Proof Theory in the Abstract

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Dedicated to Anne Troelstra on the occasion of his 60th birthday: with great affection and respect, this small tribute to his influence.

1 Background

In the Introduction to the recent text Troelstra and Schwichtenberg [44], the authors contrast structural proof theory on the one hand with interpretational proof theory on the other. They write thus.

> Structural proof theory is based on a combinatorial analysis of the structure of formal proofs; the central methods are cut elimination and normalization.
>
> In interpretational proof theory the tools are (often semantically motivated) syntactic translations of one formal theory in another.

We are left in no doubt that proof theory as currently practised is essentially syntactic in nature. Indeed proof theory has been the poor relation of logic, at least partly for this very reason. So the reference to semantic motivation is tantalising and should give one pause.

Reflecting on the notion of semantic motivation, one might crudely distinguish between philosophical and mathematical motivation. In the first case one tries to convince with a telling conceptual story; in the second one relies more on the elegance of some emergent mathematical structure. If there is a tradition in logic it favours the former, but I have a sneaking affection for the latter. Of course the distinction is not so clear cut. Elegant mathematics will of itself tell a tale, and one with the merit of simplicity. This may carry philosophical weight. But that cannot be guaranteed: in the end one cannot escape the need to form a judgement of significance.

Let us first consider interpretational proof theory. I urge an understanding of the notion of interpretation in logic along the lines of the following slogan.

INTERPRETATION = MODEL + CODING.

By coding I mean that aspect of logic which deals with representability (whether of functions or of mathematical arguments) in a formal system. If one leaves that aside, one has just the model which is the mathematical idea in the interpretation. Such an idea may well arise from philosophical considerations, but it may equally constitute or give rise to interesting mathematical structure.

Gödel's Dialectica interpretation [23] represents a particularly interesting example of the contrast between philosphical and mathematical motivation. The metamathematical applications mentioned in Gödel's 1941 Princeton lecture 'In what sense is intuitionistic logic constructive?' (reproduced in [16]) may have been the original motivation, but Gödel himself appears to have moved towards the position that his interpretation was fundamentally of philosophical interest.[1] Be that as it may, the metamathematical and the philosophical issues are distinct from the abstract mathematical properties of the interpretation. These mathematical properties are themselves curious and I shall devote much of this paper to explaining them.

Turning now to structural proof theory, one may wonder whether it is really so free of semantic content. Experience of constructive proofs suggests otherwise. Two different points of view are represented by [22] and [31]: certainly there is a syntactic approach, and issues that demand it, but there appears to be more than that. Things become more problematic in the case of classical proof but still it is natural to hope for some semantic understanding of some of the central methods. While there has been much stress lately on studying the 'dynamics of proofs', there are still things to be done at a simpler level. Semantic motivation in the mathematical sense challenges structural proof theory to give a mathematically interesting account of the structure of propositions and proofs.

Categorical Proof Theory is one modern approach to the issue of mathematical structure in the two branches of proof theory. The subject arose out of the well known connections between constructive logic and typed lambda calculus[2] (Girard et al [22] give a succinct account) and typed lambda calculus and cartesian closed categories (see Lambek and Scott [30] for example). There is little systematic in the literature, though Girard's Linear Logic has succesfully been treated from this point of view (see amongst others Seely [36] de Paiva [12] Benton et al [1] [2], Bierman [3] [4]). Probably the best overall impression of Categorical Proof Theory is given by work in the related area of Categorical Type Theory (for which see Crole [9], Jacobs [26], Taylor [40]). The main idea of Categorical Proof Theory is to represent propositions and proofs in some logical system as the objects and maps in some structured category. Any interesting version requires some non-trivial notion of equality of proofs. With that in place, the proof theory itself corresponds to the initial structured category. This perspective has been reasonably successful in the case of constructive and linear proofs, providing both a clear explanation of structure and clean approaches to the proof of metamathematical results. At the end of this

[1] For more on this issue see Troelstra's detailed Introductory Note to the two versions of the Dialectica interpretation paper in [17]. It is worth comparing these versions one published and one prepared for publication with the 1941 lecture reproduced with a brief note by Troelstra in [16].

[2] This is sometimes called the Curry-Howard isomorphism or (better) correspondence.

paper I make a suggestion as to how to regard classical proofs from this point of view.

Of course besides the initial syntactic model there are many other structured categories which can be read as models for a given style of proof theory. We can think of these as providing other worlds of propositions and proofs. Since interpretational proof theory itself provides structured categories corresponding to the mathematical models, the question of semantics of proofs brings these two sides of proof theory together. That is one of the merits of Categorical Proof Theory: it is a kind of 'Proof Theory in the Abstract'.

This paper contains a number of loosely linked sections. I start by discussing aspects of the Dialectica interpretation from the point of view of Categorical Proof Theory. I thereby pay tribute to Troelstra's work on this interpretation over many years. His contributions are technical as in [42] and conceptual as in the contributions to Gödel's Collected works [17] and [16]. In respect of both, the community of logicians is much in his debt. Another reason for discussing the Dialectica interpretation is that of all the standard functional interpretations it is the most riddling and hence the one for which the possibility of the abstract analysis of the kind I wish to promote is least obvious. So I hope to give prominence to the existence of such an analysis by the discussion in Section 2. An abstract view of proofs is particularly revealing for the so-called Diller-Nahm variant of the Dialectica interpretation. I consider this in some detail in Section 3, as the excellent structural properties of this interpretation deserve wider recognition.

I then go on to consider some possibilities for a theory of classical proof. From the point of view of structural proof theory, the picture today is something like this. On the one hand constructive proof theory looks in good shape. The syntactic representations of proofs can be read as intelligible mathematical descriptions: at one simple level as maps in a cartesian closed category. So we have a good categorical proof theory.[3] On the other hand it used to be said that classical proof theory does not really exist. Two distinct thoughts lie behind this conclusion.

- Suppose that $\mathbb{C}$ is a cartesian closed category with initial object 0 such that the contravariant functor $(- \Rightarrow 0)$ is an involution. Then $\mathbb{C}$ is equivalent to a boolean algebra. (This elementary point is attributed to Joyal.) Thus any attempt to build on the theory of constructive proof in terms of a cartesian closed category by adding a strict notion of falsity and requiring an involution (that is $\neg\neg A \cong A$) is doomed to triviality: proof reduces to provability.

- In the classical sequent calculus the process of cut elimination cannot usefully be taken as a computation process preserving meaning of proofs. (I believe this is discussed in Lafont's dissertation [27].) Given any sequent

[3] Of course it does not tell the whole story. The equality in the natural categorical formulation is $\beta\eta$-equality, not β-equality and so corresponds to some contextual equality on proofs. Thus it does not capture all the computational distinctions which naturally arise in proof theory. But nobody's perfect.

calculus proofs π_1 and π_2 of $\Gamma \vdash \Delta$ one has a proof

$$\dfrac{\dfrac{\dfrac{\pi_1}{\Gamma \vdash \Delta}}{\Gamma \vdash \Delta, A}\,\text{WEAK} \qquad \dfrac{\dfrac{\pi_2}{\Gamma \vdash \Delta}}{A, \Gamma \vdash \Delta}\,\text{WEAK}}{\Gamma \vdash \Delta}\,\text{CUT}$$

using Weakening and Cut. (There are also some implicit Contractions.) Eliminating the Cut one gets either π_1 or π_2. So if cut elimination preserved meaning all proofs of $\Gamma \vdash \Delta$ would be equal.

In response to the first point it is natural to ask what happens if we relax the conditions given. Double negation translations already provide one answer, albeit one which relies on a coding into constructive proof theory. I give the standard category theoretic account of these in Section 4, and investigate what happens when one applies the ideas to the usual Dialectica and Diller-Nahm interpretations. The disadvantages with this approach are that the symmetry of classical logic is lost, and that the interpretation appears coding dependent.

The second of the points is usually glossed by referring to the essential non-determinism of classical proof. There are two natural kinds of response to that. On the one hand one can regard classical propositions as being inherently ambiguous, that ambiguity being resolved by a finer analysis of propositions. Many finer propositions will correspond to one coarse classical proposition, and the proof theory of the more refined propositions will be deterministic. This approach has been investigated in the context of Linear Logic (Girard [19]) by Girard [20] [21], and by Schellinx [35] and others (see in particular [10]). On the other hand one can embrace the non-determinism, and regard the sequent calculus as a kind of process calculus with implicit choice (as between π_1 and π_2 above). This idea is obvious enough, but it is hard to give it shape: recent work on structural proof theory by Bierman and his student Urban does just that ([46], see also their [47] and Urban's PhD dissertation [45]). At the end of Section 5, I briefly consider this approach from the point of view of Categorical Proof Theory. Linear Logic is again involved though in a rather different way. Troelstra himself was quick to appreciate the potential of Linear Logic (as witnessed by the books [43], [44] and also the important dissertation of Schellinx [35]), so I first try to indicate why a categorical analysis of the sequent calculus makes Linear Logic inevitable by sketching an approach to some very basic theorems in Categorical Proof Theory. I close by explaining what I think is the crucial difficulty in the semantic analysis of classical proof.

I am happy to acknowledge the use of (an old version of) Paul Taylor's diagram macros and of his prooftree macros in the preparation of this paper.

2 The Dialectica Interpretation

Gödel lectured on his Dialectica interpretation at Yale in 1941, but the published paper [23] did not appear until 1958.[4] In this section I explain an abstract form of the Dialectica interpretation and analyze its mathematical structure. The main points come from de Paiva's PhD dissertation [11] (for a succinct account see de Paiva [12] and for related work de Paiva [13]). It is appropriate to remark at this time that the analysis given by de Paiva owes more than may be apparent to Troelstra [42]. In a recent survey paper [48] van Oosten expresses his feeling that the categorical analyses of realizability owe a lot to the systematic treatment of realizability interpretations in Troelstra [42]. I certainly think that is correct.[5] Similarly, without the stimulus of Troelstra's treatment in [42] of the Dialectica interpretation in parallel with other forms of functional interpretation, the motivation to study the structure from the point of view of Categorical Proof Theory would have been entirely lacking.

2.1 The Dialectica Category

Suppose that we have a category $\mathbb{T}$ which we can think of as interpreting some type theory; and suppose that over the category $\mathbb{T}$ we have a pre-ordered fibration $p : \mathbb{P} \to \mathbb{T}$, which we can regard as providing for each $I \in \mathbb{T}$ a pre-ordered collection of (possibly non-standard) predicates $\mathbb{P}(I) = (\mathbb{P}(I), \vdash)$. Starting with this data we construct a new category **Dial** = **Dial**(p) which we regard as a category of propositions and proofs. We do this as follows.

- The objects A of **Dial** are $U, X \in \mathbb{T}$ together with $\alpha \in \mathbb{P}(U \times X)$. We write this in text as $A = (U \overset{\alpha}{\longleftarrow} X)$ and in displays as

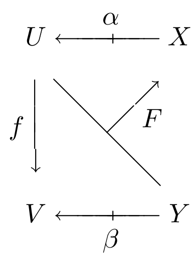
  $$A = (U \overset{\alpha}{\longleftarrow\!\!\!+\!\!\!\longrightarrow} X)\,.$$

  Our understanding of the predicate α is not symmetric as regards U and X: we read $U \overset{\alpha}{\longleftarrow} X$ as $\exists u.\forall x.\alpha(u,x)$, in accord with the form of propositions in the image of the Dialectica interpretation.

- Maps of **Dial** from $A = (U \overset{\alpha}{\longleftarrow} X)$ to $B = (V \overset{\beta}{\longleftarrow} Y)$ are diagrams of the form

[4] For more historical background see Troelstra's Introductory notes in [17] and [16].

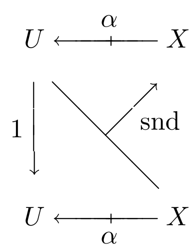
[5] Indeed in my first lectures on the Effective Topos I started the course by explaining a categorical view of the realizability interpretation of higher order arithmetic ($\mathbf{HA}_\omega$) using the effective operations along Troelstra's lines. And after that, the passage from Troelstra's systematic treatment of functional interpretations of **HAS** to the topos theoretic point of view is really one of relatively straightforward abstraction.



with $\alpha(u, F(u,y)) \vdash \beta(f(u), y)$ in $\mathbb{P}(U \times Y)$.

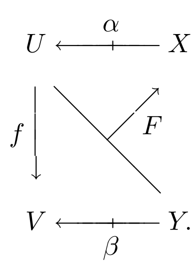
Thus maps $A \to B$ of **Dial** consist of maps $f : U \to V$ and $F : U \times Y \to X$ in $\mathbb{T}$ such that $\alpha(u, F(u,y)) \vdash \beta(f(u), y)$ holds in $\mathbb{P}(U \times Y)$. One should think of this as saying that a proof of $\exists u \forall x \alpha \to \exists v \forall y \beta$ is obtained by transforming to $\forall u \exists v \forall y \exists x(\alpha \to \beta)$ and then Skolemizing along the lines explained by Troelstra [42] (see also [17]).

- The identity on $A = (U \overset{\alpha}{\longleftarrow} X)$ is given by

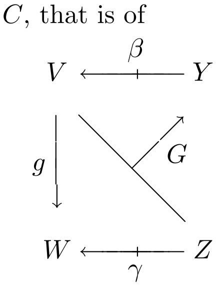


where $\mathsf{snd} : U \times X \to X$ is the second projection.

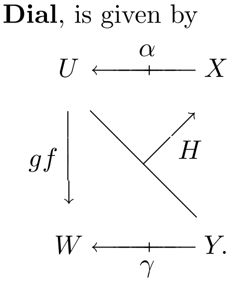
- Composition of maps $A \to B$ and $B \to C$, that is of



and



in **Dial**, is given by



where $\quad H(u,z) = F(u, G(f(u), z))\,.$

One sees at once that if

$$\alpha(u, F(u,y)) \vdash \beta(f(u), y) \quad \text{and} \quad \beta(v, G(v,z)) \vdash \gamma(g(v), z),$$

then

$$\alpha(u, F(u, G(f(u), z))) \vdash \gamma(g(f(u)), z).$$

So the composite is indeed a map $A \to C$ of **Dial**.

It is straightforward to check the associativity and identity laws, so we get the following.

Proposition 2.1 **Dial** *forms a category.*

Let us call **Dial** the *Dialectica category*; it encapsulates in a pure form the basic mathematical feature of the Dialectica interpretation, namely its interpretation of implication.[6] The point of view of Categorical Proof Theory is that to treat the Dialectica interpretation as a theory of proofs is to study the categorical structure of **Dial**.

2.2 Natural Structure

In this section I describe structure on the Dialectica category **Dial** which is an easy consequence of simple assumptions about the propositional fibration $p : \mathbb{P} \to \mathbb{T}$. This resulting structure is natural in the categorical sense that it is given by natural transformations; and it is the structure of Intuitionistic Linear Logic. (The basic references are Girard [19], [22] and [43]; for categorical analyses see Seely [36], Hyland and de Paiva [25], and in particular for the Intuitionistic version Benton et al [1] [2], and Bierman [3] [4]). In the next section I shall discuss that further structure which on the one hand is needed for the applications to constructive logic, but on the other is not (at least in the technical sense) so natural.

2.2.1 Propositional Logic

Let us start with the simplest condition. Suppose that $p : \mathbb{P} \to \mathbb{T}$ is a product fibration, that is $\mathbb{T}$ has finite products and the fibres $\mathbb{P}(I)$ have finite products preserved by reindexing.[7] Then we have the following.

Proposition 2.2 **Dial** *carries the structure of a symmetric monoidal category.*

Proof. The tensor product $A \otimes B$ of $A = (U \stackrel{\alpha}{\longleftarrow} X)$ and $B = (V \stackrel{\beta}{\longleftarrow} Y)$ is

$$A \otimes B \;=\; (U \times V \stackrel{\alpha \wedge \beta}{\longleftarrow} X \times Y)$$

where $\alpha \wedge \beta$ is the obvious relation $\alpha(u, x) \wedge \beta(v, y) \in \mathbb{P}(U \times V \times X \times Y)$. The unit I for this tensor is $I = (1 \stackrel{\mathsf{true}}{\longleftarrow} 1)$. The unit and associativity structure is

[6] In fact as first pointed out by Girard one can regard **Dial** as arising (as a category of coalgebras) from a more primitive category. So at a formal level there is yet more to say. For details see de Paiva [12].

[7] So $p : \mathbb{P} \to \mathbb{T}$ is a map of finite product categories, and the fibration models simple conjunctive logic.

obvious.

To go beyond simple tensor logic we need some function spaces. From now on we shall assume at least the following structure.

- $\mathbb{T}$ is cartesian closed. We write 1 for the terminal object, $X \times Y$ for the product of X and Y, and $(Y \Rightarrow Z) = Z^Y$ for the object of functions from Y to Z (usually called the function space).
- $p : \mathbb{P} \to \mathbb{T}$ is a fibration over $\mathbb{T}$ of (preordered) cartesian closed categories: that is, each fibre has terminal object, products and function spaces and reindexing preserves the structure. We use the standard logical notation $\top, \wedge, \to$ for this structure.

Thus our ruling assumption is that $\mathbb{T}$ models the simply typed lambda calculus, and each $\mathbb{P}(I)$ models $\top, \wedge, \to$ logic.

Theorem 2.3 *With our ruling assumption,* **Dial** *is symmetric mononoidal closed.*

Proof. The object $B \multimap C$ of functions from $B = (V \overset{\beta}{\longleftarrow} Y)$ to $C = (W \overset{\gamma}{\longleftarrow} Z)$ (that is, the linear function space) is given by

$$B \multimap C = \ (V \Rightarrow W) \times (V \times Z \Rightarrow Y) \overset{\rho}{\longleftarrow\!\!\!\!\!\!\!+\!\!\!\!\!\!\!\longrightarrow} V \times Z$$

where ρ is the relation $\rho((g, G), (v, z)) = \beta(v, G(v, z)) \to \gamma(g(v), z)$. It is easy to check the adjunction

$$\mathbf{Dial}(A \otimes B, C) \cong \mathbf{Dial}(A, B \multimap C)\,.$$

According to this proposition we automatically have a model for the multiplicative fragment of intuitionistic linear logic.

In fact one can get the additive conjunction of intuitionistic linear logic without much further assumption on $p : \mathbb{P} \to \mathbb{T}$. Suppose that $\mathbb{T}$ has finite coproducts;[8] and suppose also that $\mathbb{P}(0) \cong 1$ and that the injections $X \to X+Y$ and $Y \to X + Y$ induce an equivalence $\mathbb{P}(X + Y) \simeq \mathbb{P}(X) \times \mathbb{P}(Y)$.

Proposition 2.4 *With these assumptions* **Dial** *has finite products.*

Proof. The product $A \times B$ of $A = (U \overset{\beta}{\longleftarrow} X)$ and $B = (V \overset{\beta}{\longleftarrow} Y)$ is

$$A \times B = (U \times V \overset{\pi}{\longleftarrow\!\!\!\!\!\!\!+\!\!\!\!\!\!\!\longrightarrow} X + Y)$$

where $\pi \in \mathbb{P}(U \times V \times (X + Y)) \cong \mathbb{P}(U \times V \times X) \times \mathbb{P}(U \times V \times Y)$ is given by the pair $(\alpha(u, x), \beta(v, y))$: the projections $A \times B \to A$ and $A \times B \to B$ are obvious. The terminal object in **Dial** is the unique relation $1 \longleftarrow 0$.

[8] In the absence of our now ruling assumption that $\mathbb{T}$ is cartesian closed we would additionally require distributivity.

2.2.2 Predicate logic

Up to now we have just considered the Dialectica category **Dial** as a category of propositions and proofs. To handle a basic form of quantification we need to index it over some category to represent the types over which we quantify. The standard Dialectica treatment of quantification comes from an indexing over $\mathbb{T}$ itself. Clearly one can define for each $I \in \mathbb{T}$ a parametrized Dialectica category $\mathbf{Dial}(I)$ by carrying I along as a simple parameter throughout. Thus objects are of the form $A = (U \overset{\alpha}{\longleftarrow} X)$ where $\alpha \in \mathbb{P}(I \times U \times X)$; and maps from $A = (U \overset{\alpha}{\longleftarrow} X)$ to $B = (V \overset{\beta}{\longleftarrow} Y)$ are diagrams in the simple slice category[9] of $\mathbb{T}$ over I of the form

$$
\begin{array}{ccc}
U & \overset{\alpha}{\longleftarrow\!\!+\!\!-} & X \\
{\scriptstyle f}\downarrow & \diagdown\!\!\nearrow {\scriptstyle F} & \\
V & \underset{\beta}{\longleftarrow\!\!+\!\!-} & Y.
\end{array}
\qquad \text{where } \alpha(i, u, F(u, y)) \vdash \beta(i, f(u), y) \text{ in } \mathbb{P}(I \times U \times Y)\,.
$$

Thus concretely maps $A \to B$ of $\mathbf{Dial}(I)$ consist of maps $f : I \times U \to V$ and $F : I \times U \times Y \to X$ in $\mathbb{T}$ such that $\alpha(i, u, F(i, u, y)) \vdash \beta(i, f(i, u), y)$ holds in $\mathbb{P}(I \times U \times Y)$. Reindexing along maps in $\mathbb{T}$ preserves the structure. So we can put the categories $\mathbf{Dial}(I)$ together to get a fibration $q : \mathbf{Dial} \to \mathbb{T}$.

We study the treatment of the quantifiers in the Dialectica interpretation by asking after the existence of adjoints to reindexing along the projections in the fibration $q : \mathbf{Dial} \to \mathbb{T}$. This is quite straightforward. Take for simplicity an object $A = (U \overset{\alpha}{\longleftarrow} X)$ in $\mathbf{Dial}(I)$ (so that $\alpha \in \mathbb{P}(I \times U \times X)$. Then we define $\exists_I A$ by

$$\exists_I A = (I \times U \overset{\alpha}{\longleftarrow} X) \quad \text{with } \alpha \in \mathbb{P}(I \times U \times X);$$

and we define $\forall_I A$ by

$$\forall_I A = (I \Rightarrow U \overset{\widehat{\alpha}}{\longleftarrow} I \times X) \quad \text{where } \widehat{\alpha}(f, (i, x)) = \alpha(f(i), x).$$

It is easy to see that these provide the required adjoints.

Theorem 2.5 *The fibration q : **Dial** $\to \mathbb{T}$ has both left and right adjoints to reindexing along product projections. These adjoints satisfy the Beck-Chevalley condition.*[10]

One can extend the abstract analysis of the Dialectica interpretation to deal with interpretations of various kinds of type theories.[11] However I leave that for another occasion.

[9] For an explanation of the simple fibration obtained from any category with products see for example Jacobs [26].

[10] That is they behave appropriately under substitution or reindexing. For a discussion of the issue see Jacobs [26] or Taylor [40].

[11] The basic idea is already in Girard [18]. There is an accessible explanation in Troelstra [42].

2.3 Interpreting intuitionistic logic

The treatment of conjunction and disjunction in the Dialectica interpretation raise quite different issues. So I deal with them separately.

2.3.1 Conjunction and true

It is well known that the tricky point in the original Dialectica interpretation is to get an interpretation for the usual rules for conjunction using the tensor defined above: we want to satisfy the rules

$$A \vdash A \wedge A \quad \text{and} \quad A \wedge B \vdash A;$$

that is, we want canonical proofs of $A \wedge A$ from A and of A from $A \wedge B$. The problem is that the tensor is certainly not the product in the category and the unit I is not the terminal object. One gets round these two difficulties in rather different ways.

Diagonals We need a map $A \to A \otimes A$ in **Dial**. For this additional structure is used. We suppose that $p : \mathbb{P} \to \mathbb{T}$ is equipped with a kind of weak definition by cases. For $\phi \in \mathbb{P}(X)$ and $f, g : X \to Y$ in $\mathbb{T}$, we suppose given a map

$$\mathsf{cases}(\phi, f, g) : X \to Y \,,$$

which we can also write suggestively as $\mathsf{if}\ \phi\ \mathsf{then}\ f\ \mathsf{else}\ g$. This construction is supposed to have the following properties.

- (Naturality on right.) If $u : Z \to X$, then

$$\mathsf{cases}(\phi, f, g).u = \mathsf{cases}(u^*\phi, f.u, g.u) \,.$$

- (Naturality on left.) If $v : Y \to W$, then

$$v.\mathsf{cases}(\phi, f, g) = \mathsf{cases}(\phi, v.f, v.g) \,.$$

- (Identity rule.) $\mathsf{cases}(\phi, f, f) = f$.
- (Cases condition.) For all $\psi \in \mathbb{P}(Y)$,

$$\psi(g(x)) \vdash \phi(x) \quad \text{implies} \quad \psi(\mathsf{cases}(\phi, f, g)) \vdash \phi(x) \wedge \psi(f(x)).$$

Observation 1. In the traditional view $\mathsf{cases}(\phi, f, g)$ depends on the decidability of predicates in $\mathbb{P}$. We think of it as arising in the following way. There is an object bool (of booleans) with $1 \to \mathsf{bool}$ acting as a classifier of predicates in $\mathbb{P}$; so there is a natural isomorphism $\mathbb{P}(X) \cong \mathbb{T}(X, \mathsf{bool})$. Further $\mathsf{bool} \cong 1 + 1$ is a coproduct, so we can construct $\mathsf{cases}(\phi, f, g) : X \to Y$ as the composite

$$X \xrightarrow{(\phi, f, g)} B \times Y \times Y \cong Y \times Y + Y \times Y \xrightarrow{[\mathsf{fst}, \mathsf{snd}]} Y \,.$$

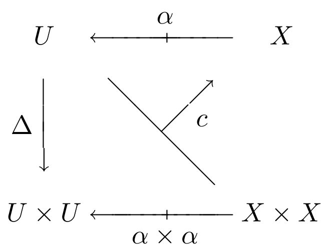
The condition $\mathsf{bool} \cong 1+1$ corresponds to the decidability of the core predicates of the Dialectica interpretation together with the existence of characteristic functions for them.[12]

Observation 2. The conditions above are, however, all we really use to define the (diagonal) comultiplication map. One can find examples of them with no connection with decidability. For example if we took for $\mathbb{T}$ a category of Scott domains and continuous maps and for $\mathbb{P}(X)$ the Scott open subsets of the domain X, then we could define

$$\mathsf{cases}(\phi, f, g) = \begin{cases} f(x) & \text{if } x \in \phi \\ f(x) \wedge g(x) & \text{otherwise.} \end{cases}$$

It is easy to see that this will satisfy the conditions we set on our weak definition by cases.

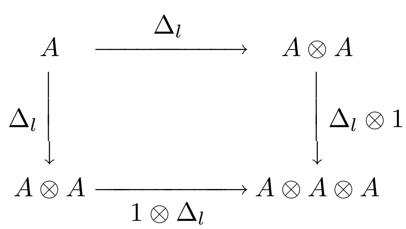
Given a notion of definition by cases, $\mathsf{cases}(\phi, f, g)$, or $\mathsf{if}\ \phi\ \mathsf{then}\ f\ \mathsf{else}\ g$, we define for $A = (U \overset{\alpha}{\longleftarrow} X)$ a comultiplication map $A \to A \otimes A$ by



$$c(u, x_1, x_2) = \mathsf{if}\ \alpha(u, x_1)\ \mathsf{then}\ x_2\ \mathsf{else}\ x_1.$$

This provides an interpretation of $A \vdash A \wedge A$ which I write $\Delta_l : A \to A \otimes A$. The following properties are easy to establish.

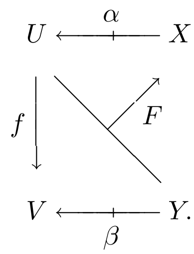
- Δ_l is a coassociative comultiplication: that is, the diagram

  commutes. This is a simple consequence of the properties we supposed for our weak definition by cases. There is however no counit for the comultiplication (see the discussion of projections below).

- Δ_l is not natural with respect to all maps; but it is natural with respect to 'information preserving' maps, that is, those of the form

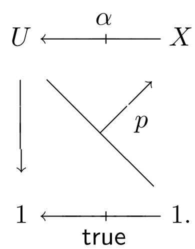
[12] For some historical background see Troelstra's Introductory Note in [17].



with $\alpha(u, F(u, y)) \dashv\vdash \beta(f(u), y)$.

- Composing Δ_l with the twist map $A \otimes A \to A \otimes A$ gives an alternative comultiplication Δ_r. This is the familiar symmetric choice of Dialectica interpretation of $A \vdash A \wedge A$.

Projections Again we need something extra, but that something is rather dull. We ask for inhabited types, or more exactly for a choice of element $x_0 \in X$ for each $X \in \mathbb{T}$. Then we have $A \to I$ given by



with $p(u, *) = x_0$.

This counit is not natural, and there are no good connections between it and the comultiplication.

The counit gives projections $\mathsf{fst} : A \otimes B \to A$ and $\mathsf{snd} : A \otimes B \to B$:[13] here fst is natural in A and snd in B. The maps fst and snd are interchanged by the twist giving rise to equations like $\mathsf{fst}.\Delta_l = \mathsf{snd}.\Delta_r$; but there are no really significant mathematical properties.

2.3.2 Disjunction and false

To interpret disjunction one needs some kind of coproduct in **Dial**. To give this assume that there is an object $\mathsf{bool} \in \mathbb{T}$ (of Booleans) with maps $1 \xrightarrow{\mathsf{true}} \mathsf{bool}$ and $1 \xrightarrow{\mathsf{false}} \mathsf{bool}$ such that the induced map $\mathbb{P}(\mathsf{bool} \times X) \to \mathbb{P}(X) \times \mathbb{P}(X)$ is an isomorphism.

Then we define the (weak) sum $A+B$ of $A = (U \xleftarrow{\alpha} X)$ and $B = (V \xleftarrow{\beta} Y)$ to be

$$A + B \;=\; (\mathsf{bool} \times U \times V \xleftarrow{\rho} X \times Y)$$

where $\rho \in \mathbb{P}(\mathsf{bool} \times U \times V \times X \times Y) \cong \mathbb{P}(U \times V \times X \times Y)^2$ is given by the pair $(\alpha(u, x), \beta(v, y))$: we might suggestively write

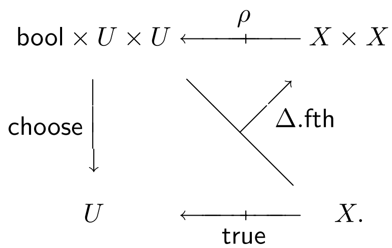
$$\rho(b, u, v, x, y) = [(b = \mathsf{true} \to \alpha(u, x)) \wedge (b = \mathsf{false} \to \beta(v, y))] \,.$$

[13] This notation is a bit overloaded but never mind.

We get the following properties.

- The operation $+$ extends to give a functor $\mathbf{Dial} \times \mathbf{Dial} \to \mathbf{Dial}$. As so often occurs with weak coproducts, this functor is not associative.

- There is a natural codiagonal $\nabla : A + A \to A$ given by

  where fth projects onto X, and

  and where choose is $\mathsf{bool} \times U \times U \cong U \times U + U \times U \xrightarrow{[\mathsf{fst},\mathsf{snd}]} U$. The naturality is easy to check.

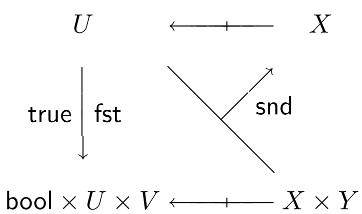
Had we not commited ourselves to considering inhabited types in our discusion of projections, we would find that if $\mathbb{T}$ has an initial object 0, then the unique relation $0 \longleftarrow X$ is initial in **Dial** for any $X \in \mathbb{T}$. With inhabited types however we are restricted to a weak initial object

$$\bot = (1 \overset{\mathsf{false}}{\longleftarrow} 1)\,.$$

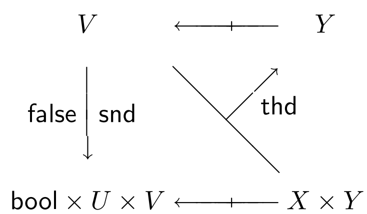
This provides an interpretation for intuitionistic falsity, but there is no naturality, nor any good properties with respect to the weak coproduct. Even $\bot + A \cong A$ fails. It follows that we have separately to give injections to interpret the rules

$$A \vdash A \vee B \quad \text{and} \quad B \vdash A \vee B\,.$$

We define $\mathsf{in}_l : A \to A + B$ and $\mathsf{in}_r : B \to A + B$ by



and



respectively. The composites

$$A \xrightarrow{\mathsf{in}_l} A + A \xrightarrow{\nabla} A \quad \text{and} \quad A \xrightarrow{\mathsf{in}_r} A + A \xrightarrow{\nabla} A$$

are the identity. And notwithstanding that in_l is not natural we have that the composite

$$A \xrightarrow{\mathsf{in}_l} A + B \xrightarrow{f+g} C + C \xrightarrow{\nabla} C$$

is equal to $f : A \to C$.

The Dialectica interpretation is usually taken at the level of provability. We can understand the fact that at that level the interpretation with inhabited types interprets constructive logic as follows.

Theorem 2.6 *The poset reflection of our indexed category* **Dial** $\rightarrow \mathbb{T}$ *of proofs is a 'first-order hyperdoctine': we get indexed Heyting algebras and good quantification.*

Of course the interpretation applies to arithmetic and stronger systems, and we have not accounted abstractly for those aspects.

3 The Diller-Nahm Variant

I now give a categorical analysis of the interpretation introduced by Diller and Nahm in [15]. One value of the categorical approach is that when one strips interpretations of their coding component one gets a fresh perspective on their mathematical content. What is usually called the Diller-Nahm variant of the Dialectica Interpretation is a striking example of this. The description of it as a 'variant' fails utterly to give credit to the elegant mathematical structure of this interpretation.

3.1 The Diller-Nahm Category

Suppose again that we have a pre-ordered set fibration $p : \mathbb{P} \rightarrow \mathbb{T}$, providing for each type $I \in \mathbb{T}$ a collection of (possibly non-standard) predicates $\mathbb{P}(I)$ over I. We need some additional structure. We suppose that $p : \mathbb{P} \rightarrow \mathbb{T}$ is equipped with a commutative monoid $(-)^\bullet$ in the following sense.

- First $\mathbb{T}$ is a category with products and $(-)^\bullet$ is a strong monad on $\mathbb{T}$ such that each algebra is equipped naturally with the structure of a commutative monoid.
- Secondly we suppose that we have an indexed extension of $(-)^\bullet$ to $\mathbb{P}$. For $\phi \in \mathbb{P}(I \times A)$ we have $\phi^\bullet \in \mathbb{P}(I \times X^\bullet)$.[14] For each $I \in \mathbb{T}$, the strength gives an action of $(-)^\bullet$ on the (simple slice) category $\mathbb{T}_I$. And the operation $\phi \rightarrow \phi^\bullet$ just described is an extension of this to the global category $\mathbb{P}_I \rightarrow \mathbb{T}_I$.

The example to have in mind here is the finite multiset monad on the category of sets; of course that is exactly the monad whose algebras are commutative monoids. This monad extends naturally to the subset lattices: if $\phi \subseteq I \times X$ then $\phi^\bullet \subseteq I \times X^\bullet$ is defined by

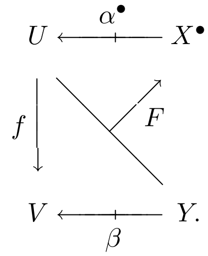
$$\phi^\bullet(i, \xi) \quad \text{if and only if} \quad \forall x \in \xi . \phi(i, x).$$

From the data just described we construct a new category **Dill** $=$ **Dill**(p) which we again regard as a category of propositions and proofs.

- The objects of **Dill** are still pairs $U, X \in \mathbb{T}$ together with $\alpha \in \mathbb{P}(U \times X)$. We continue to use the notation introduced for **Dial**.

[14] In the notation we suppress the dependence on the parameter I.

- Maps of **Dill** from $A = (U \xleftarrow{\alpha} X)$ to $B = (V \xleftarrow{\beta} Y)$ are now diagrams of the form

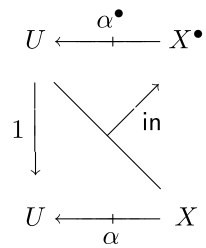


$$\alpha^\bullet(u, F(u, y)) \vdash \beta(f(u), y) \quad \text{in} \quad \mathbb{P}(U \times Y)\,.$$

That is they consist of maps $f : U \to V$ and $F : U \times Y \to X^\bullet$ such that $\alpha^\bullet(u, F(u, y)) \vdash \beta(f(u), y)$ holds in $\mathbb{P}(U \times Y)$. In accord both with the basic example and with the traditional formulation of the interpretation (see for example Diller [14]) we read this as

$$\forall x \in F(u, y).\alpha(u, x) \vdash \beta(f(u), y)\,.$$

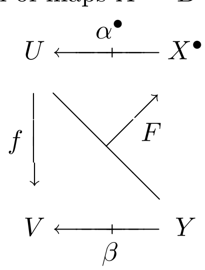
- The identity on $A = (U \xleftarrow{\alpha} X)$ is given by



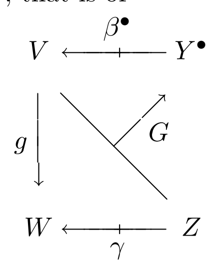
where $\mathsf{in}(u, x) = \eta(x) \in X^\bullet$,

and with η the unit for the monad $(-)^\bullet$.

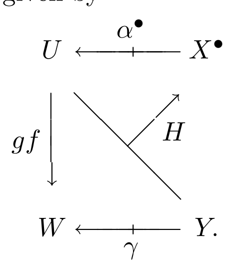
- Composition of maps $A \to B$ and $B \to C$, that is of



and



is given by



where $\quad H(u, z) = F^{(\bullet)}(u, G(f(u), z))\,.$

Here we use a construction familiar from models for the computational lambda calculus. If $F : I \times Y \to X^\bullet$ then we write $F^{(\bullet)} : I \times Y^\bullet \to X^\bullet$ for the composite

$$I \times Y^\bullet \longrightarrow (I \times Y)^\bullet \xrightarrow{F^\bullet} X^{\bullet\bullet} \xrightarrow{\mu} X^\bullet$$

One checks easily that if

$$\alpha^\bullet(u, F(u,y)) \vdash \beta(f(u), y) \quad \text{and} \quad \beta^\bullet(v, G(v,z)) \vdash \gamma(g(v), z),$$

then

$$\alpha^\bullet(u, F^{(\bullet)}(u, G(f(u), z))) \vdash \gamma(g(f(u)), z).$$

So the composite is indeed a map $A \to C$ of **Dill**.

Again it is straightforward to check the associativity and identity laws so we get the following.

Proposition 3.1 **Dill** *is a category.*

Let us call **Dill** the *Diller-Nahm category*. It encapsulates the basic mathematical structure of the Diller-Nahm variant of the Dialectica interpretation. We investigate what properties the category **Dill** has under some natural assumptions.

3.2 Natural structure

3.2.1 Propositional Logic

To define **Dill** we used a strong monad $(-)^\bullet$ on $p : \mathbb{P} \to \mathbb{T}$. Now we make our usual assumption that $\mathbb{T}$ is cartesian closed, and that $\mathbb{P} \to \mathbb{T}$ models $\top, \wedge, \to$ logic. In addition we assume (as we did to get the product in **Dial**) that $\mathbb{T}$ has finite coproducts and that we have natural isomorphisms

$$\mathbb{P}(0) \cong 1\,, \quad \text{and} \quad \mathbb{P}(X+Y) \cong \mathbb{P}(X) \times \mathbb{P}(Y)\,,$$

the latter induced by the injections. Finally we assume the natural isomorphisms of commutative monoids

$$0^\bullet \cong 1\,, \quad \text{and} \quad (X+Y)^\bullet \cong X^\bullet \times Y^\bullet\,,$$

the latter again induced by the injections.

Theorem 3.2 *With the above assumptions,* **Dill** *is a cartesian closed category.*

Proof. We give details of the structure. The categorical product of the objects $A = (U \xleftarrow{\alpha} X)$ and $B = (V \xleftarrow{\beta} Y)$ is

$$A \times B = (U \times V \xleftarrow{\alpha \times \beta} X + Y)$$

where $\alpha \times \beta \in \mathbb{P}(U \times V \times X + Y) \cong \mathbb{P}(U \times V \times X) \times \mathbb{P}(U \times V \times Y)$ is given by the pair of relations $(\alpha(u,x), \beta(v,y)) \in \mathbb{P}(U \times V \times X) \times \mathbb{P}(U \times V \times Y)$. The terminal object is the unique predicate $I = (1 \longleftarrow 0)$. The function space $(V \xleftarrow{\beta} Y) \Rightarrow (W \xleftarrow{\gamma} Z)$ is given by

$$(V \Rightarrow W) \times (V \times Z \Rightarrow Y^{\bullet}) \xleftarrow{\rho} V \times Z$$

where $\rho((g,G),(v,z))$ is $\beta^{\bullet}(v, G(v,z)) \to \gamma(g(v), z)$. We leave the details of the adjunction to the reader.

What this result shows is that with the Diller-Nahm variant, the $\top, \wedge, \to$ fragment of intuitionistic logic automatically gets an interpretation simply because **Dill** is a cartesian closed category. One can regard the fact that the natural deduction view of proof and the Diller-Nahm interpretation both give rise to cartesian closed categories as a delightful confirmation of the intuitions behind the interpretation as described for example in the introduction to Diller [14].

Weak coproducts Note first that if $\mathbb{T}$ has an initial object 0 then **Dill** has an initial object $(0 \longleftarrow 0)$. (In fact all objects of the form $(0 \longleftarrow X)$ are isomorphic in **Dill**!)

We can define a weak coproduct $A \boxplus B$ of the objects $A = (U \xleftarrow{\alpha} X)$ and $B = (V \xleftarrow{\beta} Y)$ by

$$A \boxplus B = (U + V \xleftarrow{\alpha+\beta} X + Y)$$

where $\alpha + \beta \in \mathbb{P}(U + V \times X + Y) \simeq \mathbb{P}(U \times V \times X) \times \mathbb{P}(U \times V \times Y)$ is given by the pair of relations $(\alpha(u,x), \beta(v,y)) \in \mathbb{P}(U \times V \times X) \times \mathbb{P}(U \times V \times Y)$. Then one can extend $\boxplus$ to maps in **Dill** in such a way that $\boxplus$ is functorial. Further there are canonical (natural) choices of codiagonal maps

$$\nabla : A \boxplus A \to A$$

and the initial object provides insertion maps

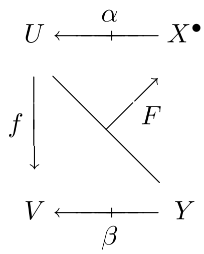
$$\mathsf{in}_l : A \to A \boxplus B \quad \text{and} \quad \mathsf{in}_r : B \to A \boxplus B$$

(The basic point is that wherever we might not be sure how to define a map, we have a canonical choice of the identity for multiplication in a standard commutative monoid provided by $(-)^{\bullet}$ to help us out.) This gives $\boxplus$ the structure of a weak coproduct in the following sense. There is a natural retraction

$$\mathbf{Dill}(A, C) \times \mathbf{Dill}(B, C) \lhd \mathbf{Dill}(A \boxplus B, C).$$

3.2.2 Predicate Logic

Just as we did for **Dial**, we can consider **Dill** as a category fibred over $\mathbb{T}$. Objects are still of the form $U \xleftarrow{\alpha} X$ where $\alpha \in \mathbb{P}(I \times U \times X)$; and maps from $U \xleftarrow{\alpha} X$ to $V \xleftarrow{\beta} Y$ are diagrams of the form



$$\alpha^{(\bullet)}(i,u,F(u,y)) \vdash \beta(i,f(u),y) \quad \text{in} \quad \mathbb{P}(I \times U \times Y)\,.$$

in the simple slice category. Thus maps of $\mathbf{Dill}(I)$ consist of maps $f : I \times U \to V$ and $F : I \times U \times Y \to X$ in $\mathbb{T}$ such that the entailment above which we read intuitively as $\forall x \in F(i,u,y).\alpha(i,u,x) \vdash \beta(i,f(i,u),y)$ holds in $\mathbb{P}(I \times U \times Y)$. Reindexing along maps in $\mathbb{T}$ preserves the structure defined above. This gives us a fibration $q : \mathbf{Dill} \to \mathbb{T}$.

We study the the quantifiers in the Diller-Nahm interpretation by asking after the existence of adjoints to reindexing along projections in $q : \mathbf{Dill} \to \mathbb{T}$. This goes through exactly as for the case of **Dial**. Take an object $A = (U \overset{\alpha}{\longleftarrow} X)$ in $\mathbf{Dill}(I)$. We have the two definitions

$$\exists_I A = (I \times U \overset{\alpha}{\longleftarrow} X) \quad (\text{with } \alpha \in \mathbb{P}(I \times U \times X))$$

$$\forall_I A = (I \Rightarrow U \overset{\widehat{\alpha}}{\longleftarrow} I \times X) \quad \text{where } \widehat{\alpha}(f,(i,x)) = \alpha(f(i),x).$$

Theorem 3.3 *The fibration $q : \mathbf{Dill} \to \mathbb{T}$ has both left and right adjoints to reindexing along product projections. These adjoints satisfy the Beck-Chevalley condition.*

Again one can extend the abstract analysis of the Diller-Nahm variant to deal with interpretations of various kinds of type theories.

The properties of the Diller-Nahm variant are already so good at the level of proofs that there is no point in stating a theorem at the level of provability. These good properties suggest that it is more flexible than Gödel's original interpretation. This seems to be confirmed by recent work of Burr [6]; it would certainly be interesting to have an abstract analysis of the interpretation of set theory studied by Burr.

4 Coding Classical Proof

I rehearse here some background to double-negation translations. There are any number of ways to prove the following basic limitative result. (This is folklore generally associated with Joyal, but I include a proof as I have no ready reference.)

Proposition 4.1 *Suppose that $\mathbb{C}$ is a cartesian closed category with initial object 0.*
(i) Any object of the form $\neg A = (A \Rightarrow 0)$ is a subobject of 1; and the subcategory of such objects forms a boolean algebra.
(ii) If $\neg\neg A \cong A$ for all $A \in \mathbb{C}$ then $\mathbb{C}$ is equivalent to boolean algebra.

Proof. First since $- \times A$ is a left adjoint, $0 \times A \cong 0$. Secondly

$$\begin{array}{ccc} A \times B & \xrightarrow{1 \times g} & A \times D \\ {\scriptstyle f \times 1}\downarrow & & \downarrow{\scriptstyle 1 \times g} \\ C \times B & \xrightarrow[1 \times g]{} & C \times D \end{array}$$

is a pullback; and applying this to $0 \to 1$ and $0 \to A$ we deduce that

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & A \end{array}$$

is a pullback and hence that $0 \to A$ is monic. So in particular $0 \to 1$ is monic. So as $(-)^A$ is a right adjoint, $0^A \to 1^A \cong 1$ is monic. This shows each $\neg A$ is a subobject of 1.
Now the subobjects of 1 form a poset modelling $\top, \wedge, \to$ logic and with a (strict[15]) initial object modelling $\bot$. The subobjects of the form $\neg A$, are the regular subobjects; and they automatically form a boolean algebra.
Finally if $\neg\neg A \cong A$, then every object is a regular subobject of 1 and $\mathbb{C}$ is equivalent to a boolean algebra.

It seems possible that if we weaken (ii) of the above by asking perhaps oddly that $\neg\neg A$ be a retract of A, we may get models with interesting proof theory.[16] I should like to have a compelling example.

4.1 Double negation translations

Here I briefly explain the mathematical context in which we can most simply explain some double negation translations. The ideal set-up is this. Suppose we have a cartesian closed category $\mathbb{C}$ with finite sums. Take any $R \in \mathbb{C}$. We can identify two 'double negation categories'.

- The full subcategory ${}_R\mathbb{C}$ of $\mathbb{C}$ on those objects $A = R^{\overline{A}} = (\overline{A} \Rightarrow R)$ which are powers of R.
- The Kleisli category $\mathbb{C}_R$ for the double negation monad $(- \Rightarrow R) \Rightarrow R$.

---

[15] Strictness of 0 is not really needed but is easy: if $X \to 0$ take the product of it and $0 \to 1$ to get the pullback $\begin{array}{ccc} 0 & \to & X \\ \downarrow & & \downarrow \\ 0 & \to & 0 \end{array}$ showing $X \cong 0$.

[16] After all in the constructive setting, realizability certainly has interest despite its crude treatment of negated formulae.

Observe that

$$_R\mathbb{C}(R^{\overline{A}}, R^{\overline{B}}) \cong \mathbb{C}(R^{\overline{A}}, R^{\overline{B}}) \cong \mathbb{C}(\overline{B}, R^{R^{\overline{A}}}) \cong \mathbb{C}_R(\overline{B}, \overline{A})\,.$$

Thus $_R\mathbb{C} \simeq (\mathbb{C}_R)^{\mathrm{op}}$, and the categories are opposites of one another.

Theorem 4.2 *$_R\mathbb{C}$ is a cartesian closed category with weak finite coproducts; and for each object $A \in_R \mathbb{C}$ there is a natural retraction $A \lhd \neg\neg A$ in $_R\mathbb{C}$ where $\neg\neg A = (A \Rightarrow R) \Rightarrow R$.*

Proof. This is well known, and I just give the structure for completeness.

Terminal Object	$1 \cong R^0$
Binary Products	$A \times B = R^{\overline{A}} \times R^{\overline{B}} \cong R^{\overline{A}+\overline{B}}$
Function space	$B \Rightarrow C = R^{\overline{B}} \Rightarrow R^{\overline{C}} \cong R^{B \times \overline{C}}$
Weak coproducts	$A \boxplus B = R^{\overline{A}} \boxplus R^{\overline{B}} \cong R^{\overline{A} \times \overline{B}}$
Weak initial object	$R \cong R^1$

The basic properties are routine. The only point worth stressing is that the weak coproduct $\boxplus$ is not functorial, but rather carries premonoidal structure in the sense of Power and Robinson [34].

Both the categories $_R\mathbb{C}$ and $\mathbb{C}_R$ model some form of classical proof theory. They can be regarded as corresponding to different choices in the cut elimination process. One can express this in terms of computation paradigms for Parigot's $\lambda\mu$-calculus (see Parigot [32]). A precise connection was given by Hofmann and Streicher [24]. As explained in detail by Selinger [37], $_R\mathbb{C}$ corresponds to the call-by-name $\lambda\mu$-calculus, and $\mathbb{C}_R$ to the call by value. All this is related to the continuation passing style (CPS); however one should be cautious as the basic content of CPS can be analyzed in more primitive terms. Thielecke [41] gives a clear account of the categorical structure of continuation passing in terms of premonoidal structure with a self adjunction; and he is able to handle real programming language features in terms of his formulation. Below we compare the models we construct with Thielecke's notion.

4.2 Shoenfield's Version of Dialectica

In his classic text Shoenfield [38] considers an interpretation which he describes in terms of formulae $A = \forall u \exists x \alpha(u, x)$. The description is purely formal and it is best for us to regard such a formula as being of the form $A = \neg\exists u \forall x \overline{\alpha}(u, x)$. More precisely we take $A = (\overline{A} \multimap R)$ in **Dial** where $R = \bot = (1 \xleftarrow{\mathsf{false}} 1)$. So A is an object of the double negation category $_R(\mathbf{Dial})$. We give a concrete description of this category.

- The objects A of the category are of the form $\overline{A} \multimap R$ for $\overline{A} \in \mathbf{Dial}$. Thus formally we have

$$A \quad = \quad ((U \Rightarrow X) \xleftarrow{\ \alpha(u,\phi(u))\ } U)$$

which we interpret by means of the formulae

$$A = ((\exists u \forall x \overline{\alpha}(u,x)) \multimap R) = \exists \phi^{(U \Rightarrow X)}.\forall u \alpha(u, \phi(u))$$

- Maps in ${}_R\mathbf{Dial}$ from an object

  $$A = (U \Rightarrow X \overset{\alpha(u,\phi(u))}{\longleftarrow} U)$$

  to an object

  $$B = (V \Rightarrow Y \overset{\beta(v,\psi(v))}{\longleftarrow} V)$$

  are given by maps

  $$f : (U \Rightarrow X) \to (V \Rightarrow Y) \quad \text{and} \quad F : (U \Rightarrow X) \times V \to U$$

  or equivalently

  $$(f, F) : (U \Rightarrow X) \times V \to U \times Y$$

  such that

  $$\alpha(F(\phi, v), \phi(F(\phi, v))) \vdash \beta(v, f(\phi, v)) \quad \text{holds in} \quad \mathbb{P}((U \Rightarrow X) \times V)\,.$$

Since the above expresses exactly the maps in ${}_R\mathbf{Dial}$, these slightly implausible maps compose associatively.[17] Following through the definitions one sees that what we have defined corresponds exactly to Shoenfield's version of the Dialectica interpretation as presented in Shoenfield [38]. This makes precise the observation of Troelstra (see [42]) that Shoenfield's interpretation results from combining the Dialectica and the double negation translations.

Of course since in **Dial** we did not start with the ideal situation of a cartesian closed category with coproducts we cannot use the general observations of the previous section to analyze the structure of the category ${}_R(\mathbf{Dial})$. It turns out however that it does have some good structure.

We suppose without further ado that we are in the indexed situation discussed in Section 2.2.2, so we deal with propositional connectives and quantifiers together. We write objects as $A = (\overline{A} \multimap R)$, $B = (\overline{B} \multimap R)$ and so on. Then we can define the following logical operations.

Disjunction	$A \vee B = (\overline{A} \otimes \overline{B} \multimap R) = R^{\overline{A} \otimes \overline{B}}$
Falsity	$\bot = R$
Implication	$A \multimap B = (A \otimes \overline{B} \multimap R) = R^{A \otimes \overline{B}}$
Negation	$\neg A = A \multimap R = R^{A}$
Universal quantification	$\forall z.A = ((\exists z \overline{A}) \multimap R) = R^{\exists z \overline{A}}$

We make a number of points about how the logic is represented in the categorical structure of ${}_R\mathbf{Dial}$.

[17] The composition more or less forces itself on one. There is just one sensible thing to do.

1. The falsity $\bot$ is a unit for the disjunction $\vee$. However $\vee$ is not monoidal. Rather it is premonoidal in the sense of Power and Robinson [34]. As usual maps $A \to B$ of the form $R^f : R^{\overline{A}} \to R^{\overline{B}}$ for $f : \overline{B} \to \overline{A}$ are central.

2. We could regard $A \multimap B = \neg A \vee B$ as a derived operation. It is functorial and we have a natural isomorphism

$$ {}_R\mathbf{Dial}(C, \neg A \vee B) \cong {}_R\mathbf{Dial}(A, \neg C \vee B)\,; $$

so $A \multimap B \cong \neg A \vee B$ is a closed structure on ${}_R\mathbf{Dial}$.

3. While $A \multimap B$ gives a closed structure, the corresponding tensor is missing as $R^{\overline{A}} \otimes R^{\overline{B}}$ is not of the form R^C for any C. In fact we can read $R^{\overline{A}} \otimes R^{\overline{B}}$ as

$$ \exists \phi : X^U \exists \psi : Y^V \forall u \forall v . \alpha(u, \phi(u)) \wedge \beta(v, \psi(v))\,, $$

that is essentially as the Henkin quantified formula[18]

$$ \begin{pmatrix} \forall u & \exists x \\ \forall v & \exists y \end{pmatrix} \alpha(u, x) \wedge \beta(v, y)\,. $$

4. We could regard negation $\neg A = A \multimap \bot$ as a derived operation. It is certainly not an involution though we have A a retract of $\neg\neg A$ for all A.

5. The correctness of the definition of universal quantification follows from the natural isomorphisms

$$ \begin{array}{rcl} \mathbf{Dial}(C, \exists z \overline{A} \multimap R) & \cong & \mathbf{Dial}(\exists z \overline{A}, C \multimap R) \\ & \cong & \mathbf{Dial}(\overline{A}, \Delta_Z(C \multimap R)) \\ & \cong & \mathbf{Dial}(\overline{A}, \Delta_Z(C) \multimap R) \\ & \cong & \mathbf{Dial}(\Delta_Z C, \overline{A} \multimap R) \\ & \cong & \mathbf{Dial}(C, \forall z(\overline{A} \multimap R)) \end{array} $$

where we have not distinguished between R and $\Delta_Z R$.

6. Finally we can consider the opposite category $\mathbf{Dial}_R$. We get a premonoidal $\wedge$ dual to $\vee$ and we get the natural isomorphism

$$ \mathbf{Dial}_R(A \wedge \neg B, C) \cong \mathbf{Dial}_R(A \wedge \neg C, B)\,. $$

One can compare this with Thielecke's notion of a $\otimes\neg$-category (see Thielecke [41]). We get much of what Thielecke requires, but $\wedge$ is emphatically not a product in the centre of $\mathbf{Dial}_R$. I do not know if there is (any interest in) a variant of the CPS calculus corresponding to the structure of $\mathbf{Dial}_R$. But at least the example shows the independence of the cartesian assumption from some other components of the structure.

[18] The identification of a sufficiently simple tensor as a Henkin quantifier is a common feature of a number of interpretations of Linear Logic.

4.3 Double negation Diller-Nahm

In this section I give a brief description of what we get by applying a generalised double negation translation to the Diller-Nahm interpretation. That is I consider the category ${}_R\mathbf{Dill}$ for $R = (1 \overset{\mathsf{false}}{\longleftarrow} 1)$.[19] Again we give a concrete description of the category.

- The objects A of the category are of the form $\overline{A} \multimap R$ for $\overline{A} \in \mathbf{Dill}$. Thus formally we have
$$A \quad = \quad ((U \Rightarrow X^\bullet) \overset{\alpha^\bullet(u,\phi(u))}{\longleftarrow\!\!\!+\!\!\!-} U)$$
which we give also in logical notation
$$A = ((\exists u \forall x \overline{\alpha}(u,x)) \multimap R) = \exists \phi^{(U \Rightarrow X^\bullet)}.\forall u.\forall x \in \phi(u).\alpha(u,x)\,.$$

- Maps in ${}_R\mathbf{Dill}$ from an object
$$A = (U \Rightarrow X^\bullet \overset{\alpha^\bullet(u,\phi(u))}{\longleftarrow} U)$$
to an object
$$B = (V \Rightarrow Y^\bullet \overset{\beta^\bullet(v,\psi(v))}{\longleftarrow} V)$$
are given by maps
$$f : (U \Rightarrow X^\bullet) \to (V \Rightarrow Y^\bullet) \quad \text{and} \quad F : (U \Rightarrow X^\bullet) \times V \to U^\bullet$$
or equivalently
$$(f, F) : (U \Rightarrow X^\bullet) \times V \to U^\bullet \times Y^\bullet$$
such that an entailment which I write
$$(\alpha^\bullet)^\bullet(F(\phi,v), \phi(F(\phi,v))) \vdash \beta^\bullet(v, f(\phi,v))$$
holds in $\mathbb{P}((U \Rightarrow X^\bullet) \times V)$. This notation is ambiguous so I give its interpretation in logical notation.
$$\forall u \in F(\phi,v).\forall x \in \phi(F(\phi,v)).\alpha(u,x) \to \forall y \in f(\phi,v).\beta(v,y)\,.$$
I wonder whether anyone has written down this interpretation of implication before.

Since the above describes the maps in the category ${}_R\mathbf{Dill}$ there is indeed an associative composition.[20]

We are now closer to the ideal situation of Section 4.1 in that **Dill** is cartesian closed. But it does not have coproducts, so we cannot deduce that ${}_R\mathbf{Dill}$

[19] The category **Dill** has an initial object 0, but in view of Proposition 4.1 we do not want to consider the double negation category ${}_0\mathbf{Dill}$.

[20] I leave it to the reader to think that through. Again there is just one sensible thing to do.

is cartesian closed. Again we suppose that we are in the indexed situation discussed in Section 3.2.2, and deal with propositional connectives and quantifiers together. We write objects of ${}_R\mathbf{Dill}$ as $A = (\overline{A} \multimap R)$, $B = (\overline{B} \multimap R)$ and so on. Then we can define the following logical operations.

Disjunction	$A \vee B = (\overline{A} \times \overline{B} \Rightarrow R) = R^{\overline{A} \times \overline{B}}$
Falsity	$\bot = R$
Implication	$A \Rightarrow B = (A \times \overline{B} \Rightarrow R) = R^{A \times \overline{B}}$
Negation	$\neg A = A \Rightarrow R = R^A$
Universal quantification	$\forall z.A = ((\exists z\overline{A}) \Rightarrow R) = R^{\exists z\overline{A}}$

We briefly compare the associated categorical structure of ${}_R\mathbf{Dill}$ with that of ${}_R\mathbf{Dial}$ which we treated in the previous section.

1. Again falsity $\bot$ is a unit for the disjunction $\vee$, and $\vee$ is not monoidal but premonoidal in the sense of Power and Robinson [34]. Maps $A \to B$ of the form $R^f : R^{\overline{A}} \to R^{\overline{B}}$ for $f : \overline{B} \to \overline{A}$ are central.

2. As before, we can regard $A \Rightarrow B = \neg A \vee B$ as a derived operation. It is functorial and we have a natural isomorphism

$$ {}_R\mathbf{Dial}(C, \neg A \vee B) \;\cong\; {}_R\mathbf{Dial}(A, \neg C \vee B)\,; $$

   so $A \Rightarrow B \cong \neg A \vee B$ is a closed structure on ${}_R\mathbf{Dill}$.

3. $A \Rightarrow B$ is a cartesian closed structure in that the corresponding tensor would be a cartesian product; but that product is missing as $R^{\overline{A}} \otimes R^{\overline{B}}$ is not of the form R^C for any C. (It is a retract of $R^{\overline{A} \boxplus \overline{B}}$.) So we do not have a control category in the sense of Selinger [37]. But we are close; for example Thielecke's proof [41] identifying the focal and central maps goes through.

4. Again we can regard negation $\neg A = A \Rightarrow \bot$ as a derived operation; it is not an involution though we have A a retract of $\neg\neg A$ for all A.

5. The correctness of the definition of universal quantification is justified exactly as for ${}_R\mathbf{Dial}$.

6. We can consider the opposite category $\mathbf{Dill}_R$ where we get a premonoidal $\wedge$ dual to $\vee$ and the natural isomorphism

$$ \mathbf{Dial}_R(A \wedge \neg B, C) \cong \mathbf{Dial}_R(A \wedge \neg C, B)\,. $$

   Now $\wedge$ is a product in the centre of $\mathbf{Dill}_R$, so we get an example of a $\otimes\neg$-category in the sense of Thielecke [41]. This is a natural example of the situation where we have the structure analyzed by Thielecke, but simply do not have to hand the additional structure of a co-control category in the sense of Selinger [37].

5 Classical Proof

In the previous section I discussed approaches to classical proof theory which depend on some form of the double negation translation. These do not respect the symmetry of classical logic which is expressed in the sequent calculus. It is natural to ask for a direct coding-free formulation of that notion of classical proof implicit in the sequent calculus. As indicated in the introduction, we have then to confront the inherent non-determinism exemplified by cutting weakened formulae: one wants a notion of equality of proof, but this equality cannot be preserved by cut elimination. One obvious idea is to regard a proof as being a non-deterministic choice between the various cut-free proofs to which it reduces. This is a bit crude as just expressed and as yet we have no satisfactory semantics for proofs along these lines. As a prelude to a suggestion as to the shape of such a semantics, I discuss first the basic semantics of the sequent calculus and its relation with Linear Logic. At least this provides an illustration of Categorical Proof Theory in action.

5.1 Polycategories

Many years ago Szabo gave a categorical formulation of the core of the sequent calculus in terms of the notion of a polycategory (Szabo [39]). More recently a number of variants of the basic idea have been considered ([28], [29], [7], [8], [5]). In the last two references one can find a syntactic proof of a conservative extension result: essentially that any polycategory embeds fully and faithfully in the free linearly distributive category which it generates.[21] The proof in particular of faithfulness is really quite involved and makes heavy use of an intricate analysis of proof nets in [5].

As an illustration of the value of abstract proof theory I give here an abstract category theoretic proof of the result. I shall do this in the simple symmetric case which is the fundamental case for traditional logic, though the reader will see that other versions follow the same lines.

I recall the idea of polycategory. First let us consider a general form of the core of the sequent calculus. Suppose we have a collection of propositions A, B, C, and so on; and suppose further that we have, for sequences Γ, Δ of propositions, collections of proofs of $\Gamma \vdash \Delta$. Suppose finally that this collection of proofs contains identity proofs, is closed under the cut rule and under exchange. So the basic structural rules are

$$\frac{}{A \vdash A}\ \text{Identity} \qquad\qquad \frac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Sigma}{\Gamma, \Delta \vdash \Pi, \Sigma}\ \text{Cut}$$

and we have in addition the general rule

$$\frac{\Gamma \vdash \Delta}{\sigma\Gamma \vdash \tau\Delta}\ \text{Exchange}$$

[21] This is not quite how Cockett and Seely [8] describe the matter, but the difference is inessential.

where σ and τ are arbitrary permutations. We can model this fragment of proof theory in a symmetric polycategory.

A *symmetric polycategory* (henceforth just *polycategory*) $\mathcal{P}$ consists in the first instance of the following data.

- A collection ob$\mathcal{P}$ of objects of $\mathcal{P}$.
- For each pair of finite sequences Γ and Δ of objects, a collection $\mathcal{P}(\Gamma;\Delta)$ of maps from Γ to Δ.

First this data is required to satisfy symmetry conditions corresponding to the exchange rule.[22] Secondly we require the following data.

- For each object A an identity $1_A \in \mathcal{P}(A;A)$.
- For each Γ, Δ, A, Π, Σ a composition

$$\mathcal{P}(\Gamma;\Delta,A) \times \mathcal{P}(A,\Pi;\Sigma) \to \mathcal{P}(\Gamma,\Pi;\Delta,\Sigma)\,.$$

This data satisfies identity and associativity laws, compatibly with the symmetries. For details of more or less the same definition and for related ideas, see [39]), [28], [7] or [8].

In [7] and more fully in [8] Cockett and Seely consider the notion of a *linearly distributive category*. In essence this is a category modelling the positive part of multiplicative linear logic: that is, multiplicative linear logic without the involutive duality. So there are (symmetric) monoidal structures $(I,\otimes)$ and $(\bot\,⅋)$ on $\mathbb{D}$ and a linear distributivity, that is, a natural transformation

$$A \otimes (B ⅋ C) \to (A \otimes B) ⅋ C$$

satisfying simple coherence conditions (for these see in particular [8]).

Obviously any linearly distributive category $\mathbb{D}$ gives rise to a polycategory $Poly(\mathbb{D}) = Poly_{\mathbb{D}}$ where

$$Poly_{\mathbb{D}}(\Gamma;\Delta) = \mathbb{D}(\otimes\Gamma, ⅋\Delta)\,.$$

There is a sensible way to make $Poly$ into a 2-functor between the obvious 2-categories: $Poly$: **LinDist** $\to$ **Poly**. Conversely given a polycategory $\mathcal{P}$ we can freely construct a linearly distributive category $LinDist(\mathcal{P}) = LinDist_{\mathcal{P}}$ generated as such by the objects of $\mathcal{P}$ and by maps $\otimes\Gamma \to ⅋\Delta$ corresponding to the polymaps $\Gamma \to \Delta$, where we require that the respective composites agree. (Cockett and Seely have a slightly different formulation of all this.) This construction provides a 2-functor $LinDist$: **Poly** $\to$ **LinDist**. It is easy to see that one has a 2-adjunction $LinDist \dashv Poly$, and one main theorem of Cockett and Seely [8] is the following conservativity result.

[22]We can express these conditions as follows. Any pair Γ, Δ can be regarded as a pair of objects in the free symmetric strict monoidal category on the collection of propositions. Maps in that free category are given by object preserving bijections between the strings. Any such pair of maps $\sigma : \Gamma \to \sigma\Gamma$ and $\tau : \Delta \to \tau\Delta$ induces a bijection $(\sigma;\tau) : \mathcal{P}(\Gamma;\Delta) \to \mathcal{P}(\sigma\Gamma;\sigma\Delta)$. And these bijections compose in the natural way.

Theorem 5.1 *In the 2-adjunction LinDist ⊣ Poly the unit*

$$\mathcal{P} \to Poly(LinDist(\mathcal{P}))$$

is full and faithful on each polycategory $\mathcal{P}$.

Fullness can be derived by a cut elimination argument, but the argument for faithfulness given in Cockett and Seely [8] depends on the careful syntactic analysis of Blute, Cockett, Seely and Trimble [5]. Here I explain how the result can be derived purely categorically, essentially by a Yoneda argument. Just as the Yoneda embedding is one of the basic constructions of pure category theory, the embedding described here seems to be a basic one for categorical proof theory.

A *bimodule* (or plain *module* as we consider no others) $\mathcal{M}$ over $\mathcal{P}$, is a family of sets $\mathcal{M}(\Gamma;\Delta)$ satisfying symmetry conditions as for $\mathcal{P}$ above, and with natural left actions

$$\mathcal{P}(\Pi;\Sigma,A) \times \mathcal{M}(A,\Gamma;\Delta) \to \mathcal{M}(\Pi,\Gamma;\Sigma,\Delta)\,,$$

and right actions

$$\mathcal{M}(\Gamma;\Delta,A) \times \mathcal{P}(A,\Pi;\Sigma) \to \mathcal{M}(\Gamma,\Pi;\Delta,\Sigma)\,,$$

which commute. The naturality conditions parallel the identity and associativity laws for $\mathcal{P}$. Note that $\mathcal{P}$ itself is a bimodule: it has both left and right actions by $\mathcal{P}$ and these commute by the associativity of polycategory composition. Let us now agree to drop the 'bi' and refer only to modules.

A map (or natural transformation) $\alpha : \mathcal{M} \to \mathcal{N}$ of modules consists of an indexed family $\alpha_{\Gamma;\Delta} : \mathcal{M}(\Gamma;\Delta) \to \mathcal{N}(\Gamma;\Delta)$ respecting the actions: for the left action one has

$$\begin{array}{ccc}
\mathcal{P}(\Pi;\Sigma,A) \times \mathcal{M}(A,\Gamma;\Delta) & \longrightarrow & \mathcal{M}(\Pi,\Gamma;\Sigma,\Delta) \\
\downarrow & & \downarrow \\
\mathcal{P}(\Pi;\Sigma,A) \times \mathcal{N}(A,\Gamma;\Delta) & \longrightarrow & \mathcal{N}(\Pi,\Gamma;\Sigma,\Delta)
\end{array}$$

commuting, and similarly for the right action.

We also have the notion of *representable modules*. The left representable module ${}_A\mathcal{P}$ is defined by ${}_A\mathcal{P}(\Pi;\Sigma) = \mathcal{P}(A,\Pi;\Sigma)$. The right representable module $\mathcal{P}_A$ is defined by $\mathcal{P}_A(\Gamma;\Delta) = \mathcal{P}(\Gamma;\Delta,A)$.

5.2 The envelope

Suppose that $\mathcal{P}$ is a polycategory. Define $Env(\mathcal{P})$, the *envelope* of $\mathcal{P}$, to be the following category.

- Objects of $Env(\mathcal{P})$ are pairs $\mathcal{U}$, $\mathcal{X}$ of modules equipped with a natural map

$$\alpha : \mathcal{U}(\Gamma; \Delta) \times \mathcal{X}(\Pi; \Sigma) \to \mathcal{P}(\Pi, \Gamma; \Sigma, \Delta)\,.$$

  We write such an object simply as $\alpha : \mathcal{U} \times \mathcal{X} \to \mathcal{P}$.

- Maps of $Env(\mathcal{P})$ from $\alpha : \mathcal{U} \times \mathcal{X} \to \mathcal{P}$ to $\beta : \mathcal{V} \times \mathcal{Y} \to \mathcal{P}$ are given by natural maps $\phi_0 : \mathcal{U} \to \mathcal{V}$ and $\phi_1 : \mathcal{Y} \to \mathcal{X}$ such that

$$\begin{array}{ccc} \mathcal{U}(\Gamma;\Delta) \times \mathcal{Y}(\Pi;\Sigma) & \xrightarrow{1 \times \phi_1} & \mathcal{U}(\Gamma;\Delta) \times \mathcal{X}(\Pi;\Sigma) \\ \phi_0 \times 1 \downarrow & & \downarrow \\ \mathcal{V}(\Gamma;\Delta) \times \mathcal{X}(\Pi;\Sigma) & \longrightarrow & \mathcal{P}(\Pi, \Gamma; \Sigma, \Delta) \end{array}$$

  commutes.[23]

We shall now show that the category $Env(\mathcal{P})$ carries the structure of a $*$-autonomous category. There is an obvious involution of the form

$$(\mathcal{U} \times \mathcal{X} \to \mathcal{P})^{\perp} = (\mathcal{X} \times \mathcal{U} \to \mathcal{P}).$$

Given that, it is enough to define a suitable tensor product. For $\alpha : \mathcal{U} \times \mathcal{X} \to \mathcal{P}$ and $\beta : \mathcal{V} \times \mathcal{Y} \to \mathcal{P}$, we define their tensor product $\alpha \otimes \beta : (\mathcal{U} \otimes \mathcal{V}) \times \mathcal{M} \to \mathcal{P}$ as follows. First we let $\mathcal{U} \otimes \mathcal{V}$ be given by

$$(\mathcal{U} \otimes \mathcal{V})(\Gamma; \Delta) = \sum \{\mathcal{U}(\Gamma_1; \Delta_1) \times \mathcal{V}(\Gamma_2; \Delta_2) \,|\, \Gamma = \Gamma_1 + \Gamma_2\,,\ \Delta = \Delta_1 + \Delta_2\}\,,$$

where the sum is taken over all ways of decomposing the strings. Then we define $\mathcal{M}$ by setting $\mathcal{M}(\Pi; \Sigma)$ to be the set of all pairs (χ_0, χ_1) with

$$\chi_0 : \mathcal{V}(\Phi; \Psi) \to \mathcal{X}(\Pi, \Phi; \Psi, \Sigma), \quad \text{and} \quad \chi_1 : \mathcal{U}(\Phi; \Psi) \to \mathcal{Y}(\Pi, \Phi; \Psi, \Sigma)\,,$$

natural in Φ and Ψ, and such that the obvious diagram of the form

$$\begin{array}{ccc} \mathcal{U}(\Phi_1;\Psi_1) \times \mathcal{V}(\Phi_2;\Psi_2) & \longrightarrow & \mathcal{U}(\Phi_1;\Psi_1) \times \mathcal{X}(\Pi, \Phi_2;\Psi_2, \Sigma) \\ \downarrow & & \downarrow \\ \mathcal{V}(\Phi_2;\Psi_2) \times \mathcal{Y}((\Pi, \Phi_1;\Psi_1, \Sigma) & \longrightarrow & \mathcal{P}(\Pi, \Phi_1, \Phi_2; \Psi_1, \Psi_2, \Sigma) \end{array}$$

commutes. The action $\alpha \otimes \beta$ is defined as follows.
Take $(u, v) \in \mathcal{U} \otimes \mathcal{V}(\Gamma; \Delta)$, and $(\chi_0, \chi_1) \in \mathcal{M}(\Pi; \Sigma)$. We have $u \in \mathcal{U}(\Gamma_1; \Delta_1)$ and $v \in \mathcal{V}(\Gamma_2; \Delta_2)$ for some decomposition; then $x = (\chi_0)_{\Gamma_2;\Delta_2}(v) \in \mathcal{X}(\Pi, \Gamma_2; \Delta_2, \Sigma)$ and $y = (\chi_1)_{\Gamma_1;\Delta_1}(u) \in \mathcal{Y}(\Pi, \Gamma_1; \Delta_1, \Sigma)$ are such that $\alpha(u, x) = \beta(v, y)$; this is the value for $(\alpha \otimes \beta)((u, v), (\chi_0, \chi_1))$.
One checks that this action is functorial; and then it is routine to work through the details of the proof of the following proposition.

[23] There is a clear sense in which the envelope construction combines the idea of a category of presheaves with that of the Chu construction.

Proposition 5.2 *With the structure just defined Env($\mathcal{P}$) is a $*$-autonomous category.*

To each object A of $\mathcal{P}$ we associate the object $\gamma_A : \mathcal{P}_A \times {}_A\mathcal{P} \to \mathcal{P}$ obtained by composing the right and left representable modules in $\mathcal{P}$. One should think of this object γ_A as the operation 'cut on a formula A'. Now any polymap $f \in \mathcal{P}(\Gamma; \Delta)$ induces an obvious map

$$\gamma_f : \bigotimes_{C \in \Gamma} \gamma_C \to \bigparr_{D \in \Delta} \gamma_D$$

in $Env(\mathcal{P})$; and γ_f depends naturally on f in the obvious sense, so that there is a map of polycategories $\mathcal{P} \to Poly(Env(\mathcal{P}))$. Then a Yoneda argument shows the following.

Proposition 5.3 *For any polycategory $\mathcal{P}$, the map $\mathcal{P} \to Poly(Env(\mathcal{P}))$ is full and faithful.*

We shall refer to the map $\mathcal{P} \to Poly(Env(\mathcal{P}))$ as the Yoneda embedding for polycategories. One can think of it as suggesting (though after the event) that the basic idea of the sequent calculus gives rise inevitably to multiplicative linear logic.

We observed that $Env(\mathcal{P})$ is $*$-autonomous and so in particular linearly distributive. Hence by the 2-adjunction we get a map $Lindist(\mathcal{P}) \to Env(\mathcal{P})$ of linearly distributive categories so that the above Yoneda embedding factors

$$\mathcal{P} \to Poly(Env(\mathcal{P})) \quad = \quad \mathcal{P} \to Poly(Lindist(\mathcal{P})) \to Poly(Env(\mathcal{P}))$$

through the unit $\mathcal{P} \to Poly(Lindist(\mathcal{P}))$. As $\mathcal{P} \to Poly(End(\mathcal{P}))$ is faithful so also is the unit $\mathcal{P} \to Poly(Lindist(\mathcal{P}))$. This proves the faithful half of Theorem 5.1.[24]

5.3 $*$-polycategories

To complete the picture I mention the notion of $*$-polycategory, which models a symmetric form of the sequent calculus.

Suppose we have a collection of propositions A, B, C, and so on, equipped with an involution $(-)^{\perp}$. Suppose further that we have for sequences Γ, Δ of propositions collections of proofs of $\Gamma \vdash \Delta$. Suppose finally that this collection of proofs contains identity proofs and is closed under cut and under exchange as before; and that in addition it is closed under the classical rules for negation.

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^{\perp}, \Delta}\,\text{Negation} - \text{right} \qquad\qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^{\perp} \vdash \Delta}\,\text{Negation} - \text{left}$$

We can model this fragment of logic in a *$*$-polycategory.*

A *symmetric $*$-polycategory* (henceforth just *$*$-polycategory*) $\mathcal{P}$ consists in the first instance of the following data.

[24] One can in fact get the full result by a use of a glueing argument along lines due to Lafont.

- A collection ob$\mathcal{P}$ of objects of $\mathcal{P}$ closed under an involutory negation $(-)^{\perp}$.
- For each pair of finite sequences Γ and Δ of objects, a collection $\mathcal{P}(\Gamma;\Delta)$ of maps from Γ to Δ.

This data is required to satisfy conditions corresponding to the exchange rule and the rules for negation.[25] Secondly we require the following data.

- For each object A an identity $1_A \in \mathcal{P}(A;A)$.
- For each Γ, Δ, A, Π, Σ a composition

$$\mathcal{P}(\Gamma;\Delta,A)\times\mathcal{P}(A,\Pi;\Sigma)\rightarrow\mathcal{P}(\Gamma,\Pi;\Delta,\Sigma)\,.$$

This data satisfies identity and associativity laws, compatibly with the symmetries. I omit the details.

Recall that the multiplicative fragment of linear logic corresponds to the notion of a $*$-autonomous category. It is easy to see that a $*$-autonomous category $\mathbb{A}$ gives rise to a $*$-polycategory $SPoly(\mathbb{A}) = SPoly_{\mathbb{A}}$, where

$$SPoly_{\mathbb{A}}(\Gamma;\Delta) = \mathbb{A}(\otimes\Gamma, \parr\Delta)$$

A duality is a contravariant structure and hence neither $*$**Aut** nor $*$**Poly** are naturally 2-categories; rather they are naturally enriched in groupoids. Then $SPoly$ extends to a groupoid enriched functor $SPoly : *\mathbf{Aut} \rightarrow *\mathbf{Poly}$. Conversely, given a $*$-polycategory $\mathcal{P}$, we can freely construct a $*$-autonomous category $SAut(\mathcal{P})$ generated by the objects and polymaps and subject to obvious identifications as before. Then $SAut : *\mathbf{Poly} \rightarrow *\mathbf{Aut}$ is a groupoid enriched functor, and we have a groupoid enriched adjunction $SAut \dashv SPoly$. There is a corresponding conservativity result.

Theorem 5.4 *In the groupoid enriched adjunction $SAut \dashv SPoly$, the unit*

$$\mathcal{P} \rightarrow SPolySAut(\mathcal{P})$$

is full and faithful for any $*$-polycategory $\mathcal{P}$.

The faithfulness is again a consequence of a Yoneda argument. $Env(\mathcal{P})$ is a $*$-autonomous category for any polycategory $\mathcal{P}$. If $\mathcal{P}$ is itself a $*$-polycategory then $A \rightarrow (\gamma_A : \mathcal{P}_A \times_A \mathcal{P} \rightarrow \mathcal{P})$ is a map of $*$-polycategories $\mathcal{P} \rightarrow SPolyEnv(\mathcal{P})$. Then the same Yoneda argument as before gives the following proposition.

Proposition 5.5 *For any $*$-polycategory $\mathcal{P}$, the map $\mathcal{P} \rightarrow SPolyEnv(\mathcal{P})$ is full and faithful.*

[25] We can express these conditions as follows. Any pair Γ, Δ gives us a sequence $\Gamma^{\perp}, \Delta$ which we can regard as an object in the free symmetric strict monoidal category on the collection of propositions. Maps in that free category are given by object preserving bijections between the strings. Each such map $\sigma : \Gamma^{\perp}, \Delta \rightarrow \Pi^{\perp}, \Sigma$ induces a map $\sigma_* : \mathcal{P}(\Gamma;\Delta) \rightarrow \mathcal{P}(\Pi;\Sigma)$. And these compose in the natural way.

This is the Yoneda embedding for $*$-polycategories. Now for any $*$-polycategory $\mathcal{P}$, the groupoid enriched adjunction gives a map $SAut(\mathcal{P}) \to Env(\mathcal{P})$ of $*$-autonomous categories so that the Yoneda embedding factors

$$\mathcal{P} \to SPolyEnv(\mathcal{P}) = (\mathcal{P} \to SPolySAut(\mathcal{P}) \to SPolyEnv(\mathcal{P})$$

through the unit $\mathcal{P} \to SPolySAut(\mathcal{P})$. As $\mathcal{P} \to SPolyEnv(\mathcal{P})$ is full and faithful, so also is the unit $\mathcal{P} \to SPolySAut(\mathcal{P})$. This proves the faithful part of Theorem 5.4.[26]

5.4 Modelling Classical Sequent Calculus

I now describe in outline an approach to the semantics of proofs in the classical sequent calculus which is intended to be faithful to its essential mathematical structure. This approach comes from a consideration of the system for annotating proofs developed by Bierman and Urban and analyzed in detail in Urban's PhD dissertation [45]. The papers [46] and [47] give a compressed account of some of the work.

So what should we regard as the essential mathematical structure of the sequent calculus? The basic idea is surely that we compose proofs by cuts. If we think of this as plugging proof modules together, the order of the plugging should not matter and we expect at least to identify sequent calculus proofs up to naturally commuting cuts. Thus we expect there to be a $*$-polycategory $\mathcal{C}$ of propositions and classical proofs. We start with the following intuitions.

- We should identify proofs up to commutative conversions as is the basic idea of proof nets.
- The key step of cut elimination, that of eliminating logical a cut at a critical formula, should be regarded as unproblematic and performing such a cut should leave the meaning of a proof unchanged.
- More generally we expect to identify proofs when the cut elimination process transforms a cut in only one way. However when the cut elimination process has a choice we get no identification.[27]

Now the $*$-polycategory $\mathcal{C}$ will contain a core category $\mathbb{C}$ of propositions and proofs. Clearly $\mathbb{C}$ must (in some sense) be equipped with the structure, *true*, *and*, *false*, *or*, *not* of classical logic: we write this structure as 1, $\wedge$, 0, $\vee$ and $(-)^{\perp}$, the last being the involutory negation.

The critical question is whether or not polymaps in $\mathcal{C}$ are represented in $\mathbb{C}$ by $\wedge$ and $\vee$: that is, in effect, whether or not $\mathbb{C}$ is a $*$-autonomous category

[26] Again the full result will follow from a more complex argument following Lafont.

[27] Many have had the obvious thought to regard the proof as a non-deterministic choice between the two results; but without careful caveats that is not faithful to the dynamics of proofs. In any case what is proposed here is more innocent in that it makes no preemptive assumptions about meaning.

with $SPoly(\mathbb{C}) \cong \mathcal{C}$.[28] The idea that we have $SPoly(\mathbb{C}) \cong \mathcal{C}$ seems appealing in terms of mathematical elegance, so let us see what it means. It amounts requiring that proofs of $\Gamma, A \wedge B \vdash \Delta$ coincide with proofs of $\Gamma, A, B \vdash \Delta$, or dually whether proofs of $\Gamma \vdash A \vee B, \Delta$ coincide with proofs of $\Gamma \vdash A, B, \Delta$. Suppose we start (for example) with a proof of $\Gamma, A, B \vdash \Delta$, form from it a proof of $\Gamma, A \wedge B \vdash \Delta$ in the obvious way, and then cut that with the simple proof of $A, B \vdash A \wedge B$. Then our intuition is surely that we are back where we started.[29] However it is not quite so plausible that all proofs of $\Gamma, A \wedge B \vdash \Delta$ are (equivalent to) ones deriving directly from proofs of $\Gamma, A, B \vdash \Delta$. Indeed some computational intuition is against that.

We can look at the issue from another point of view. Note that given proofs $A \vdash^f B$ and $C \vdash^g D$, there is a canonical proof $A \wedge C \vdash^{f \wedge g} B \wedge D$ given by the following

$$\cfrac{\cfrac{A \vdash^f B \qquad C \vdash^g D}{A, C \vdash B \wedge D}\ \wedge\text{−right}}{A \wedge C \vdash B \wedge D}\ \wedge\text{−left}\,.$$

Similarly we have $f \vee g$ a proof of $A \vee C \vdash B \vee D$. Thus we have operations $\wedge$ and $\vee$ on maps and the question then is whether or not these operations on maps are functorial. If we take the full range of possibilities in the sequent calculus seriously then simple experiments will show that making $\wedge$ and $\vee$ functorial would force us to identify proofs which are intuitively distinct; so we should not assume functoriality. Now if $\wedge$ and $\vee$ represent multimaps then they will be functorial.[30] So by the same token $\wedge$ and $\vee$ should not represent multimaps.

Suppose we agree that $\wedge$ and $\vee$ are not be functorial on $\mathbb{C}$: then we are left with the question of exactly what properties these operations should have. Since the problems arise out of the structural rules, it seems natural to require that linear proofs should play a special role. So without further ado, here is a modest proposal for the abstract definition of a semantics for classical proof. A *static model for classical propositional logic* consists of the following data.

- An identity on objects faithful functor $\mathbb{L} \to \mathbb{C}$. (We think of $\mathbb{C}$ as being a category of propositions and classical proofs; and of $\mathbb{L}$ as being a subcategory of the same propositions but with linear proofs.)
- The structure of a $*$-autonomous category on $\mathbb{L}$. We write the structure as 1, $\wedge$, 0, $\vee$ and the involution as $(-)^\perp$.
- An extension of $(-)^\perp$ to an involution on $\mathbb{C}$.

[28] Of course we cannot have $\mathbb{C}$ $*$-autonomous and the diagonal $A \vdash A \wedge A$ and codiagonal $A \vee A \vdash A$ natural. For then $\mathbb{C}$ would be a self dual cartesian closed category and so by Proposition 4.1 a Boolean algebra.

[29] Thus, despite the difficulties we are about to discuss, it seems possible to give a model in purely categorical rather than polycategorical terms.

[30] With natural assumptions there is in fact a converse.

- Extensions of $\wedge$ and $\vee$ to act on maps of $\mathbb{C}$, preserving the duality by $(-)^\perp$. These extensions (are not functorial, but) satisfy naturality conditions with respect to the maps of $\mathbb{L}$. Thus if

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} \\ A' & \xrightarrow[f']{} & B' \end{array} \quad \text{and} \quad \begin{array}{ccc} C & \xrightarrow{g} & D \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle u} \\ C' & \xrightarrow[g']{} & D' \end{array}$$

  commute in $\mathbb{C}$ with r, s, t and u in $\mathbb{L}$, then

$$\begin{array}{ccc} A \wedge C & \xrightarrow{f \wedge g} & B \wedge D \\ {\scriptstyle r \wedge t}\downarrow & & \downarrow{\scriptstyle s \wedge u} \\ A' \wedge C' & \xrightarrow[f' \wedge g']{} & B' \wedge D' \end{array} \quad \text{and} \quad \begin{array}{ccc} A \vee C & \xrightarrow{f \vee g} & B \vee D \\ {\scriptstyle r \vee t}\downarrow & & \downarrow{\scriptstyle s \vee u} \\ A' \vee C' & \xrightarrow[f' \vee g']{} & B' \vee D' \end{array}$$

  commute.
  In addition the linear distributivity $A \wedge (B \vee C) \to (A \wedge B) \vee C$ from $\mathbb{L}$ is natural in $\mathbb{C}$.

- Maps $A \to A \wedge A$, $A \to 1$, and $A \vee A \to A$ and $0 \to A$ in $\mathbb{C}$, again dual with respect to $(-)^\perp$. These again satisfy naturality conditions with respect to $\mathbb{L}$. If $r : A \to B$ in $\mathbb{L}$, then the diagrams

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle r}\downarrow & & \downarrow \\ B & \longrightarrow & 1 \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & A \wedge A \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle r \wedge r} \\ B & \longrightarrow & B \wedge B \end{array}$$

$$\begin{array}{ccc} 0 & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle r} \\ 0 & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} A \vee A & \longrightarrow & A \\ {\scriptstyle r \vee r}\downarrow & & \downarrow{\scriptstyle r} \\ B \vee B & \longrightarrow & B \end{array}$$

  all commute.

I stress that this is very much a preliminary and tentative suggestion.[31] Its most definite merit is formal: it fits well with the philosophy proposed by Power [33]. According to that categorical semantics should be formulated in terms of some

[31] Recent investigations by Christian Urban suggest at least one refinement.

kind of algebraic structure in a suitable enriched setting.[32] The definition is certainly crude in one probably inessential respect: it fails to account for the intuition that a proof $\Gamma \vdash \Delta$ may be linear in some arguments and not in others. However the main problems relate to semantics. I am still trying to find a non-trivial natural mathematical semantics, so for that if no other reason the value of the suggestion must be in doubt.

The key features of the definition are that $\wedge$ and $\vee$ are not functorial with respect to all maps, and that the diagonal $A \rightarrow A \wedge A$ and multiplication $A \vee A \rightarrow A$ are not natural with respect to all maps. The second of these 'negative' features does not seem too bad. However the existence of an extension of $\wedge$ and $\vee$ to all maps natural with respect to composition with linear maps, but not generally functorial seems beyond the scope of current models.

Let us review the situation in the language of polycategories. We distinguish between linear and classical proofs in the sequent calculus. Thus we expect to have two $*$-polycategories, one $\mathcal{L}$ of linear proofs and the other $\mathcal{C}$ of classical proofs. There will be an embedding $\mathcal{L} \rightarrow \mathcal{C}$. Now we aim for some relation between the initial category theoretic model $\mathbb{L} \rightarrow \mathbb{C}$ and $\mathcal{L} \rightarrow \mathcal{C}$. Cut elimination is unproblematic for linear logic, and this finds expression in the natural $\mathcal{L}(\Gamma, \Delta) \cong \mathbb{L}(\wedge\Delta, \vee\Gamma)$. We can read this as saying $\mathcal{L} \cong SPoly(\mathbb{L})$. It would be mathematically appealing to ask also that $\mathbb{C}$ be $*$-autonomous with $\mathcal{C} \cong SPoly(\mathbb{C})$. Prima facie there is nothing to say against this. Models given by a $*$-autonomous embedding $\mathbb{L} \rightarrow \mathbb{C}$ would be relatively easy to understand. Unease arises from the fact that this simple idea conflicts with some kind of (admittedly tenuous) computational intuition about the sequent calculus. The alternative suggestion made here, roughly based on a proof theoretic analysis ([46] and [47]), is perhaps equally unsatisfactory in that we have no good mathematical models.

We began the paper with reflections on the contrast between philosophical and mathematical ways of giving semantic motivation for a theory of proofs; but I have focussed throughout on mathematical motivation arising from elegance of structure. So it is instructive that our brief look at a possible semantics for classical proof finds tension between different kinds of mathematical motivation. Such tension may not be easy to resolve at the technical level. But if that means we need a better conceptual analysis, we shall I fear be falling back on philosophical motivation in the last resort.

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[32] Indeed the exact form of the definition above arose in discussion with Power as to how to explain the basic intuitions of classical proof in terms of exactly the same enrichment that he uses in Power [33].

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