Lineales

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The first aim of this note is to describe an algebraic structure, more primitive than lattices and quantales, which corresponds to the intuitionistic flavour of Linear Logic we prefer. This part of the note is a total trivialisation of ideas from category theory and we play with a toy-structure a not distant cousin of a toy-language.

The second goal of the note is to show a generic categorical construction, which builds models for Linear Logic, similar to categorical models *GC* of [de P], but more general. The ultimate aim is to relate different categorical models of linear logic.

The first part of the note consists of two sections. The first section introduces lineales; the second adds some structure to lineales, compares our work to other approaches and show the main result of this part.

The second part of the note consists of four sections, which run along similar lines to part I. In section 3 we define our basic categorical construction, section 4 adds the extra structure corresponding to section 2 and shows the main result of part II. Section 5, adding the modalities « ! » and « ? », has no corresponding section in part I, as we have not even tried to find the right notion of « ! » in the restricted set-up of lineales. Section 6 describes some preliminary conclusions and further work.

1. Introducing lineales

We start by considering a very familiar structure, a commutative monoid in the category of posets. We are thinking of posets as a restriction of the

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general notion of categories. That is the opposite of what people normally d in CS when they explain the notion of a category as a generalization of a poset We call a commutative (or symmetric) monoid in the category **Posets** a pre-lineale. [In the more general set-up we're thinking of a monoid object in the category of categories.]

ordered monoid or monoidal poset

Definition 1 *A pre-lineale is a poset* $(L, \leq)$ *with a given compatible symmetric monoidal structure* $(L, \circ, e)$*. That is, a set L equipped with a binary relation « ≤ » satisfying :*

- $a \leq a$ *for all a in L (reflexivity)*
- $a \leq b$ *and* $b \leq c \Rightarrow a \leq c$ *(transitivity)*
- $a \leq b$ *and* $b \leq a \Rightarrow a = b$ *(antisymetry)*

together with a monoid structure $(\circ, e)$ *consisting of a « multiplication »* $\circ : L \times L \to L$ *and a distinguished object « e » of L, such that the following hold :*

- $(a \circ b) \circ c = a \circ (b \circ c)$ *(associativity)*
- $a \circ e = e \circ a = a$ *(identity)*
- $a \circ b = b \circ a$ *(symmetry)*

The structures are compatible in the sense that, if $a \leq b$*, we have* $a \circ c \leq b \circ c$*, for all c in L.*

We write a quadruple $(L, \leq, \circ, e)$ for a pre-lineale. Note that, even if we want to think of «∘» as a form of conjunction, we do not have $a \circ a = a$ (idempotency) nor $a \leq e$ for all a in L. Thus the relation between the order structure and the multiplication is not as tight as in a sup-lattice.

But a pre-lineale is not the toy-structure we want to play with. A pre-lineale corresponds, in the more general set-up of categories, to a *symmetric monoidal category* and we are interested in *symmetric monoidal closed categories.* To trivialise this notion we first define :

Definition 2 *Suppose L is a pre-lineale and* $a, b \in L$*. If there exists a largest* $x \in L$ *such that* $a \circ x \leq b$ *then this element is denoted* $a \multimap b$ *and it is called the* relative pseudocomplement of a wrt b.

Thus, by definition, if $a \multimap b$ exists in a pre-lineale L then

- $a \circ (a \multimap b) \leq b$
- *if* $a \circ y \leq b$ for some y, then $y \leq a \multimap b$

Definition 3 *A* lineale *is a pre-lineale* $(L, \leq, \circ, e)$ *such that* $a \multimap b$ exists for all a and b in L.
Since we defined a lineale to be a simplification of the notion of a symmetric monoidal closed category, we have an obvious proposition :

Proposition 1 *A lineale* $(L, \leq, \circ, e, \multimap)$ *has the following properties :*

1. *If* $a \leq b$, *for any* c *in* L, $c \multimap a \leq c \multimap b$ and $b \multimap c \leq a \multimap c$;

2. $a \circ b \leq c \Leftrightarrow a \leq b \multimap c$

The proof is very easy, it only uses the definition of $\multimap$ and L. Observe that the item *1* in the proposition says, in the more general set-up of categories, that $\multimap : L \times L \to L$ is a « bifunctor », contravariant in its first coordinate and covariant in the second coordinate, while item 2 says there is an adjunction between functors $() \circ b : L \to L$ and $b \multimap () : L \to L$.

Another observation is that as $e \circ a \leq a$ for any $a \in L$, we know $e \leq a \multimap a$ and $a \leq a \multimap e$ for any $a \in L$.

Note that if we denote by $\bot$ *any* element of L and write $(a)^\bot$ for $(a \multimap \bot)$ we have :

(i) $a \leq b \Rightarrow b^\bot \leq a^\bot$ by prop 1.1.

(ii) $a \circ (a \multimap \bot) \leq \bot \Leftrightarrow a \circ a^\bot \leq \bot$ implies $a \leq a^\bot \multimap \bot \equiv a^{\bot\bot}$ by prop 1.2.

Properties (i) and (ii) are called by Dunn the Intuitionistic Contraposition.

Definition 4 *A* Heyting lineale *is a lineale* $(L, \leq, \circ, e, \multimap)$ *equipped with a given compatible symmetric monoidal structure* $(\Box, \bot)$ *weakly de Morgan-dual to « ∘ ». That means that*

- *the given structure* $(\Box, \bot)$ *satisfies*
  - *(associativity)* $a \Box (b \Box c) = (a \Box b) \Box c$
  - *(symmetry)* $a \Box b = b \Box a$
- *the structure* $(\Box, \bot)$ *is compatible with* $(L, \leq, \circ, e)$ *means that, as before, if* $a \leq b$ *then for any* c *in* L, $a \Box c \leq b \Box c$
- *the object* $\bot$ *is the identity for* $\Box$

$$a \Box \bot = \bot \Box a = a$$

- *we have associative (or absorptive) laws :*

$$(a \,\Box\, b) \circ c \leq a \,\Box\, (b \circ c)$$

$$a \circ (b \,\Box\, c) \leq (a \circ b) \,\Box\, c$$

Note that, if we write $(a)^{\perp}$ for $(a \multimap \perp)$, $\perp$ the identity for $\Box$, we ca show

$$a^{\perp} \,\Box\, b \leq a \multimap b$$

simply using symmetry of « ∘ » and the distributive law above, as follows :

$$a \circ (a^{\perp} \,\Box\, b) \leq (a \circ a^{\perp}) \,\Box\, b =$$

$$(a \circ a \multimap \perp) \,\Box\, b \leq \perp \Box\, b = b$$

With definition 4 we are trying to capture the (intuitionistic !) notion tha conjunction and disjunction are not de Morgan dual — as they are in Classica Logic, but instead, we have :

- $(a \,\Box\, b)^{\perp} = a^{\perp} \circ b^{\perp}$,
- $a^{\perp} \,\Box\, b^{\perp} \leq (a \circ b)^{\perp}$

We can prove,

Proposition 2 *A Heyting lineale L satisfies :*

(a) $a^{\perp} \circ b^{\perp} \leq (a \,\Box\, b)^{\perp}$,

(b) $a^{\perp} \,\Box\, b^{\perp} \leq (a \circ b)^{\perp}$

To show *(b)*, as $(a \circ b)^{\perp} = (a \circ b) \multimap \perp$, it is enough to show $(a^{\perp} \,\Box\, b^{\perp}) \circ (a \circ b) \leq \perp$, easy as

$$(a^{\perp} \,\Box\, b^{\perp}) \circ (b \circ a) = a^{\perp} \,\Box\, (b^{\perp} \circ b \circ a) \leq$$

$$a^{\perp} \,\Box\, (\perp \circ\, a) = (a^{\perp} \,\Box \perp) \circ a = a^{\perp} \circ a \leq \perp$$

To show *(a)* $a^{\perp} \circ b^{\perp} \leq (a \,\Box\, b)^{\perp}$ we use the same kind of reasoning, as it is enough to show $(a^{\perp} \circ b^{\perp}) \circ (a \,\Box\, b) \leq \perp$.

Note that as e is the identity for $\circ$ and $\perp$ the identity for $\Box$,

$$a^{\perp} \circ \perp^{\perp} \leq (a \Box \perp)^{\perp} = a^{\perp} = a^{\perp} \circ e \Rightarrow \perp^{\perp} \leq e$$

By proposition 1.2, as $e \circ \perp \leq \perp$ we have $e \leq \perp \multimap \perp = \perp^{\perp}$, thus $\perp^{\perp} = e$. But in the weakly-dual case we cannot guarantee that $e^{\perp} = \perp$ as we only know that

$$a^{\perp} \Box e^{\perp} \leq (a \circ e)^{\perp} = a^{\perp} = a^{\perp} \Box \perp \Rightarrow e^{\perp} \leq e$$

Note that the condition of compatibility says in the more general set-up of categories that $\Box$ is a covariant bifunctor.

We would call the symmetric monoidal structure ($\Box$, $\perp$) de Morgan-dual to « $\circ$ » if we had equality in condition *(a)* and *(b)*. In that case we would call the lineale a *strong Heyting lineale.*

One may think that names were badly chosen as a lineale already satisfies what maybe be called a Heyting condition, namely

$$a \circ b \leq c \Leftrightarrow a \leq b \multimap c$$

ut lineales have no notion of disjunction whatsoever, while Heyting lineales an be restricted to Heyting algebras if $\circ$ satisfies a universal property (cf. elow in def. 5).

. Additive lineales

A (Heyting) lineale is characterized by its « multiplicative » structure given y ($\leq$, $\circ$, $\multimap$, e) (perhaps also ($\Box$, $\perp$)). But we can have another « layer » of ructure, called its *additive* structure.

efinition 5 *A* semi-additive (Heyting) lineale *is a (Heyting) lineale equipped ith an extra symmetric monoidal structure, notation* ($\times$, 1) *such that given a and b* L, $a \times b$ satisfies

- $a \times b \leq a$ *and* $a \times b \leq b$
- *If* m *is such that* $m \leq a$ *and* $m \leq b$ *then* $m \leq a \times b$

Note that $a \times b$ is defined as a binary greatest lower bound; that having ary glb's we can easily define finite n-ary ones and that 1 is the empty-set), which means that for all $a \in L$, $a \leq 1$. In particular $e \leq 1$ (and $\perp \leq 1$, if it is esent). Also ($\times$, 1) being a symmetric monoidal structure means

- $(a \times b) \times c = a \times (b \times c)$
- $a \times b = b \times a$
- $a \times 1 = 1 \times a = a$

A semi-additive lineale corresponds to a symmetric monoidal clos category with products in the more general framework.

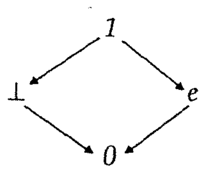
Definition 6 *An* additive (Heyting) lineale *is a semi-additive (Heyting) line equipped with an another symmetric monoidal structure, notation* $(\oplus, 0)$ *such t given* a *and* b *in* L, $a \oplus b$ *satisfies :*

- $a \leq a \oplus b$ *and* $b \leq a \oplus b$
- *if* $a \leq n$ *and* $b \leq n$ *then* $a \oplus b \leq n$

Dually, $0 \leq a$ for any $a \in L$, in particular, $0 \leq \perp$, $0 \leq e$ and $0 \leq 1$.

Observe that the conditions in the definition 5 and 6 above are t restrictions to the poset set-up of the conditions on the existence of produ and coproducts. They could be described in terms of adjunctions, in this ca Galois connections, to a diagonal functor, $\Delta : L \to L \times L$. Note that they determine a lattice structure in L.

If the four constants $\perp$, e, 0 and 1 are distinct we have a picture like



but they may coincide.

Trivial examples of additive Heyting lineales are Heyting algebras (whe $\circ$ and $\times$ and $\Box$ and $+$ coincide and $0 = \perp$ and $1 = e$) and Boolean algebras (before plus $a^{\perp\perp} = a$).

Proposition 3 *In an additive Heyting lineale we have the distributive laws :*

- $a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c)$
- $a \Box (b \times c) \leq (a \Box b) \times (a \Box c)$

Notice that the first law is a direct consequence of the fact that th

« category » L is a symmetric monoidal closed one, as $\oplus$ is a coproduct and coproducts are preserved by functors which have right-adjoints. The semi-law is a consequence of $\times$ being a categorical product, as $b \times c \leq b$ and $b \times c \leq c$ implies $a \square (b \times c) \leq a \square b$ and $a \square (b \times c) \leq a \square c$, so

$$a \square (b \times c) \leq (a \square (b \times c)) \times (a \square (b \times c)) \leq (a \square b) \times (a \square c)$$

Comparison with other approaches

It seems reasonable to compare the approach taken here with the one by Hesselink using *Girard monoids*. Quite apart from the fact that Girard monoids are based on/par the linear connective less amenable to intuitive explanations, Hesselink's approach is based on the classical equivalence between $A \rightarrow B$ and $\neg A \vee B$. It seems to us that one should strive for the more general set-up — in this case the intuitionistic one — as that allows us to restrict ourselves to the classical case, when (and if) wanted.

A strong Heyting lineale can be seen as a Girard monoid wrt $\square$ and a Girard monoid restricts to a phase structure, the model for linear logic provided by Girard himself in [tcs60]. Also a Girard monoid is a generalization of the de Morgan monoids in Dunn, the semantical model for relevance logic.

The definition of a Heyting additive lineale is also very similar to some work done by Ginsberg and also Fitting on bilattices. Again the difference is that the structure on the horizontal direction need not be a lattice. The conditions forced on us by the (categorical) adjunction are not strong enough for that, but of course a bilattice is a rather special case of an additive Heyting lineale.

Rules and axioms of Linear Logic

Axioms :

$$A \vdash A \quad (identity)$$
$$\vdash I \qquad \bot \vdash$$
$$\Gamma \vdash 1, \Delta \qquad \Gamma, 0 \vdash \Delta$$

Structural Rules :

$$\frac{\Gamma \vdash \Delta}{\sigma \Gamma \vdash \tau \Delta} \ (permutation) \qquad \frac{\Gamma \vdash A, \Delta \quad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta', \bot} \ (cut)$$

Logical Rules :

$$(var_l) \ \frac{\Gamma \vdash B, \Delta}{\Gamma, B^{\perp} \vdash \Delta}$$

Multiplicatives :

$$(unit_l)\ \frac{\Gamma \vdash \Delta}{\Gamma, I \vdash \Delta} \qquad (unit_r)\ \frac{\Gamma \vdash \Delta}{\Gamma \vdash \bot, \Delta}$$

$$(\otimes_l)\ \frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \qquad (\otimes_r)\ \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \otimes B, \Delta, \Delta'}$$

$$(\Box_l)\ \frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Box B \vdash \Delta, \Delta'} \qquad (\Box_r)\ \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \Box B, \Delta}$$

$$(\multimap_l)\ \frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \multimap B \vdash \Delta', \Delta} \qquad (\multimap_r)\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ *$$

Additives :

$$(\&_r)\ \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta} \qquad (\&_l)\ \frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta} \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash}$$

$$(\oplus_l)\ \frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta} \qquad (\oplus_r)\ \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B, \Delta} \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B,}$$

* Observe that in rule ($\multimap_r$) we only deal with one formula on the right-ha side of the turnstyle, according to our intuitionistic flavour of Linear Logic

Then we have another obvious proposition

Proposition 4 *An additive Heyting Lineale* (L, $\leq$, $\circ$, $\multimap$, $\Box$, e, $\bot$, $+$, $\times$, 1, 0) *is algebraic model of Linear Logic, as described above.*

Just read atomic propositions in LL as elements of L, $\vdash$ as *leq*, $\otimes$ as $\circ$ a the other connectives and constants for their homonimous.

Note that the poset reflection of GC is a lineale, the simplest non-collaps one (see figure above).

3. A categorical construction

Suppose **C** is a concrete *linear* category with products, by that we mean concrete symmetric monoidal closed category with products. And suppose th L is an object of **C** endowed with a (Heyting) lineale structure ($\leq$, $\circ$, $\multimap$, (perhaps also ($\Box$, $\bot$)). To make notation manageable we write :

- $[U, V]$ for the internal hom in C,

- $U \otimes V$ for the tensor product in **C**, with identity I;
- $U \times V$ for the categorical product in **C**, with identity 1.

Then we can construct the category $M_L\mathbf{C}$, which has as objects morphisms of **C** of the form $U \otimes X \xrightarrow{\alpha} L$. One such object is written as $(U \xleftarrow{\alpha} X)$ and called A.

Given two objects, says $A = (U \xleftarrow{\alpha} X)$ and $B = (V \xleftarrow{\beta} Y)$, the morphisms of $M_L\mathbf{C}$ are pairs of morphisms of **C**, $f : U \to V$ and $F : Y \to X$ such that the following diagram is satisfied,

$$\begin{array}{ccc} U \otimes Y & \xrightarrow{U \otimes F} & U \otimes X \\ {\scriptstyle f \otimes Y}\downarrow & & \downarrow{\scriptstyle \alpha} \\ V \otimes Y & \xrightarrow[\beta]{} & L \end{array}$$

where the diagram being satisfied means that given $u \otimes y$ in $U \otimes Y$, the composite morphism $\alpha \bullet (U \otimes F)$ applied to $(u \otimes y)$ as an element of L is smaller than $\beta \bullet (f \otimes Y)$ applied to $(u \otimes y)$. Simplifying, morphisms are pairs of maps in **C** (f, F), $f : U \to V$ and $F : Y \to X$ such that

$$\alpha\,(u, Fy) \leq \beta\,(f u, y)$$

It is easy to verify that $M_L\mathbf{C}$ is a category with an abundance of symmetric monoidal structures.

Proposition 5 *The construction above really defines a category* $M_L\mathbf{C}$.

Clearly identities are pairs of identities of **C**, composition is composition in each coordinate and associativity is an immediate consequence of the associativity in **C**.

Linear structure of $M_L\mathbf{C}$

One of the possible symmetric monoidal structures of $M_L\mathbf{C}$ is :

Definition 7 *Given two objects* $A = (U \xleftarrow{\alpha} X)$ *and* $B = (V \xleftarrow{\beta} Y)$ *in* $M_L\mathbf{C}$ *we define* $A \otimes B$ *their tensor product as follows :*

$$A \otimes B = (U \otimes V \xleftarrow{\alpha \otimes \beta} [V, X] \times [U, Y])$$

The morphism « $\alpha \otimes \beta$ » intuitively says $\alpha \otimes \beta\,(u, v, f, g) = \alpha\,(u, fv) \circ \beta\,(v, gu)$.

To define the morphism $\alpha \otimes \beta$ consider the following map, which we call $\overline{\alpha}$:

$$(U \otimes V) \otimes ([V, X] \times [U, Y]) \xrightarrow{U \otimes V \otimes \pi_1} U \otimes V \otimes [V, X] \xrightarrow{U \otimes eval} U \otimes X \xrightarrow{\alpha} L$$

Similarly we define $(U \otimes V) \otimes ([V, X] \times [U, Y]) \xrightarrow{\overline{\beta}} L$. Then to get $\alpha \otimes \beta$ we pair $\overline{\alpha}$ and $\overline{\beta}$ and use the multiplication « $\circ$ » of L, as follows :

$$(U \otimes V) \otimes ([V, X] \times [U, Y]) \xrightarrow{\overline{<\alpha, \beta>}} L \times L \xrightarrow{\circ} L$$

Proposition 6 *The construction above induces a bifunctor, covariant in both coordinates, with identity* I_M *given by* $(I \overset{e}{\longleftarrow\!\!+\!\!-} 1)$, *where the morphism* $I \otimes 1 \cong 1 \xrightarrow{e} L$ *just picks up the object « e » from L.*

Note that $\otimes$ is not a categorical product, for instance we have no projections, even if C is a Cartesian closed category.

Definition 8 *Given two objects* $A = (U \overset{\alpha}{\longleftarrow\!\!+\!\!-} X)$ *and* $B = (V \overset{\beta}{\longleftarrow\!\!+\!\!-} Y)$ *in* $M_L\mathbf{C}$ *we define* $[A, B]$ *their internal hom as follows :*

$$[A, B] = ([U, V] \times [Y, X] \overset{\alpha \multimap \beta}{\longleftarrow\!\!+\!\!-} U \otimes Y)$$

The morphism « $\alpha \multimap \beta$ » intuitively says $(\alpha \multimap \beta)(f, F, u, y) = \alpha(u, Fy) \multimap \beta(fu, y)$. The definition of the morphism $\alpha \multimap \beta$ is similar to the definition of $\otimes$ above. First consider maps $\overline{\alpha}$ and $\overline{\beta}$:

$$([U, V] \times [Y, X]) \otimes (U \otimes Y) \xrightarrow{\pi_1 \otimes U \otimes Y} [U, V] \otimes U \otimes Y \xrightarrow{eval \otimes Y} V \otimes Y \xrightarrow{\beta} L$$

$$([U, V] \times [Y, X]) \otimes (U \otimes Y) \xrightarrow{\pi_2 \otimes U \otimes Y} [Y, X] \otimes U \otimes Y \xrightarrow{U \otimes eval} U \otimes X \xrightarrow{\alpha} L$$

Then, to obtain $\alpha \multimap \beta$ we pair $\overline{\alpha}$ and $\overline{\beta}$ and compose the result with $\multimap$, considered as a map from $L \times L$ to L :

$$([U, V] \times [Y, X]) \otimes (U \otimes Y) \xrightarrow{\overline{<\alpha, \beta>}} L \times L \xrightarrow{\multimap} L$$

Note that if we consider the internal hom $[A, A]=([U, U]\times[X, X] \overset{\alpha \multimap \alpha}{\longleftarrow\!\!+\!\!-} U \otimes X)$, there is always a morphism from I_M to it,

$$\begin{array}{ccc} I & \overset{e}{\longleftarrow\!\!+\!\!-} & 1 \\ \downarrow & & \uparrow \\ [U, U] \times [X, X] & \overset{\alpha \multimap \alpha}{\longleftarrow\!\!+\!\!-} & U \otimes X \end{array}$$

as C is symmetric monoidal closed with products and $e \leq \alpha(u, x) \multimap \alpha(u, x)$.

Proposition 7 *The construction above induces a bifunctor, contravariant in its first coordinate and covariant in its second coordinate.*

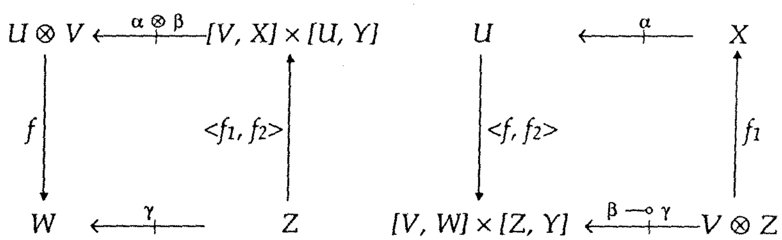
Having defined both a tensor product and an internal hom, we want to prove that they provide us with a symmetric monoidal closed category.

Proposition 8 *The category M_LC is a symmetric monoidal closed category.*

The proof is simple, one has to verify the natural isomorphism :

$$Hom\ (A \otimes B, C) \equiv Hom\ (A, [B, C])$$

This can be done by looking at the diagram



If the morphism $(f, <f_1, f_2>)$ is a *Hom* $(A \otimes B, C)$, then given (u, v) in $U \otimes V$ and z in Z, we know $(\alpha \otimes \beta)\ (u, v, f_1z, f_2z) \leq \gamma\ (f\ (u, v), z)$.

That means, by definition of tensor, that $\alpha(u, f_1zv) \circ \beta(v, f_2zu) \leq \gamma\ (f\ (u, v), z)$. But as L is a lineale,

$$\alpha\ (u, f_1zv) \circ \beta\ (v, f_2zu) \leq \gamma\ (f\ (u, v), z) \Leftrightarrow \alpha\ (u, f_1zu) \leq \beta\ (v, f_2zv) \multimap \gamma\ (f\ (u, v), z)$$

Now to show that $(<f, f_2>, f_1)$ is in *Hom* $(A, [B, C])$ we have to show

$$\alpha\ (u, f_1\ (v, z)) \leq (\beta \multimap \gamma)\ (fu, f_2u, v, z)$$

But $(\beta \multimap \gamma)\ (fu, f_2u, v, z) = \beta\ (v, f_2uz) \multimap \gamma\ (fuv, z)$ which we know, if ransposing is allowed.

If we have a *Heyting lineale* we can also define another bifunctor « □ » of bjects in M_LC.

efinition 9 *Given two objects $A = (U \xleftarrow{\alpha} X)$ and $B = (V \xleftarrow{\beta} Y)$ in M_LC we define □ B their /par operator as follows :*

$$A \,\square\, B = ([X, V] \times [Y, U] \xleftarrow{\alpha \,\square\, \beta} X \otimes Y)$$

The morphism « $\alpha \Box \beta$ » intuitively says $(\alpha \Box \beta)(f, g, x, y) = \alpha(x, gy) \Box \beta(fx, y)$.
The definition of the morphism $\alpha \Box \beta$ is similar to the definitions of $\otimes$ and $[-, -]$ above. First consider maps $\bar{\alpha}$ and $\bar{\beta}$:

$$([X, V] \times [Y, U]) \otimes (X \otimes Y) \xrightarrow{\pi_1 \otimes X \otimes Y} [X, V] \otimes X \otimes Y) \xrightarrow{eval \otimes Y} V \otimes Y \xrightarrow{\beta} L$$

$$([X, V] \times [Y, U]) \otimes (X \otimes Y) \xrightarrow{\pi_2 \otimes X \otimes Y} [Y, U] \otimes X \otimes Y) \xrightarrow{X \otimes eval} U \otimes X \xrightarrow{\alpha} L$$

Then to obtain $\alpha \Box \beta$ we pair $\bar{a}$ and $\bar{\beta}$ and compose the result with $\Box$, considered as a map from $L \times L$ to L :

$$([U, V] \times [Y, X]) \otimes (U \otimes Y) \xrightarrow{\overline{<\alpha, \beta>}} L \times L \xrightarrow{\Box} L$$

Proposition 9 *The operation $A \Box B$ defines a bifunctor $\Box : M_L C \times M_L C \to M_L C$ with identity given by the object $\bot_M = (\longleftarrow\!\!\!\!\!+\!\!\!\!\!\longrightarrow 1 \perp I)$, where the map $\bot : 1 \otimes I = 1 \to L$ picks up the object $\bot$ from L.*

4. Additive structure of $M_L C$

Now we want to define products and coproducts in $M_L C$. To do that we need at least

- a *semi-additive* (Heyting) lineale
- (disjoint ?) coproducts in C.

Note that it is not necessary to add products and coproducts to $M_L C$ at the same time.

Suppose **C** is a linear category with coproducts. Then a form of distributivity holds, namely :

$$U \otimes (V + W) \cong U \otimes V + U \otimes W$$

As **C** is symmetric monoidal closed, the functor $U \otimes (-)$ has a right-adjoint, $[U, -]$, hence it preserves colimits and, in particular, initial objects and coproducts.

Definition 10 *Given two objects* $A = (U \xleftarrow{\alpha} X)$ *and* $B = (V \xleftarrow{\alpha} Y)$ *in* $M_L C$ we define *A & B their categorical product as follows :*

$$A \,\&\, B = (U \times V \xrightarrow{\alpha \,\&\, \beta} X + Y)$$

The morphism « α & β » intuitively says $(\alpha \,\&\, \beta)\,(u, v, \binom{x,\,0}{y,\,1}) = \alpha\,(u, x) \asymp \beta\,(v, y)$.

But note that, despite the similarity with previous definitions, the multiplication « $\times$ » is not used, what is used is the structure on **C**, as an element of $X + Y$ is either $(x, 0)$ or $(y, 1)$ but not both.

Proposition 10 *The operation « & » above defines a bifunctor* $\& : M_L\mathbf{C} \times M_L\mathbf{C} \to M_L\mathbf{C}$, *with identity given by* $1_M = (1 \xleftarrow{1} 0)$ *and* $A \,\&\, B$ *is really a categorical product in* $M_L\mathbf{C}$.

To define the morphism $(U \times V) \otimes (X + Y) \xrightarrow{\alpha \,\&\, \beta} L$ in **C**, which corresponds to the object $A \,\&\, B$ in $M_L\mathbf{C}$, we do :

$$(U \times V) \otimes (X + Y) \cong (U \times V) \otimes X + (U \times V) \otimes Y \xrightarrow{\pi_1 \otimes 1 \,+\, \pi_2 \otimes 1} U \otimes X + V \otimes Y \xrightarrow{\binom{\alpha}{\beta}} L$$

Projections are trivially given by projections in **C** in the first coordinate and :anonical injections in the second coordinate.

$$\begin{array}{ccc} U \times V & \xleftarrow{\alpha \,\&\, \beta} & X + Y \\ {\scriptstyle \pi_1}\downarrow & & \uparrow{\scriptstyle i_1} \\ U & \xleftarrow{\alpha} & X \end{array}$$

e have a diagonal functor $\Delta : M_L\mathbf{C} \to M_L\mathbf{C} \times M_L\mathbf{C}$

$$\begin{array}{ccc} U & \xleftarrow{\alpha} & X \\ \downarrow & & \uparrow \\ U \times U & \xleftarrow{\alpha \,\&\, \alpha} & X + X \end{array}$$

en by the diagonal in **C** in the first coordinate and the canonical folding ɔ in the second coordinate.

o show the universal property of products we consider an object $C = (W \xleftarrow{\gamma} Z)$ ı that there are maps in $M_L\mathbf{C}$ of the form

$$\begin{array}{ccc} W & \xleftarrow{\gamma} & Z \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle F} \\ U & \xleftarrow{\alpha} & X \end{array} \qquad\qquad \begin{array}{ccc} W & \xleftarrow{\gamma} & Z \\ {\scriptstyle g}\downarrow & & \uparrow{\scriptstyle G} \\ V & \xleftarrow{\beta} & Y \end{array}$$

Then there is a unique map in $M_L\mathbf{C}$ from C to $A \,\&\, B$,

$$\begin{array}{ccc} W & \xleftarrow{\;\gamma\;} & Z \\ {\scriptstyle \langle f,g\rangle}\downarrow & & \downarrow{\scriptstyle \binom{F}{G}} \\ U \times V & \xleftarrow{\;\alpha \,\&\, \beta\;} & X + Y \end{array}$$

Dually we can define

Definition 11 *Given two objects $A = (U \xleftarrow{\alpha} X)$ and $B = (V \xleftarrow{\beta} Y)$ in $M_L\mathbf{C}$ we define $A \oplus B$ their categorical coproduct as follows :*

$$A \oplus B = (U + V \xleftarrow{\;\alpha \oplus \beta\;} X \times Y)$$

The morphism « $\alpha \oplus \beta$ » intuitively says $(\alpha \oplus \beta)\,(\binom{u,0}{v,1}, x, y) = \alpha\,(x, gy) \oplus \beta\,(fx, y)$.

It is another easy proposition to show that $A \oplus B$ is a bifunctor with identity $0_M = (0 \xleftarrow{0} 1)$ and $A \oplus B$ is a categorical coproduct. Note that as morphisms of $\mathbf{C}$ 0_M and 1_M are isomorphic, but not as objects of $M_L\mathbf{C}$. Note also that the additive structure of the lineale L is not used at all.

The category $M_L\mathbf{C}$ was defined following the pattern of $\mathbf{GC}$, so it is no surprise that

Proposition 11 *The category $M_L\mathbf{C}$ is a model of Linear Logic as described before.*

The last observation in this section is that we can describe another useful monoidal structure in $M_L\mathbf{C}$.

Definition 12 *Given two objects $A = (U \xleftarrow{\alpha} X)$ and $B = (V \xleftarrow{\beta} Y)$ in $M_L\mathbf{C}$ we define $A \circ B$ another tensor product as follows :*

$$A \circ B = (U \otimes V \xleftarrow{\;\alpha \circ \beta\;} X \otimes Y)$$

The morphism « $\alpha \circ \beta$ » intuitively says $(\alpha \circ \beta)\,(u \otimes v, x \otimes y) = \alpha\,(u, x) \otimes \beta\,(v, y)$. Its usefulness will become apparent in the next section.

5. Modalities in $M_L\mathbf{C}$

Now the intention is to define a comonad in $M_L\mathbf{C}$ to provide an interpretation of the modality or exponential « ! » of Linear Logic.

We start by recalling the rules for the modality « ! ». These are :

I. $\dfrac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$ (*dereliction*) II. $\dfrac{\Gamma \vdash B}{\Gamma, !A \vdash B}$ (*weakening*)

III. $\dfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$ (*contraction*) IV. $\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}$ (!)

But as observed by several people, the four rules for the modality « ! » fall neatly into two pairs. The pair (II, III) has to do with putting back into the logic, in a controlled way, contraction and weakening and the pair (I, IV) makes « ! » look like the $\Box$ modal operator of **S4**.

Suppose **C** is a linear category which has countable coproducts (instead of finite ones as in the last section). Then using the well-known construction of MacLane ([CWM] p. 168 theorem 2) we can show that **C** has free (commutative ?) monoids, as **C** being symmetric monoidal closed the other condition in MacLane's theorem is automatically satisfied. Having free monoids means that there exists a functor $F : \mathbf{C} \to \mathbf{Mon\ C}$, which is left-adjoint to the forgetful functor $U : \mathbf{Mon\ C} \to \mathbf{C}$. In other words, there is an adjunction $<F, U, \eta, \varepsilon> : \mathbf{C} \to \mathbf{Mon\ C}$, which we write simply as $F \dashv U$.

The adjunction says that every map on **C** of the form,

$$X \xrightarrow{f} U\,(Y, \eta_Y, \mu_Y)$$

corresponds, by a natural isomorphism, to a monoid homomorphism $\bar{f}$ of the form

$$(X^*, \eta_{X^*}, \mu_{X^*}) \xrightarrow{\bar{f}} (Y, \eta_Y, \mu_Y)$$

We write $(\ )^*$ for the composite functor $U \bullet F : \mathbf{C} \to \mathbf{C}$, recall from MacLane that $X^* = \coprod_{i \in N} X^i$ and denote by $(*, \eta, \mu)$ the corresponding monad in **C**.

Note that the unit of the adjunction $F \dashv U$, the natural transformation $\eta : \mathbf{C} \to \mathbf{C}$ takes any object X of **C** to the carrier of the free monoid X^*. Also the co-unit of the adjunction $e : \mathbf{Mon\ C} \to \mathbf{Mon\ C}$ takes any free monoid (X^*, η^*, μ^*) arising from an arbitrary monoid (X, η, μ) to itself. Thus

$$e : FU\,(M, \eta, \mu) = (M^*, \eta^*, \mu^*) \to (M, \eta, \mu)$$

where the morphism e corresponds to « iteration » of the original multiplication μ.

Now, in this stronger version of the existence of monoids, the monad $(*, \eta, \mu)$ is easily proved a *strong monad*, so there are morphisms

$$[X, Y]_C \xrightarrow{st\,(X,\,Y)} [X^*, Y^*]_C$$

and using these we can define the endofunctor below.

Definition 13 *The endofunctor $S : M_L\mathbf{C} \to M_L\mathbf{C}$ takes an object $(U \overset{\alpha}{\longleftarrow\!\!+\!\!-} X)$ of $M_L\mathbf{C}$ to the object $(U \overset{S\alpha}{\longleftarrow\!\!+\!\!-} X^*)$, where intuitively $uS\alpha\,(x_1, x_2, \ldots, x_n)$ means $u\alpha x_1$ and $u\alpha x_2$ and ... and $u\alpha x_n$.*

The object $S\alpha$ of $M_L\mathbf{C}$ is defined by the sequence of morphisms

$$\frac{\dfrac{U \otimes X \xrightarrow{\alpha} L}{U \to [X, L] \xrightarrow{st} [X^*, L^*]}}{U \otimes X^* \to L^* \xrightarrow{\mu} L}$$

So far so good and very similar to what happens in GC. But if we try to make another definition

Definition 14 *The endofunctor $T : M_L\mathbf{C} \to M_L\mathbf{C}$ takes an object $(U \overset{\alpha}{\longleftarrow\!\!+\!\!-} X)$ of $M_L\mathbf{C}$ to the object $(U \overset{T\alpha}{\longleftarrow\!\!+\!\!-} [U, X])$, where intuitively $uT\alpha f$ means $u\alpha fu$.*

But to give the morphism in $U \otimes [U, X] \xrightarrow{T\alpha} L$ we would need to « duplicate » U, so that

$$U \otimes [U, X] \xrightarrow{\delta \otimes 1} U \otimes U \otimes [U, X] \xrightarrow{1 \otimes eval} U \otimes X \xrightarrow{\alpha} L$$

Also to obtain comonoids in $M_L\mathbf{C}$, which would satisfy rules (*contraction*) and (*weakening*), for instance

$$\begin{array}{ccc} U & \longleftarrow\!\!+\!\!- & X \\ \downarrow & & \uparrow \\ U \otimes U & \longleftarrow\!\!+\!\!- & [U, X^*] \times [U, X^*] \end{array}$$

we need U's with some kind of structure.

Thus the proposal at the moment is to take $\mathbf{C}$ with free comonoids, having ree comonoids means that there exists a functor $F_1 : \mathbf{C} \to$ **Common C**, which s left-adjoint to the forgetful functor U_1 : **Common C** $\to$ **C**. In other words, here is an adjunction $<F_1, U_1, \eta, \varepsilon> : \mathbf{C} \to$ **Mon C**, which we write simply as $_1 \dashv U_1$.

The adjunction says that every map on $\mathbf{C}$ of the form,

$$X \xrightarrow{f} U\,(Y, !_Y, \delta_Y).$$

rresponds by a natural isomorphism, to a comonoid homomorphism $\bar{f}$ of the form

$$(X^*, \eta_{X^*}, \delta_{X^*}) \xrightarrow{\bar{f}} (Y, \eta_Y, \delta_Y)$$

We write $(\,)_*$ for the composite functor $U \bullet F : \mathbf{C} \to \mathbf{C}$, and denote by $((\,)_*^*, \eta, \mu)$ the corresponding monad in **C**.

Definition 15 *The endofunctor* $F : M_L\mathbf{C} \to M_L\mathbf{C}$ *takes an object* $(U \overset{\alpha}{\leftarrow\!\!+\!\!-} X)$ *of* $M_L\mathbf{C}$ *to the object* $(U_* \overset{F\alpha}{\leftarrow\!\!+\!\!-} X^*)$*, where intuitively* $uF\alpha\,(x_1, x_2, \ldots, x_n)$ *means that* u *can be shared out between* $u_1, u_2,$ *as many times as necessary so that* $u_1 \alpha x_1$ *and* $u_2 \alpha x_2$ *and . . . and* $u_n \alpha x_n$.

But this definition of F has to be shown to work and this is work in progress.

6. Further work

Apart from making sure that the definition of the modality « ! » works properly, which seems to be clear from previous work on Hopf Algebras by Sweedler and others, it seems that the main work that remains to be done is to get things at the right level of generality. The one adopted here seems clearly inadequate, as one would like to « change basis » on doing the construction of $M_L\mathbf{C}$, *i. e.* one would like to have constructions $M_L\mathbf{C}$, with different L's.

It is worth mentioning that there is some joint work in progress with Carolyn Brown from LFCS, Edinburgh connecting the quantales models for Linear Logic arising from Petri Nets to the dialectica-like ones proposed in Brown/Gurr, Lics'90, see [H&dP] for the extension that allows Petri Nets with multiplicities >2.

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