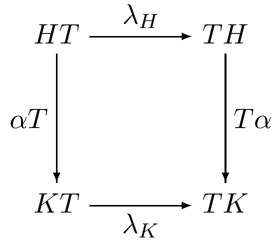
We extend the notion of an extension of an endofunctor to $Kl(T)$ and the equivalence with a distributive law to the situation of a natural transformation between endofunctors as follows.

Definition 3.7 Given a monad T on a category C, given endofunctors H and K on C and extensions $\tilde{H}$ and $\tilde{K}$ of H and K to $Kl(T)$, and given a natural transformation $\alpha : H \Rightarrow K$, an *extension* of α to $Kl(T)$ is a natural transformation $\tilde{\alpha} : \tilde{H} \Rightarrow \tilde{K}$ such that $\tilde{\alpha} J = J\alpha$.

An extension of a natural transformation is unique if it exists: the data for $\tilde{\alpha}$ is determined by the fact that $J : C \longrightarrow Kl(T)$ is the identity-on-objects, so the only question is whether the data satisfies the naturality axiom with respect to $\tilde{H}$ and $\tilde{K}$. Observe that we make mild abuse of notation here: we speak of α extending to

$Kl(T)$ whereas its extension really relies upon a pre-existing choice of extensions of H and K to $Kl(T)$, of which there may be many.

Definition 3.8 Given endofunctors H and K on a category C, a natural transformation $\alpha : H \Rightarrow K$, and a monad T on C, and given distributive laws $\lambda_H : HT \Rightarrow TH$ and $\lambda_K : KT \Rightarrow TK$, we say α *distributes* over T if the diagram



commutes.

Proposition 3.9 *Given distributive laws $\lambda_H : HT \Rightarrow TH$ and $\lambda_K : KT \Rightarrow TK$, and a natural transformation $\alpha : H \Rightarrow K$, the natural transformation α distributes over T if and only if there is an extension (necessarily unique) of α to $Kl(T)$.*

The proof is routine.

We can combine the above propositions to yield easy proofs of several results. We do not spell out the definitions of a distributive law of each of a pointed endofunctor, a copointed endofunctor, a monad, and a comonad, over a monad: the four definitions are routine consequences of the above definitions and propositions (see [16] for the cases involving a comonad). But writing down the result of primary interest to us that flows from the above analysis, we have the following.

Definition 3.10 Given monads (S, η, μ) and T on a category C, an *extension* of S to $Kl(T)$ is a monad $(\tilde{S}, \tilde{\eta}, \tilde{\mu})$ on $Kl(T)$ such that $\tilde{S}$, $\tilde{\eta}$, and $\tilde{\mu}$ extend S, η and μ respectively.

Theorem 3.11 *Given monads S and T on a category C, to give a distributive law of S over T is equivalent to giving an extension of the monad S to $Kl(T)$.*

This immediately allows us to develop Example 3.4.

Example 3.12 Let C be a cocomplete symmetric monoidal closed category, and let T be a commutative monad on C. Then the monad M_f for commutative monoids in C extends to $Kl(T)$.

Example 3.13 By dint of Theorem 2.1 and Example 2.4, Example 3.12 does not induce an extension of the monad P_f from *Set* to *Rel*: the latter is $Kl(P)$ and we know that the distributive law of M_f qua monad over P does not quotient to one of P_f over P, as otherwise the lifting of P to $CMon(C)$ would restrict to $ICMon(C)$, as discussed in Example 2.4. However, by Example 3.5, there is a distributive law of P_f qua endofunctor over P. Moreover, it is routine to verify that the natural transformation given by the multiplication of P_f distributes over P too. So the functor P_f together with its multiplication natural transformation extend to *Rel*.

Our development in this section extends readily to the situation of 2-categories and pseudo-monads. Considerably more care is required there with coherence issues, making our perhaps apparently slow progress through this section vital to providing rigorous proofs at that increased level of generality. We shall return to the two-dimensional case in future work, as it coheres with the work of Winskel and colleagues on domain theory for concurrency, where Set is replaced by Cat, the powerset monad is replaced by the presheaf construction, which is a pseudo-monad in every way except for size, and where commutative monoids are replaced by symmetric monoidal categories [4]. The notion of commutative monad generalises to that of pseudo-commutative 2-monad, and our axiomatic development of commutative monads and their relationship with commutative monoids extends to pseudo-commutative 2-monads and symmetric monoidal categories [12]. Details of the definitions and constructions involved with pseudo-distributive laws will appear in [22] but see also [5] for an outline.

4 Symmetric Monoidal Adjunctions and Commutative Monoids

Our attention so far has focused on a commutative monad T on a symmetric monoidal closed category C, and we have considered the relationship between the monad M_f for commutative monoids on C and T, equivalently the extension of M_f to $Kl(T)$. The latter is a statement about the monad structure of M_f. In this section, we address issues that more specifically involve the commutative monoid structure of M_f: but the following does still apply to symmetric operads more generally. The commutative monoid structure of M_f involves the symmetric monoidal structure of C more directly, together with the notion of commutative monoid in C and in $Kl(T)$. We recall some elementary definitions from [7].

Given symmetric monoidal categories C and D, a *symmetric monoidal functor* from A to B is a functor $H : A \longrightarrow B$ together with natural transformations with components $HX \otimes HY \longrightarrow H(X \otimes Y)$ and $I \longrightarrow HI$ subject to four coherence conditions to the effect that the associativity, left and right unit, and symmetry isomorphisms are respected. A symmetric monoidal functor is called *strong* if the structural natural transformations are invertible. A *symmetric monoidal natural transformation* between symmetric monoidal functors H and K is a natural transformation $\alpha : H \Rightarrow K$ that respects the rest of the structure for a symmetric monoidal functor. Small symmetric monoidal categories, symmetric monoidal functors, and symmetric monoidal natural transformations form a 2-category $SymMon$. A *symmetric monoidal adjunction* is an adjunction in the 2-category $SymMon$.

The following result is straightforward to prove and is implicit in [7].

Theorem 4.1 *Every symmetric monoidal adjunction from A to B lifts to an adjunction from $CMon(A)$ to $CMon(B)$.*

That is the result we need, providing we can obtain a symmetric monoidal adjunction between a base category C and $Kl(T)$ for a commutative monad on C.

In fact, we can do that, via the following line of argument. The following result is not in its most general form (see [3]) but is in the form we need here [13].

Theorem 4.2 *Given symmetric monoidal categories A and B and given an ordinary adjunction $F \dashv G : A \longrightarrow B$, to extend the adjunction to an adjunction of symmetric monoidal categories is equivalent to giving a strong symmetric monoidal structure on the ordinary functor $F : B \longrightarrow A$.*

We can combine that characterisation of symmetric monoidal adjunctions with the following result in [19].

Theorem 4.3 *Given a symmetric monoidal category C and a monad T on it, to give a commutative strength for T is equivalent to giving a symmetric monoidal structure on $Kl(T)$ such that the Kleisli adjunction is a symmetric monoidal adjunction.*

The significance of this for us is that, given a commutative monad T on a cocomplete symmetric monoidal closed category C, it yields a characterisation of the extension $\tilde{M}_f$ of M_f to the category $Kl(T)$.

Corollary 4.4 *For any cocomplete symmetric monoidal closed category C and for any commutative monad T on C, the extension $\tilde{M}_f$ of M_f is the monad describing the free commutative monoid in the symmetric monoidal category $Kl(T)$.*

Proof. By Theorem 4.2 and Theorem 4.3, the Kleisli adjunction extends canonically to a symmetric monoidal adjunction between C and $Kl(T)$. By Theorem 4.1, the adjunction lifts to an adjunction between $CMon(C)$ and $CMon(Kl(T))$. In particular, the diagram of categories

$$\begin{array}{ccc} CMon(Kl(T)) & \longrightarrow & CMon(C) \\ \big\downarrow & & \big\downarrow \\ Kl(T) & \longrightarrow & C \end{array}$$

commutes, with the evident labelling of functors. The left adjoint of the composite is $JM_f = \tilde{M}_f J$, and the counit of the symmetric monoidal adjunction extends from $Kl(T)$ to $CMon(Kl(T))$. So $\tilde{M}_f$ must act as the free commutative monoid in $Kl(T)$. □

We have one final axiomatic result here. In some categories with finite products and finite coproducts, notably Set, the monad M_f admits the property that the canonical comparison map

$$M_f(X + Y) \longrightarrow M_f X \times M_f Y$$

is always an isomorphism. For those categories C for which that is true, we seek conditions under which, given a commutative monad T, that fact lifts to $\tilde{M}_f$, by which we mean that the isomorphism induces an isomorphism from $\tilde{M}_f(X + Y)$

to $\tilde{M}_f X \otimes \tilde{M}_f Y$, using the fact that coproducts extend to $Kl(T)$ and using the symmetric monoidal structure on $Kl(T)$ induced by Theorem 4.3. The only issue is one of naturality of the comparison map with respect to $Kl(T)$.

Proposition 4.5 *For any commutative monad T on a cocomplete cartesian closed category C, the extension $\tilde{M}_f$ of M_f sends coproducts in $Kl(T)$ to the symmetric monoidal structure of $Kl(T)$ if the diagram*

$$\begin{array}{ccccc}
M_f(TX+TY) & \longrightarrow & M_fTX \times M_fTY & \longrightarrow & TM_fX \times TM_fY \\
\downarrow & & & & \downarrow \\
M_fT(X+Y) & \longrightarrow & TM_f(X+Y) & \longrightarrow & T(M_fX \times M_fY)
\end{array}$$

with all maps given by the canonical choices, commutes.

Evidently, this result extends beyond M_f to any symmetric operad and to some non-operadic monads such as that for Abelian groups.

In order to analyse Engeler-style models of the untyped λ-calculus as we shall do in the next section, we should like to find axiomatic conditions under which the category $Kl(\tilde{M}_f)^{op}$ for a commutative monad T on *Set* is cartesian closed: we know it has finite products, as $Kl(\tilde{M}_f)$ has finite coproducts. And the above proposition helps, as it means that a map out of a binary product in $Kl(\tilde{M}_f)^{op}$, i.e., a map into a binary coproduct in $Kl(\tilde{M}_f)$, can be seen as a map into a tensor product in $Kl(T)$, but from that point, at present, we do not see how to proceed without using the self-duality and closedness of our leading example, that where T is P and therefore $Kl(T)$ is *Rel*. We can, however, generalise in a non-trivial way to 2-categories: the bicategory $Prof$ of small categories and profunctors is not quite self-dual in the sense we use here, but it has enough self-dual structure to allow us to mimic our argument.

5 Engeler Models

In this section, we finally link our axiomatic development of previous sections with Engeler's construction. We first consider the variant given by finite multisets, as that provides easier and more elegant category-theoretic models of untyped λ-calculus. We then proceed to P_f, as Engeler originally considered. That involves more complex category theory, but it also allows us to make a more precise statement of the relationship.

The following easy result has been fundamental to the semantics of linear logic over many years [2].

Theorem 5.1 *Let C be a symmetric monoidal closed category with finite products, and let G be a comonad on C that sends finite products to the symmetric monoidal structure. Then the Kleisli category $Kl(G)$ is cartesian closed, with exponential given by the linear exponential $[GX, Y]$ in C.*

Example 5.2 The category *Rel* is symmetric monoidal closed, in fact compact closed, with the symmetric monoidal structure given by the product of sets, and it has both products and coproducts given by the coproduct of sets. Moreover, it is self-dual and the linear exponential is given, most unusually, by the product of sets. We know from previous sections that M_f extends to a monad on Rel, hence equally to a comonad on Rel, and the extension sends coproducts to tensors. Thus, if we denote the comonad corresponding to $\tilde{M}_f$ by $\hat{M}_f$, it follows that $Kl(\hat{M}_f)$, equally $Kl(\tilde{M}_f)^{op}$, is cartesian closed. The closed structure is given by $M_f X \times Y$.

Example 5.2 gives us a cartesian closed category. To give a reflexive object of the category is to give a set D together with data to exhibit $M_f D \times D$ as a retract of D in $Kl(\hat{M}_f)$. But retractions are preserved by all functors, and there is a canonical composite inclusion functor from Set to $Kl(\hat{M}_f)$. So, given any retraction in Set, i.e., any set D together with an inclusion of $M_f D \times D$ into D, application of the canonical inclusion yields a reflexive object of $Kl(\hat{M}_f)$. One can obtain sets satisfying the conditions required of D by solving the evident domain equation. Thus the variant of Engeler models given by replacing P_f by M_f may be seen as reflexive objects in the cartesian closed category $Kl(\hat{M}_f)$.

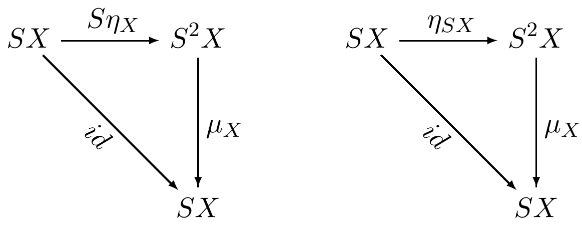
We now consider exactly Engeler's models by replacing finite multisets by finite subsets. In principle, we wish to remain as axiomatic in our development as reasonably possible. But, as discussed in Example 3.13, the natural extension of the monad P_f to Rel does not satisfy the axioms for a monad on Rel, and so does not yield a Kleisli category for a monad on Rel. This leads us to relax the definition of category to consider semicategories: a semicategory consists of the data for a category except for the existence of identity maps, the only axiom being that composition is associative. Semifunctors are graph morphisms that preserve composition, see for instance [11].

We do not assert definitiveness of the following definition, but it is convenient for us in expressing the results of this section.

Definition 5.3 A *near-monad* on a category C consists of an endofunctor S on C, a natural transformation $\mu : S^2 \Rightarrow S$ satisfying the associativity condition

$$\begin{array}{ccc} S^3 & \xrightarrow{\mu S} & S^2 \\ {\scriptstyle S\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ S^2 & \xrightarrow[\mu]{} & S \end{array}$$

and an ObC-indexed family of maps $\eta_X : X \longrightarrow SX$ such that the following diagrams commute:



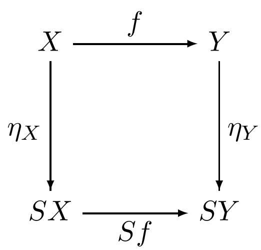
Example 5.4 Our leading example is given by the extension to $Rel = Kl(P)$ on *Set* of the endofunctor P_f and its multiplication natural transformation as in Example 3.13, together with the data for the unit of P_f.

Definition 5.5 Let S be a near-monad on C. Denote by $Kl(S)$ the semicategory whose objects are those of C, with the homset $Kl(S)(X, Y)$ defined to be $C(X, SY)$, and with composition defined by

$$X \xrightarrow{f} SY \xrightarrow{Sg} S^2Z \xrightarrow{\mu_Z} SZ$$

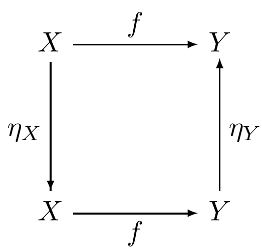
The construction $Kl(S)$ is limited as a generalisation of the Kleisli construction: observe that the canonical graph morphism from C to $Kl(S)$ need not even satisfy the axiom for a semifunctor as it need not preserve composition. So in order to make any axiomatic progress, we need to make ancillary constructions as follows.

Definition 5.6 Given a near-monad S on C, denote by C_η the subcategory of C with the same objects as C, with a map $f : X \longrightarrow Y$ of C lying in C_η if the diagram



commutes.

Definition 5.7 Given a near-monad S on C, denote by $Kl(S)_\eta$ the subsemicategory of $Kl(S)$ determined by all objects of $Kl(S)$ together with those maps f : $X \longrightarrow Y$ for which the following diagram in $Kl(S)$ commutes:



There is a key difference between the above two definitions residing in the direction of the right-hand vertical arrow: in the former definition, it points downward, while in the latter, it points upward. In the former, the arrow can only point downward as there is no natural arrow upward. But in the latter, the direction of the arrow is exactly that required to make $Kl(S)_\eta$ the full subcategory of the idempotent splitting of $Kl(S)$ determined by the idempotents (X, η_X) in $Kl(S)$. The idempotent splitting is the cofree category on the semicategory $Kl(S)$.

Note that the maps η_X need not lie in the category C_η: they do not form a natural transformation, so there is no reason to believe that they are natural with respect to themselves. But they do lie in $Kl(S)_\eta$, where they act as identity maps. And they do lie in C, allowing us to prove the following result.

Proposition 5.8 *Given a near-monad S on C, the canonical graph morphism from C to $Kl(S)$ restricts to a functor $J : C_\eta \longrightarrow Kl(S)_\eta$.*

We can extend this proposition to deal with coproducts in C: in our leading class of examples, C is of the form $Kl(T)$, which always has coproducts if the base category does, and we are trying to make a further Kleisli-like construction on $Kl(T)$ that is functorial, so preserves retracts, and preserves coproducts: the latter become products in the presence of self-duality. In fact, preservation of coproducts is routine.

Proposition 5.9 *Suppose C has finite coproducts. Let S be a near-monad on C for which the coprojections $X_i \longrightarrow X_0 + X_1$ lie in C_η. Then C_η has and $J : C_\eta \longrightarrow Kl(S)_\eta$ preserves finite coproducts. Hence, $Kl(S)_\eta$ has finite coproducts.*

From this point, an axiomatic development seems forced. So we shall restrict to the particular example of primary interest to us, where C is *Rel* and S is given by $\tilde{P}_f$ together with its multiplication and the data for the unit of P_f, i.e., the singleton maps $X \longrightarrow P_f X$.

By Proposition 5.9, the category $Kl(\tilde{P}_f)_\eta$ has finite coproducts, and, as η is natural on *Set*, and using the self-duality of *Rel*, we have a canonical functor from *Set* into $Kl(\tilde{P}_f)_\eta^{op}$. The latter category need not in general be cartesian closed, but, as in [9], its idempotent-splitting is, with the exponential Y^X of sets X and Y given by splitting an evident idempotent on $P_f X \times Y$, exactly analogously to the situation for finite multisets in Example 5.2. So, analogously to Example 5.2 but with one further step, any set D with $P_f D \times D$ exhibited as a subset of D yields the structure making D a reflexive object of the cartesian closed category determined by the idempotent-splitting of $Kl(\tilde{P}_f)_\eta^{op}$: for the exponential D^D is a retract of $P_f D \times D$, which is in turn a retract of D.

Once again, as was the case for finite multisets, in order to obtain a particular reflexive object D, one simply needs solve in *Set* the domain equation $D = A + P_f D \times D$ for countable $A \neq \emptyset$. The least fixpoint yields a countable set, with the binary operation on $P_f(D)$ given by the relation R from $P_f(D) \times P_f(D)$ to D defined as follows:

$$(u, v)Rb \text{ if } (u = \{(u_1, b), \cdots, (u_n, b)\}) \wedge (v = \textstyle\bigcup_i u_i).$$

This generates a set-theoretic λ-model on any homset with codomain D, and, considering the hom of 0 into D yields exactly Engeler's original model [8].

By way of conclusions we say something about the connection between our view of Engeler's model and that current in the literature. (A good survey of constructions from this domain-theoretic point of view is in Plotkin [18].) The simplest story starts from the theory of semifunctors (Hayashi [9]). As shown by Hoofman [10] (but see also [11]) this theory allows the construction of the category POW of power sets and continuous maps using a quite different structure associated to finite subsets. Hoofman observed that Engeler's model can be regarded as lying in (the idempotent completion of) POW which is a category of domains. For us Engeler's model lies naturally in $Kl(\tilde{P}_f)^{op}_\eta$. This latter category is emphatically not a category of domains; specifically it does not have enough points. What we wish to stress then is that Engeler's model works for a combinatorial reason independent of domain theory. However POW is locally a retract of $Kl(\tilde{P}_f)^{op}_\eta$ and in a way which is a bijection on points. In this way one can retrieve the domain-theoretic point of view.

We close by explaining the programme of work which we initiate here. Filter lambda models as introduced by the Torino school (see [6] for example) are usually taken to amount to a presentation of domain theoretic models. Certainly they are lambda *models* as they appear to come from categories with enough points. But our analysis of the Engeler *model* is that it naturally arises from a category without enough points. So in our formulation it is naturally a lambda *algebra* in the established terminology. Now the Engeler model is taken to be part of the general family of filter models. So this raises (at least for us) questions along the following lines. Which filter models really are domain models (after all nobody doubts Scott's D_∞), and which are naturally something else? There seems much more to understand about concrete constructions of models for the lambda calculus (that is, in general about lambda algebras).

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