Algebraic Types in PER Models

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ABSTRACT

Huet has conjectured that the interpretations of a class of types (the "algebraic types") in the PER model on the natural numbers for the second-order lambda calculus are in a certain sense the initial algebras. In this paper we examine several different PER models, and show that Huet's conjecture holds in each.

Introduction

If you are given a model of the polymorphic lambda calculus (or of anything else for that matter), the first question you are likely to ask is "how good is it?" For many programming languages this might translate straight into a technical question about whether the model is fully abstract with respect to some operational semantics. For the strongly normalizing second-order lambda calculus, this particular question degenerates, and becomes simply the problem of characterizing the equational theory of the model. Seen from a slightly different point of view, it is also a question about

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how close the model you are given is to the term model for the theory. Specifically, how close it is to the term model for the bare theory, with no additional types, and no extra equations between terms. In this case there is, however, another important question that we can ask (even if we cannot yet precisely formulate it): "Are all the polymorphic values parametric?" (*cf.* [Rey83, Fre89b]).

One rather crude way of measuring this is by examining the interpretations of the polymorphic natural numbers

$$\Pi X.\,[X \to X] \to [X \to X]$$

and the polymorphic booleans

$$\Pi X.\,[X \to [X \to X]].$$

In a model close to the term model or in a parametric model, one might expect that these interpretations would contain, in some suitable sense, only the closed terms of given type in the calculus. (Indeed Freyd, following Reynolds, has proposed this as part of a series of tests for the inherent parametricity of a model [Fre89b]). It is a straightforward consequence of normalization (*cf.* [Gir72, Gir71]), that the only closed terms of these types are the polymorphic Church numerals

$$\Lambda X.\ \lambda f x.\ f^n(x)$$

in the first case, and the two elements

$$\Lambda X.\ \lambda xy.\ x, \qquad \Lambda X.\ \lambda xy.\ y$$

in the second. In the first case the result goes back to [Gir72], but the second seems to be folklore, and the earliest explicit reference to it that we can find is in [BB85]! A few remarks seem to be in order:

1) This little syntactic result is irrelevant to the major proof-theoretic concerns of [Gir72]. Even the results on *representability* of functions (which have been reworked recently by [Sta81], [SFO83], and [Lei83]) do not require one to show explicitly that there are no "non-standard" terms of type $\Pi X.\,[X \to X] \to [X \to X]$ (in much the same way that results on the representability of numeric functions in the untyped lambda calculus are not invalidated by the existence of lambda terms other than numerals).

2) In so far as these results are relevant to the implementation of programming languages (*cf. e.g.* [Fai86]), it is the syntactic result which matters and not the semantic ones we present here. However, as we will explain in section 2, we can give a semantic proof of the syntactic result, by considering the special case of the PER-model generated by an open term algebra.

3) We do not in this paper explicitly consider the other obvious way of comparing a model with the term model; we do not consider the theory generated by the model. It is however worth noting that the results in this area are somewhat ambivalent.

Results such as those presented in this paper show that certain types contain only elements corresponding to closed terms, and that no closed terms of these types are unnecessarily identified. However, the types involved are all of rather a low level, and at higher levels certain non-trivial equations hold in all PER models. An example is given at the end of the paper.

The syntactic result of [BB85] is considerably more general than the two results mentioned above. If we rewrite the polymorphic natural numbers and booleans in the extended calculus with product, then the connection between the two types becomes clearer. The natural numbers become $\Pi X. [X \times [X \to X] \to X]$, and the booleans $\Pi X. [X \times X \to X]$. We can see that they both fall into the same pattern: given certain operations concerning the parametric type X we have to produce a value of type X. In the first case the data we have are a value of type X and a unary function $X \to X$, and in the second two values of type X.

Böhm and Berarducci extend the syntactic characterization above to types derived from a general many-sorted algebraic signature (and, though this will not concern us, slightly beyond). Let Σ be a signature in the sense of many-sorted algebra. Thus Σ is given by a collection of basic sorts $A_1, \ldots, A_n$, and a collection of basic operations $f_1, \ldots, f_m$, corresponding to functions

$$f_i : A_{i1} \times \ldots \times A_{iN(i)} \to A_{r(i)}.$$

We assume that constants are given by nullary operations. Closed terms of each type are built up inductively in the usual way: if $\sigma_1, \ldots, \sigma_{N(i)}$ are closed terms of types $A_1, \ldots, A_{N(i)}$ respectively, then $f_i(\sigma_1, \ldots, \sigma_{N(i)})$ is a closed term of type $A_{r(i)}$. We recall that the closed terms form the initial algebra for the signature.

Corresponding to each basic sort A_i we define types $\mathcal{A}_i^\times$ and $\mathcal{A}_i$ in the extended and the pure second-order lambda calculus respectively:

$$\mathcal{A}_i^\times = \Pi A_1 \ldots A_n. \; (\, \Sigma^\times(A_1, \ldots, A_n) \to A_i \,),$$

(we have overloaded the sort symbols, using them also as type variables).
Here $\Sigma^\times(A_1, \ldots, A_n)$ is a type which encodes the signature Σ:

$$\begin{aligned} &(A_{11} \times \ldots \times A_{1N(1)} \to A_{r(1)}) \\ &\times (A_{21} \times \ldots \times A_{2N(2)} \to A_{r(2)}) \\ &\times \ldots \\ &\times (A_{m1} \times \ldots \times A_{mN(m)} \to A_{r(m)}) \end{aligned}$$

$\mathcal{A}_i$ is the curried version of $\mathcal{A}_i^\times$

$$\begin{aligned} \Pi A_1 \ldots A_n. \;\; &[\, (A_{11} \to \ldots \to A_{1N(1)} \to A_{r(1)}) \\ &\to \; (A_{21} \to \ldots \to A_{2N(2)} \to A_{r(2)}) \\ &\to \; \ldots \\ &\to \; (A_{m1} \to \ldots \to A_{mN(m)} \to A_{r(m)}) \\ &\to \;\; A_i \,]. \end{aligned}$$

Note that there are terms in the extended calculus which define isomorphisms between the types $\mathcal{A}_i$ and the types $\mathcal{A}_i^\times$, and hence that $\mathcal{A}_i$ is isomorphic to $\mathcal{A}_i^\times$ in any model. Note also that a value of type $\mathcal{A}_i^\times$ is a function which takes a Σ-algebra $B = (B_1, \ldots, B_n, e_1, \ldots, e_m)$ as parameter, and returns a value of type B_i. Thus we shall refer to these types as *algebraic types*. In this paper, we use the types $\mathcal{A}_i$ of the pure calculus rather than the types $\mathcal{A}_i^\times$. The only reason for taking this option is that the use of this representation rather than the other seems to make our calculations slightly simpler.

Böhm-Berarducci show that any closed term of type $\mathcal{A}_i$ is reducible to one of the form $\Lambda A_1, \ldots, A_n\ \lambda f_1, \ldots, f_m.\ \sigma$, where σ is a closed Σ-term of type A_i (this time overloading function symbols as value variables).

In particular we obtain the results above about $\Pi X.\,[X \to X] \to [X \to X]$ and $\Pi X.\,[X \to [X \to X]]$ as special cases. Both correspond to one-sorted theories. The polymorphic natural numbers arise from the theory with one constant 0, and one unary operation s, and the polymorphic booleans arise from a signature with no operations but with two constants, 0 and 1. We can also see that there are no closed terms of type $\Pi X.\,X$, and that the only closed term of type $\Pi X.\,X \to X$ is the polymorphic identity.

In this paper we shall look at two different "PER" models for the second-order calculus, and show that for any signature Σ the algebraic types $\mathcal{A}_i$ are interpreted by the carriers for the initial Σ-algebra in PER. In particular, the elements of the algebraic types correspond to the closed terms of the free calculus. Our major result is for the standard PER model on the natural numbers, thus proving a conjecture of Huet, as well as showing that the PER model satisfies at least this portion of Freyd's criteria for inherent parametricity.

As is by now well-known, a PER model can be viewed as a small complete category of sets inside a realizability topos (*cf.* [Hyl87], [HRR89]). From this point of view, the polymorphic types are interpreted as a product in the topos. In the case of the algebraic types this product is quite simply the product of all Σ-algebras in the category of types.

1 A recap on PER models

Let A be a partial combinatory algebra (also called a partial Schönfinkel algebra). Thus A is given by a set A, together with a partial binary operation $\cdot$, representing functional application, together with two elements S and K. (As usual, we shall adopt the convention that application associates to the left, and drop the use of the symbol $\cdot$. Thus instead of $((\mathsf{S} \cdot X) \cdot Y) \cdot Z$, we shall simply write $\mathsf{S}XYZ$.) In order to have a partial combinatory algebra, we require also that for all X, Y in A, both $\mathsf{K}XY$ and $\mathsf{S}XY$ are always defined, and that for all X, Y, Z

$$\begin{aligned} \mathsf{K}XY &= X \\ \text{and}\quad \mathsf{S}XYZ &= (XZ)(YZ) \end{aligned}$$

where the equality means that if one side is defined, then so is the other, and they are equal.

The instances with which we shall be particularly concerned are the partial combinatory algebra N of integers with Kleene application, and the total combinatory algebra Λ of β-equivalence classes of open untyped lambda terms (the "free" lambda algebra on countably many generators). However, there are many other interesting Schönfinkel algebras around. These include algebras of closed lambda terms, and algebras of functions recursive with respect to some oracle, as well as the algebras arising from domain-theoretic models of the untyped lambda calculus, such as D_∞ or $\mathsf{P}\omega$.

Since A contains combinators S and K it enjoys a form of combinatory completeness, and in particular has a notion of pairing. Specifically, there is an element Pair of A such that $\mathsf{Pair}XY$, for which we shall write $\langle X,Y\rangle$, is always defined. Moreover there are elements P_0 and P_1 such that $\mathsf{P}_0\langle X,Y\rangle = X$ and $\mathsf{P}_1\langle X,Y\rangle = Y$.

Given such an A we can construct a model PER(A) of the second-order lambda calculus. The types of the model are the partial equivalence relations (or *per's*) on A. That is to say they are symmetric and transitive, but not necessarily reflexive, relations. Given a per R, we shall refer to the set of x such that for some y, xRy as the *domain* of R.

In the course of the paper we shall make considerable use of per's P with the property that xPy iff x is in the domain of P and $x = y$. We shall call such per's *canonically projective* (*cf.* [RR88]). In particular if $U \subseteq \mathsf{A}$, then we shall refer to *the canonically projective per on domain* U.

Given two per's R and S, a map from R to S is given by an element ϕ of A, such that

1. ϕx is defined for all x in the domain of R,

2. for all x, y in A, if xRy then $\phi x\, S\, \phi y$.

The intuition is that ϕ induces a map between the quotients of A by R and S. Accordingly, we specify that two elements ϕ and ψ induce the same map from R to S if (they both induce maps from R to S and) for all x, y such that xRy, $\phi x\, S\, \psi y$.

Given types (per's) R and S we can now interpret the product type $R \times S$ as the per whose domain is the set of pairs $\langle x,y\rangle$ where x is in the domain of R and y is in the domain of S, and where

$$\langle x,x'\rangle\ R \times S\ \langle y,y'\rangle \text{ iff } xRy \text{ and } xSy'$$

The function space type $[R \to S]$ is interpreted as the per whose domain is the set of ϕ inducing a function from R to S, and where

$$\phi[R \to S]\psi \text{ iff } \phi \text{ and } \psi \text{ induce the same map from } R \text{ to } S.$$

These definitions give the cartesian closed category structure on PER(A), and hence we have an interpretation of the first-order typed lambda calculus. Given a type expression $F[X]$, we interpret it as the family of types $F[R]$ as R varies through PER(A), and similarly for type expressions with more than one free variable. Finally, the polymorphic types are interpreted by intersection:

$$x\ \{\Pi X.\,F[X]\}\ y \text{ iff for all } R \text{ in } \mathrm{PER}(\mathsf{A}),\ x\ F[R]\ y.$$

There is more than one way to handle the presentation of the interpretation of terms in this model, and we leave it to the reader to pick his or her favourite approach.

It is by now well-known that a PER model constructed in this way can also be viewed as a small complete category of sets inside the realizability topos $\mathcal{A}$ generated from A (*cf.* [Hyl87], [HRR89]). From this point of view, the polymorphic types are interpreted by a product in the topos. The results we shall present below can also be read in this topos-theoretic setting. We shall show that for the toposes $\mathcal{L}$ (constructed from Λ) and $\mathcal{E}\!f\!f$, algebraic types form the carriers of the appropriate initial algebra in the topos.

2 PER(Λ)

Let Λ be the combinatory algebra of open untyped lambda terms (with respect to $\beta\eta$-equivalence). Let us suppose that the algebraic type $\mathcal{A}_i$ is interpreted in PER(Λ) by the per $[\![\mathcal{A}_i]\!]$. The characterization of these types is due to Eugenio Moggi.

2.1 PROPOSITION (Moggi, *cf.* [BC88]) $[\![\mathcal{A}_i]\!]$ is the canonically projective per (*cf.* section 1) whose domain is

$$\{\lambda f_1 \ldots f_m.\,\sigma \mid \sigma \text{ is a closed } \Sigma\text{-term of type } A_i\ \}.$$

Proof. Given τ in the domain of $[\![\mathcal{A}_i]\!]$, choose distinct variables $x_1 \ldots x_m$ not free in τ, and for $i = 1 \ldots n$ define the per X_i to be the canonically projective per on

$$\{\sigma[\vec{x}/\vec{f}] \mid \sigma \text{ is a closed } \Sigma\text{-term of type } A_i\ \}.$$

These types carry a Σ-algebra structure in which, when we follow through the implications of currying all our functions, the interpretation of f_i is induced by x_i.

If we specialize τ at $X_1, \ldots, X_n$, and then apply the resulting function to the variables $x_1, \ldots, x_m$, we must get some element $\sigma[\vec{x}/\vec{f}]$ of X_i. Thus

$$\tau x_1 \ldots x_m = \sigma[\vec{x}/\vec{f}]$$

and hence

$$\tau = \lambda f_1 \ldots f_m.\,\sigma.$$

In particular τ is closed.

We have thus established that the per $[\![\mathcal{A}_i]\!]$ is on the correct domain. It remains to show that it is canonically projective. Suppose $\tau\ [\![\mathcal{A}_i]\!]\ \tau'$, then

$$\begin{aligned}\tau x_1 \dots x_m &= \sigma[\vec{x}/\vec{f}]\\ \tau' x_1 \dots x_m &= \sigma'[\vec{x}/\vec{f}].\end{aligned}$$

Since X_i is canonically separated, $\sigma = \sigma'$, and thus by the η rule $\tau = \tau'$. □

2.2 COROLLARY The algebra $[\![\mathcal{A}(\Sigma)]\!]$ is the initial Σ-algebra in the realizability topos associated to Λ, and hence is also the initial Σ-algebra in PER(Λ).

The proof of this is left to the reader. Those not interested in topos theory may wish to note that the second statement admits of a simple direct proof.

If, instead of PER(Λ), we take the per model on the algebra Λ^β of β-equivalence classes of open untyped terms, then we can still obtain a result analogous to proposition 2.1.

2.3 PROPOSITION If the algebraic type $\mathcal{A}_i$ is interpreted in PER(Λ^β) by the per $[\![\mathcal{A}_i]\!]_\beta$, then $[\![\mathcal{A}_i]\!]_\beta$ is isomorphic to the canonically projective per on

$$\{\lambda f_1 \dots f_m.\ \sigma \mid \sigma \text{ is a closed } \Sigma\text{-term of type } A_i\ \}.$$

Proof. The proof is much as before, except for a slight difficulty introduced by the fact that we can no longer use

$$\tau x_1 \dots x_m = \sigma[\vec{x}/\vec{f}]$$

to conclude that

$$\tau = \lambda f_1 \dots f_m.\ \sigma.$$

Note, however, that if $\tau x_1 \dots x_m = \tau' x_1 \dots x_m$, then $\tau [\![\mathcal{A}_i]\!]_\beta \tau'$, and hence that

$$\lambda x.\ \lambda f_1 \dots f_m.\ x f_1 \dots f_m$$

induces an isomorphism from $[\![\mathcal{A}_i]\!]_\beta$ to the per required. □

To conclude the section we give an alternative proof of the result of Böhm-Berarducci.

2.4 PROPOSITION (Böhm-Berarducci) Any closed term of type $\mathcal{A}_i$ is $\beta\eta$-equivalent to one of the form

$$\Lambda X_1 \dots X_n.\ \lambda f_1 \dots f_m.\ \sigma$$

where σ is a closed Σ-term of type A_i.

Proof. Let T be a closed term of type $\mathcal{A}_i$, and τ the lambda term obtained from it by erasing the type information. Consider the interpretation of T in PER(Λ) at the algebra on types $X_1 \ldots X_n$ defined as above. This is $\tau x_1 \ldots x_m$. Hence $\tau x_1 \ldots x_m$ reduces to $\sigma[\vec{x}/\vec{f}]$. But any reduction of $\tau x_1 \ldots x_m$ lifts to a reduction of $T[X_1 \ldots X_n] x_1 \ldots x_m$ (where the X_i are type variables). Thus we conclude that

$$T[X_1 \ldots X_n]\, x_1 \ldots x_m \quad \text{reduces to} \quad \sigma[\vec{x}/\vec{f}]$$

and hence that

$$T = \Lambda X_1 \ldots X_n.\, \lambda f_1 \ldots f_m.\, \sigma$$

as required. $\square$

It is interesting to compare the proof given by Böhm and Berarducci with this one. Böhm and Berarducci use the strong normalization of polymorphic lambda terms, and then a simple argument as to the structure of a normal form of the required type. Strong normalization is itself proved by a kind of realizability argument (essentially realizability using the combinatory algebra of strongly normalizable untyped terms), as was pointed out by Tait ([Tai75]). Here, we use a less sophisticated realizability to show normalizability, and to characterize normal forms, for a very restricted class of types.

3 A simple case

We now try to prove results analogous to those in the previous section for PER models over more general partial combinatory algebras. Such an algebra does not necessarily contain anything that we can use as a variable, and so we have to attempt a different line of proof.

We shall begin by considering the simplest non-trivial case—the polymorphic booleans, $\Pi X.\,[X \to [X \to X]]$. We recall that this corresponds to a one-sorted signature with no basic operations but two constants 0 and 1.

Let θ be an element of our algebra A contained in the interpretation of the type $\mathcal{A}_i$, and let us suppose that we are given a per B, together with two elements b_0 and b_1. We want to examine the interpretation of $\theta[B]b_0b_1$. The first stage of the proof is to realize that this has to be either b_0 or b_1. Indeed, since the interpretation of $\theta[B]b_0b_1$ is given by the value of $\theta b_0 b_1$, we can look at the per B' whose only elements are b_0 and b_1, and in which elements are related iff they are equal. We now use the fact that $\theta b_0 b_1 = \theta[B']b_0b_1$ must be an element of B'. Hence it is either b_0 or else b_1.

We shall see later that this stage of the proof generalizes nicely to arbitrary algebraic types. The next stage is however more problematic. We have to show that θ is uniformly defined. If $\theta[F]f_0f_1$ is f_0, for one (suitable) per F and elements f_0 and f_1, then $\theta[B]b_0b_1$ is b_0 for all per's B and elements b_0 and b_1.

To do this, we first fix our reference per F to have only the elements 0 and 1, $0 \neq 1$, and in which 0 is not related to 1. We shall suppose, with no real loss of generality, that $\theta[F]01 = 0$. Now let's turn our attention back to B, b_0 and b_1. In order to show that θ is uniform we have to show that $\theta b_0 b_1 = b_0$. The easy case is when the sets $\{b_0, b_1\}$ and $\{0, 1\}$ are disjoint.

In this case, we glue the per's B' and F together, to get a per B'' in which 0 is related to b_0, and 1 is related b_1. Now, as far as B'' is concerned, 0 is the same element as b_0, and 1 is the same element as b_1. It follows that $\theta b_0 b_1 = \theta[B'']b_0 b_1$ is the same element of B'' as $\theta 01 = \theta[B'']01 = 0$. So $\theta b_0 b_1 = b_0$, as required.

Now, if $\{b_0, b_1\}$ and $\{0, 1\}$ are not disjoint, we simply pick $b'_0 \neq b'_1$ disjoint from both, and apply the argument above twice.

This shows that in any PER model the interpretation of the polymorphic booleans contains only the two polymorphic projections. (The result of course generalizes to all algebraic signatures which contain only constants).

To recap, the first stage of the proof was to cut down a large per (or in general, algebra) B to a small algebra B' which still contained all the elements we were interested in (in general this is the subalgebra of reachable elements). This stage generalizes, as we shall see in the next section. The second stage was to show that as well as behaving as expected on each individual algebra, our function behaved uniformly on all algebras. We managed this via a gluing construction involving disjoint per's. This construction does not generalize well. There seem to be two separate problems. One is getting the algebra operations on the glued per's, and the other is finding a disjoint algebra to glue with in the first place.

4 Some general remarks

This section contains an account of that fragment of the theory for arbitrary algebraic types that holds in general.

As before, let Σ be a signature, and $(\mathcal{A}_i)_{i\in\{1\ldots n\}}$ its associated family of algebraic types. Note that if $\mathcal{M}$ is any model of the second-order lambda calculus whatever, then the interpretations $[\![\mathcal{A}_i]\!]$ in $\mathcal{M}$ of the types $\mathcal{A}_i$ form in a canonical way the carriers for a Σ-algebra structure. In this structure the operation ϕ_i is given by the interpretation in $\mathcal{M}$ of the term

$$\begin{array}{l}\lambda\, x_1 : \mathcal{A}_{i1}\ \ldots\ x_{N(i)} : \mathcal{A}_{iN(i)}.\quad \Lambda\, B_1 \ldots B_n.\ y_1 \ldots y_m. \\ \quad y_i\quad (x_1[B_1 \ldots B_n]\, y_1 \ldots y_m) \ldots (x_{N(i)}[B_1 \ldots B_n]\, y_1 \ldots y_m).\end{array}$$

Let us call this algebra $\mathcal{A}(\Sigma)$. Now, if B is any other Σ-algebra in $\mathcal{M}$ (with carriers $B_1, \ldots, B_n$ and operations $\psi_1, \ldots, \psi_m$ say), then there is a canonical Σ-algebra homomorphism from $\mathcal{A}(\Sigma)$ to B. In general, this is given by the functions $\theta_i : [\![\mathcal{A}_i]\!] \to B_i$, where θ_i is the interpretation in $\mathcal{M}$ of

$$\lambda a : \mathcal{A}_i.\ a[B_1 \ldots B_n]\, \psi_1 \ldots \psi_m.$$

In other words $\theta_i(a)$ is the interpretation of a at the Σ-algebra B. In the case of a PER model, where we know that $\mathcal{A}(\Sigma)$ is the product of all Σ-algebras, this homomorphism is just the projection at B.

If $\mathcal{M}$ should happen to contain an initial Σ-algebra ($\mathcal{F}$ say), then we have also a map $\mathcal{F} \to \mathcal{A}(\Sigma)$, exhibiting $\mathcal{F}$ as a retract of $\mathcal{A}(\Sigma)$. We want to show that this retraction is an isomorphism. For this we use the following lemma.

4.1 LEMMA Suppose $\mathcal{M} = \mathrm{PER}(\mathsf{A})$ is an arbitrary PER model for the second-order lambda calculus, and that Σ is a signature in the sense of many-sorted logic. Then

(i) $\mathcal{M}$ contains an initial Σ-algebra $\mathcal{F}$. (Moreover, $\mathcal{F}$ is also the initial Σ-algebra in the associated realizability topos.)

(ii) $\mathcal{A}(\Sigma)$ is canonically isomorphic to $\mathcal{F}$ iff for any $a \in \mathcal{A}_i$ there is a Σ-term σ of type A_i such that
$$a = \Lambda A_1 \ldots A_n.\ \lambda\phi_1 \ldots \phi_m.\ \sigma.$$

Proof. To prove the first assertion, take some suitable Gödel encoding of the syntax of Σ into the Church numerals of the partial combinatory algebra, and take the carriers of $\mathcal{F}$ to be (the canonically projective per's whose underlying sets are) the images of the closed terms of the appropriate sort under this encoding. The operations of the algebra are now given by the functions which construct compound terms out of their components, and are therefore recursive. Given any other Σ-algebra B in the topos, there is a homomorphism from $\mathcal{F}$ to B, given by decoding the elements of $\mathcal{F}$ as closed Σ-terms, and then giving the interpretation of the term in B. This operation is recursive in codes for the operations on B, and hence is an internal homomorphism. It is the unique homomorphism from $\mathcal{F}$ to B since the carriers of $\mathcal{F}$ are canonically separated, and their underlying sets give the free algebra in *Sets*.

To prove the second assertion, note that if a is any element of $\mathcal{A}_i$, then if there is a closed Σ-term σ such that $a = \Lambda A_1 \ldots A_n.\ \lambda\phi_1 \ldots \phi_m.\ \sigma$, then since the intepretation of $a[\mathcal{F}]$ is essentially σ, such a σ is necessarily unique. We thus have $\mathcal{F}$ as a retract of $\mathcal{A}_i$, where the inclusion sends σ to $\Lambda A_1 \ldots A_n.\ \lambda\phi_1 \ldots \phi_m.\ \sigma$, and the retraction sends a to $a[\mathcal{F}]$. The second assertion of the lemma now says that this retraction is an isomorphism if and only if it is surjective. □

To sum up, if $\mathcal{M}$ is a PER model, PER(A), then we have the following situation (treating the model as an internal category of sets in the realizability topos):

1. The product of all Σ-algebras in PER(A), is itself a Σ-algebra in PER(A), and is given by taking the canonical Σ-algebra structure on the interpretations of the algebraic types associated to Σ.

2. PER(A) also contains an initial Σ-algebra.

3. There is thus a canonical map from the initial algebra to the product algebra.
4. There is also a canonical map from the product algebra to the initial algebra (the projection).

We want to show that these two maps are inverse isomorphisms.

We recall that in any model $\mathcal{M}$, a value of type $[\![\mathcal{A}_i]\!]$ can be regarded as a function taking a Σ-algebra B as parameter, and producing a value of type B_i. An important feature of PER models is that the value produced does not depend on the carriers $B_1 \ldots B_n$, but only on the names of the operations of B. More formally, we have

4.2 Lemma Suppose B and C are Σ-algebras in a PER model $\mathcal{M} = \mathrm{PER}(\mathsf{A})$, where $B = (B_1, \ldots, B_n, e_1, \ldots, e_m)$ and $C = (C_1, \ldots, C_n, e_1, \ldots, e_m)$. Then if $a \in [\![\mathcal{A}_i]\!]$,

$$\begin{aligned} a[B_1 \ldots B_n]\, e_1 \ldots e_m &= a[C_1 \ldots C_n]\, e_1 \ldots e_m \\ &= a e_1, \ldots, e_m. \end{aligned}$$

Recall that we want to show that a is $\Lambda A_1 \ldots A_n.\ \lambda\langle \phi_1, \ldots, \phi_m \rangle.\ \sigma$, where σ is a closed Σ-term of type A_i. The lemma above allows us to restrict the class of Σ-algebras at which we have to look.

Suppose $\mathcal{M} = \mathrm{PER}(\mathsf{A})$ is the PER model on a partial combinatory algebra A. Given elements $e_1, \ldots, e_m$ of A which induce the operations on some Σ-algebra $B = (b_1, \ldots, b_n, e_1, \ldots, e_m)$ in $\mathcal{M}$, we define the Σ-algebra $\mathcal{O}(e_1, \ldots, e_m)$ (the *orbit* of $e_1 \ldots e_m$), with carriers $\mathcal{O}_i(e_1, \ldots, e_m)$, where $\mathcal{O}_i$ is the subset of the carrier of B_i consisting of the interpretations of closed Σ-terms, and in which the operations are given by $e_1, \ldots, e_m$. If $e_1 \ldots e_m$ do not induce the operations on any Σ-algebra, then $\mathcal{O}(e_1, \ldots, e_m)$ is undefined.

It is easy to see that the definition of $\mathcal{O}(e_1, \ldots, e_m)$ does not depend on the algebra B; $\mathcal{O}_i$ consists of the elements $(\lambda\phi_1 \ldots \phi_m.\ \sigma)\, e_1 \ldots e_m$ such that σ is a Σ-term of type A_i. Moreover $\mathcal{O}$ is defined iff all such elements are defined. It is also clear that $\mathcal{O}$ is in some sense a minimal Σ-algebra. This intuition can be made precise using the notion of *inclusion* ([CFS87]). We recall that an inclusion is a map of per's induced by the identity function on A. The category of Σ-algebras in $\mathrm{PER}(\mathsf{A})$ with homomorphisms given by inclusions is a poset, and the minimal elements are the algebras $\mathcal{O}(e_1, \ldots, e_m)$.

4.3 Lemma If $B = (B_1, \ldots, B_n, e_1, \ldots, e_m)$ is a Σ-algebra in $\mathrm{PER}(\mathsf{A})$, and $a \in [\![\mathcal{A}_i]\!]$, then

$$a[B_1 \ldots B_n]\, e_1 \ldots e_m = a[\mathcal{O}_1 \ldots \mathcal{O}_n]\, e_1 \ldots e_m,$$

and in particular $a[B_1 \ldots B_n]\, e_1 \ldots e_m \in \mathcal{O}_i$.

As a direct consequence we have:

4.4 COROLLARY Given an element a of $[\![\mathcal{A}_i]\!]$, then for any Σ-algebra B, where $B = (B_1, \ldots, B_n, e_1, \ldots, e_m)$ there is a closed Σ-term σ of type A_i (possibly depending on $e_1, \ldots, e_m$, but certainly independent of $B_1, \ldots, B_n$), such that

$$a[B_1 \ldots B_n]\, e_1 \ldots e_m = (\lambda\phi_1 \ldots \phi_m.\ \sigma)\ e_1 \ldots e_m.$$

This corollary is the analogue of showing that if θ is a polymorphic boolean, then $\theta[B]b_0b_1$ has to be either b_0 or b_1. It is the local version of the result we really want. So, it now remains to show that the σ whose existence is guaranteed by this corollary can be chosen independently of $e_1 \ldots e_m$. Note that in order to do this we need only look at the Σ-algebras $\mathcal{O}(x_1, \ldots, x_m)$. The crucial technical result we shall need is, however best expressed for more general algebras. We first make a definition.

Definition Suppose A is a partial combinatory algebra, and U and V are subsets of A, then we say that U and V are *recursively disjoint* if there is some element ϕ of A such that for all $u \in U$, $\phi u = \ulcorner 0 \urcorner$, and for all $v \in V$, $\phi v = \ulcorner 1 \urcorner$.

4.5 LEMMA Suppose that $\theta : B \to C$ is a homomorphism of Σ-algebras, where $B = (B_1, \ldots, B_n, e_1, \ldots, e_m)$, and that $C = (C_1, \ldots, C_n, d_1, \ldots, d_m)$, and for each i the domains of the per's B_i and C_i are recursively disjoint. Then for any $a \in [\![\mathcal{A}_i]\!]$

$$\{a[C_1 \ldots C_n]\, d_1 \ldots d_m\} \quad C_i \quad \{\theta_i\, (a[B_1 \ldots B_n]\, e_1 \ldots e_m)\},$$

i.e. a evaluated at C is the image under θ of a evaluated at B.

Proof. We define for each i, a per $B_i \otimes C_i$, whose domain is $B_i \cup C_i$, and where $n\ B_i \otimes C_i\ m$ iff either

(*i*) $n\ B_i\ m$ or else $n\ C_i\ m$

or (*ii*) $(\theta_i n)\ C_i\ m$ or else $n\ C_i\ (\theta_i m)$.

(We are gluing B_i to C_i along θ_i.) Since B_i and C_i are recursively disjoint for each i, the types $B_i \otimes C_i$ carry a natural Σ-algebra structure. We can define this in two different ways, either by means of functions $e'_1 \ldots e'_m$, which restrict to $e_1 \ldots e_m$ on B, or else via $d'_1 \ldots d'_m$, which restrict to $d_1 \ldots d_m$ on C. We cannot do both simultaneously, since any constants must come either from B or from C.

Since $e'_1 \ldots e'_m$ and $d'_1 \ldots d'_m$ both define the same algebra structure on $B \otimes C$ we have

$$a[B_1 \otimes C_1 \ldots B_n \otimes C_n]\ e'_1 \ldots e'_m \ =_{B_i \otimes C_i}\ a[B_1 \otimes C_1 \ldots B_n \otimes C_n]\ d'_1 \ldots d'_m.$$

Since e'_i restricts to e_i on B, we have

$$a[B_1 \ldots B_n]\, e_1 \ldots e_m \quad =_{B_i} \quad a[B_1 \ldots B_n]\, e'_1 \ldots e'_m,$$

and similarly

$$a[C_1 \ldots C_n]\, d_1 \ldots d_m \quad =_{C_i} \quad a[C_1 \ldots C_n]\, d'_1 \ldots d'_m.$$

Now apply lemma 4.2, and we obtain

$$a[B_1 \dots B_n]\, e_1 \dots e_m \quad =_{B_i \otimes C_i} \quad a[C_1 \dots C_n]\, d_1 \dots d_m,$$

from which the result follows. $\square$

This lemma is only useful if we can find an algebra disjoint from the one which we wish to study. However, for the case $\mathsf{A} = \mathsf{N}$ it is easy to find algebras which fill up the whole space available. Consider the polymorphic integers, corresponding to a one-sorted algebra with one constant and a single unary function. If for our constant we take 0, and for our function the successor function, then we obtain an algebra which takes up the whole of $\mathbb{N}$. To get round this we use a continuity argument which will allow us to compress our algebra so that it takes up only a finite amount of space, thus allowing us easily to find algebras disjoint from it. Unfortunately our argument does not work for general combinatory algebras, and so fails for general PER models.

5 PER(N)

We can express the continuity argument we need in very simple, though slightly abstract terms. For readers unhappy with this, a direct proof of the central proposition is given in an appendix. Again, let Σ be any many-sorted algebraic signature, and A any partial combinatory algebra. Then the Σ-algebras in PER(A) form a metric space (Σ-*Alg*) in which

$$d(B, C) \leq 1/n$$

if and only if the interpretation of σ in B is the same as the interpretation of σ in C for all Σ-terms σ of size not greater than n. This is true also in the internal topos-theoretic sense, with a slight delicacy—the metric function

$$d\colon \Sigma\text{-}Alg \times \Sigma\text{-}Alg \to \mathsf{Q}$$

is not represented internally. However, the predicate

$$d(B, C) \leq q$$

on Σ-*Alg* $\times$ Σ-*Alg* $\times$ Q is, and this will suffice for our purposes.

It is easy to see that Σ-*Alg* is complete with respect to this "metric", and that it is the completion of its subspace of finite Σ-algebras. Indeed, suppose that $B = (B_1, \dots, B_n, b_1, \dots, b_m)$ is an arbitrary Σ-algebra, then for each i such that the orbit $\mathcal{O}_i(b_1, \dots, b_m)$ is non-empty we can pick a "canonical" element c_i of $\mathcal{O}_i(b_1, \dots, b_m)$. Now, given an integer l, we can find elements b_{li} of A, such that

$$b_{li}\, z_1 \dots z_{N(i)} \quad = \quad \begin{cases} b_i\, z_1 \dots z_{N(i)} & \text{if each } z_i \text{ is } [\![\sigma]\!]_b (\text{where } [\![\sigma]\!]_b = \\ & (\lambda f_1 \dots f_m.\ \sigma)\ b_1 \dots b_m),\ \text{for} \\ & \text{some closed term } \sigma \text{ of size less} \\ & \text{than } l \\ c_i & \text{otherwise} \end{cases}$$

Moreover, we can find the b_{l_i} recursively in l. Note that if for some i the orbit $\mathcal{O}_i(b_1, \ldots, b_m)$ is empty, then no closed term of any type can depend on a term of type A_i, and hence that empty orbits cause no problems. Clearly, for each l, the orbits $\mathcal{O}_i(b_{l_1}, \ldots, b_{l_m})$ are defined, and are finite subsets of the $\mathcal{O}_i(b_1, \ldots, b_m)$. Furthermore, the Σ-algebras $\mathcal{O}(b_{l_1}, \ldots, b_{l_m})$ form a Cauchy sequence (even an internal Cauchy sequence) whose limit is B.

In the case $\mathsf{A} = \mathsf{N}$, this means that Σ-Alg is a complete separable metric space.

We can also represent the collection of closed terms of sort A_i internally in the topos, via Gödel enumeration. Abusing notation, let A_i be the canonically projective per whose domain is the set of codes for closed Σ-terms of type A_i. We regard A_i as a discrete metric space. Whatever partial combinatory algebra A we use, it is always separable, and has a linear order (for example that inherited from the order on $\mathbb{N}$).

Now let a be a value of type $[\![\mathcal{A}_i]\!]$. We can use a to define a map

$$\phi_a : \Sigma\text{-}Alg \to A_i.$$

$\phi_a(B)$ is the least σ in A_i such that a evaluated on B gives the intepretation of σ (*cf.* corollary 4.4). For a suitable choice of the linear order on A_i, the function ϕ_a is represented internally in the topos.

In the case of the effective topos ($\mathsf{A} = \mathsf{N}$), we can use a well-known theorem due to Čeitin and independently Moschovakis ([Če62], *cf.* [Bee85]) on the continuity of effective operations in the presence of Markov's principle. Expressed in somewhat more topos-theoretic terms than usual, this states

5.1 THEOREM If in the effective topos $\mathcal{E}\!f\!f$, f is a function from a complete separable metric space X to a separable metric space Y, then f is pointwise continuous, *i.e.* given a fixed $x \in X$, the formula

$$\forall \epsilon > 0.\ \exists \delta > 0.\ d_X(x, x') < \delta \ \to\ d_Y(f(x), f(x')) < \epsilon$$

is satisfied.

The continuity result we need is an immediate corollary of this theorem:

5.2 PROPOSITION Suppose a is in the domain of $[\![\mathcal{A}_i]\!]$, and $\mathcal{O}(x_1, \ldots, x_m)$ is defined. Then there is a number h, such that for all $l \geq h$ we have $a\, x_{l_1} \ldots x_{l_m} = a\, x_1 \ldots x_m$.

(Here, we are using juxtaposition to denote Kleene application, and associating to the left as usual; thus $a x_1 \ldots x_m$ means $\{\ldots\{\{a\}\, x_1\}\ldots\}\, x_m$.)

In the appendix we shall give a proof of this proposition which does not depend on Čeitin's theorem. Unfortunately, both this more concrete proof, and the proof of Čeitin's theorem itself, seem to depend on properties of the partial combinatory algebra N which are not completely general.

We now conclude our proof that the algebraic types are interpreted as initial Σ-algebras in the case $\mathsf{A} = \mathsf{N}$.

We first note that $\mathbb{N}$ is recursively isomorphic to $\{n \in \mathbb{N} \mid n = k \bmod m\}$ for any m, and k, and hence that we can find a copy of the initial Σ-algebra disjoint from any finite set.

Let $\mathcal{F} = (F_1, \ldots, F_n, y_1, \ldots, y_m)$ be our designated initial algebra, and suppose that

$$a[F_1 \ldots F_n]\, y_1 \ldots y_m = [\![\sigma]\!]_y.$$

Then, given a Σ-algebra $B = (B_1, \ldots, B_n, b_1, \ldots, b_m)$ (without loss of generality we can suppose that $B = \mathcal{O}(b_1, \ldots, b_m)$), we can find an h such that

$$a[F_1 \ldots F_n]\, y_{h1} \ldots y_{hm} = a[F_1 \ldots F_n]\, y_1 \ldots y_m$$

and also

$$a[B_1 \ldots B_n]\, b_{h1} \ldots b_{hm} = a[B_1 \ldots B_n]\, b_1 \ldots b_m.$$

Now pick a copy $\mathcal{F}'$ of the initial algebra avoiding both the finite sets $\mathcal{O}(y_{h1}, \ldots, y_{hm})$ and $\mathcal{O}(b_{h1}, \ldots, b_{hm})$, and apply lemma 4.5 twice to show that

$$a[B_1 \ldots B_n]\, b_1 \ldots b_m = [\![\sigma]\!]_b.$$

Thus we have shown:

5.3 THEOREM When $\mathsf{A} = \mathsf{N}$, the algebra $\mathcal{A}(\Sigma)$ is the initial Σ-algebra in $\mathcal{E}\!f\!f$, and hence $\mathcal{A}(\Sigma)$ is also the initial Σ-algebra in PER(N).

6 Conclusion

We have achieved our purpose in proving the theorem above. However, some concluding remarks appear to be in order. The proof we have given seems to use rather a lot of machinery, for a fairly small result. In particular, it is disturbing that it will only work for a relatively small class of Schönfinkel algebras. It is possible, however, that these results do inevitably depend on the algebra chosen. The small amount of experimental evidence available would tend to support this view. Apart from the proof for the algebra of open lambda terms, due to Moggi and outlined above, we have a proof due to Peter Freyd of the result for the polymorphic natural numbers. This proof avoids the explicit use of a general continuity principle by using an elegant combinatorial trick. However, it also works only for the PER model on the natural numbers. More recently, Freyd has also been able to show that the interpretation of the types $\Pi X.[[A \to X] \to X]$ for an arbitrary per A, is isomorphic to A. The proof he has given here also works only for PER models based on a relatively restricted class of algebras (again including the Kleene algebra on the natural numbers) [Fre89a].

Finally, we should return to the relationship between PER models and the free term model. We stated above that any PER model satisfies equations which are not provable in the calculus, and which therefore do not hold in the free term model. We can now give an example. If we let P be the type $\Pi X.X \to X$, $Q = P \to P$ be the type of endomorphisms of P, and S be the "double negation" $\Pi X.[[Q \to X] \to X]$ of Q, then it is well-known that there is a closed term of type S which is not formally evaluation at any closed term of type Q (the term $\Lambda X. \lambda f : Q \to X.\ f(\lambda a : P.\ (a[X \to P](\lambda x : X.\ \Lambda Y.\ \lambda y : Y.\ y)(f(\lambda p : P.p))))$ will do). It follows that in any extensional model, either we violate parametricity, in that S is not isomorphic to Q, or there is a non-standard element of Q, or else some equation holds in the model which does not hold in the term model. In the particular case of PER models, we know that P contains only the polymorphic identity. It follows that Q and S are also singletons, and that the term we have given is equated in the model to evaluation at the identity map on P.

Appendix

In this appendix we give a direct proof of the crucial continuity result proposition 5.2. We have chosen to use a proof style modelled on Gandy's proof of the Kreisel-Lacombe-Shoenfield theorem. This gives a very slick proof in which we use the second recursion theorem to pull assorted useful integers out of a definition whose construction is about as obvious as a magician's hat. There is always an alternative to using the recursion theorem in this way. Most often, it is a longer and equally unintuitive combinatorial proof. Readers are referred to the proof of Čeitin's theorem in [Bee85] for an example.

The structure of this proof is such that it is more convenient to use the interpretations of the type $\mathcal{A}_i^\times$ of the extended calculus, rather than the types $\mathcal{A}_i$. Recall that if $B = (B_1, \ldots, B_n, b_1, \ldots, b_m)$ is a Σ-algebra, then a value a of type $[\![\mathcal{A}_i^\times]\!]$ interpreted at $B_1, \ldots, B_n$ takes the tuple $\langle b_1, \ldots, b_m \rangle$ as parameter, and produces a value of type B_i as result. (We contrast $a\,[B_1 \ldots B_n]\,\langle b_1, \ldots, b_m \rangle$ with $a[B_1 \ldots B_n]\,b_1 \ldots b_m$). As before, we can define a series of finite approximants $\mathcal{O}^\times(b_{l1}, \ldots, b_{lm})$ to B, and our proposition becomes:

5.2 Proposition Suppose a is in the domain of $[\![\mathcal{A}_i^\times]\!]$, and $\mathcal{O}^\times(x_1, \ldots, x_m)$ is defined. Then there is a number h, such that for all $l \geq h$ we have

$$a\,\langle x_{l1}, \ldots, x_{lm} \rangle = a\,\langle x_1, \ldots, x_m \rangle.$$

Proof. Let $\Phi(g, y, y', m)$ be the formula

$$\exists w, w' \leq m[T(g, y, w) \ \wedge\ T(g, y', w') \ \wedge\ Uw = Uw']$$

where T is Kleene's T-predicate, and U its output function. Use the recursion theorem to find an integer e_j such that

$$\{e_j\}\langle z_1,\ldots,z_{N(j)}\rangle = \begin{cases} \{x_j\}\langle z_1,\ldots,z_{N(j)}\rangle & \text{if some } z_i = [\![\sigma]\!]_x \text{ for some } \sigma \text{ of size } r \text{, where } \Phi(a,\langle x_1,\ldots,x_m\rangle,\langle e_1,\ldots,e_m\rangle,r) \\ \{x_k\}\langle z_1,\ldots,z_{N(j)}\rangle & \text{if for some } r\text{, some } z_i \text{ is of the form } [\![\sigma]\!]_x\text{, where } \sigma \text{ is a term of size } r\text{, but where any such } r \text{ satisfies } \Phi(a,\langle x_1,\ldots,x_m\rangle,\langle e_1,\ldots,e_m\rangle,r) \text{ and where } k \text{ is the least integer such that } \Phi(a,\langle x_1,\ldots,x_m\rangle,\langle e_1,\ldots,e_m\rangle,k) \text{ and } \{a\}\langle x_1,\ldots,x_m\rangle \neq \{a\}\langle x_{k1},\ldots,x_{km}\rangle \end{cases}$$

First note that there must be an r such that $\Phi(a,\langle x_1,\ldots,x_m\rangle,\langle e_1,\ldots,e_m\rangle,r)$. If we suppose that there is not, then $\{e_i\} = \{x_i\}$ on $\mathcal{O}$. This however implies that $\{a\}\langle x_1,\ldots,x_m\rangle = \{a\}\langle e_1,\ldots,e_m\rangle$, using the functionality of a on the Σ-algebra $\mathcal{O}$. Thus there is an r such that $\Phi(a,\langle x_1,\ldots,x_m\rangle,\langle e_1,\ldots,e_m\rangle,r)$, a contradiction. We can now let h be the least integer such that
$\Phi(a,\langle x_1,\ldots,x_m\rangle,\langle e_1,\ldots,e_m\rangle,h)$.

To conclude the proof, let n be the least integer greater than or equal to h such that $\{a\}\langle x_{n1},\ldots,x_{nm}\rangle \neq \{a\}\langle x_1,\ldots,x_m\rangle$, if such exists. For such an n we have that for each i, $\{x_{ni}\} = \{e_i\}$ on $\mathcal{O}(x_{n1},\ldots,x_{nm})$. This implies that $\{a\}\langle x_{n1},\ldots,x_{nm}\rangle = \{a\}\langle e_1,\ldots,e_m\rangle$, again using the functionality of a. But since we know that the formula $\Phi(a,\langle x_1,\ldots,x_m\rangle,\langle e_1,\ldots,e_m\rangle,h)$ actually holds, we also have $\{a\}\langle e_1,\ldots,e_m\rangle = \{a\}\langle x_1,\ldots,x_m\rangle$, which thus leads to a further contradiction. We conclude that for all $n \geq h$, we have $\{a\}\langle x_{n1},\ldots,x_{nm}\rangle = \{a\}\langle x_1,\ldots,x_m\rangle$, as required. □

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