Function spaces in the category of locales

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Introduction:

A number of authors (in particular Day & Kelly [1]) have observed that the topological spaces X which have function spaces (i.e. such that $() \times X$ has a right adjoint in Top the category of topological spaces) are just those whose lattice $\Omega(X)$ of open sets is a continuous lattice. The main result of this paper is an extension of this characterization to the category Loc of locales. Many similar results (one might say "everything but the result in this paper") are given in Isbell [5], and I imagine that others will have considered the problem. But I have never seen an exposition of the material.

This paper is written for those who like their abstract mathematics concrete. In particular, the proofs are entirely constructive, and so the results are available for applications in categorical logic. Indeed, two examples of function spaces considered here, Cantor space $2^{\mathbb{N}}$ and Baire space $\mathbb{N}^{\mathbb{N}}$, can be used to analyze the constructive force of completeness theorems (-an account should appear in Fourman [2]). Note that by [5], classically a locale which is a continuous lattice is spatial; so we do not change the objects with function spaces when passing from the category Sp of sober spaces to Loc. Constructively the situation is very different, but I will not go into that here.

This paper's dual intellectual debt is to André Joyal and Dana Scott whose insights into locales and continuous lattices, respectively, are fundamental to it. Special thanks are due to Peter Johnstone, who both encourage me to write this paper and exposed me to a civilized treatment of locales which will appear in his forthcoming book [6].

Preliminaries:

All the information needed to understand this paper is contained in Johnstone [6] Ch. 2 (for locales) and in Gierz, Hofmann &&&& Scott [4] Chs. 1 & 2 (for continuous lattices). What follows is a brief sketch.

The category Loc of locales is the opposite of the category with objects complete Heyting algebra's and with $\wedge, \bigvee$-preserving maps. The category of sober spaces is embedded (fully and faithfully) in Loc, and this embedding has a right adjoint. Loc^{op} would be thought of as the category of geometric propositional theories, where the maps correspond to interpretations: for a locale A, the theory on the elements of A is generated by the axioms $\vdash T$ (T top element), $a \vdash a'$ whenever $a \leq a'$ in A, $a \wedge a' \vdash a''$ whenever a'' is the meet of a and a' and $a \vdash \bigvee a_i$ whenever a is the join of the a_i's in A. To obtain the product $A \times B$ of two locales in Loc, we take their sum in Loc^{op} and so amalgamate the two theories and take the corresponding (classifying) complete Heyting algebra. Thus $A \times B$ consists of ideals in the lattice product "closed under the axioms for A and B" (i.e. in Johnstone's terminology, C-ideals for a suitable coverage C). Write $a \wedge b$, for the pair (a,b) in the lattice product and also for the corresponding principle ideal. Then the elements $a \wedge b$ are dense in $A \times B$ and for an arbitrary element x of $A \times B$,

$$a \wedge b \leq x \text{ if and only if } a \wedge b \in x.$$

An important part is played in the following by two locales: the terminal object 1 in Loc which is the locale corresponding to the one point space, and the cogenerator S which is the locale corresponding to Sierpinski space. Classically 1 is the two element totally-ordered set and S is the three element totally-ordered set. Constructively, 1 is P(1) as a lattice and can be identified with the ideals of the two element totally-ordered set: similarly, S is the lattice of (Scott) open sets of the continuous lattice (and sober space) P(1) and so can be identified with the ideals of $\{\perp < m < T\}$ the three element totally-ordered set. It follows from this last observation that an $\wedge\bigvee$-map $S \to A$ from S to an arbitrary locale A, is completely determined by image of m = ideal generated by m.

In a complete lattice (or more generally) the relation <u>way below</u>, $\ll$, defined by

$$a \ll b \text{ if and only if whenever a directed union } \bigvee S \geq b, \text{ then for some } s \in S \quad s \geq a$$

(equivalently if and only if for any ideal I, $\bigvee I \geq b$ implies $a \in I$). A <u>continuous lattice</u> is a complete lattice in which every element is the sup of the (directed) set of elements way below it. As with many arguments involving this concept, we will make considerable use of the density property of the relation $\ll$ in a continuous lattice:

$$\text{if } a \ll b \text{ then for some } c.\ a \ll c \ll b.$$

A locale is <u>locally compact</u> if and only if it is a continuous lattice.

<u>Function Spaces:</u>

In a category, an object A is said to <u>have function spaces</u> if and only if the functor $() \times A$ has a right adjoint. There <u>is a function space</u> B^A for two objects A and B if and only if the function Hom $(\ \times A, B)$ is representable.

We can now state our

<u>(Main) Theorem 1:</u> The following are equivalent for a locale A;

(i) A is locally compact;

(ii) A has function spaces in Loc;

(iii) The function space S^A exists in Loc.

Clearly (ii) implies (iii). We will show (iii) implies (i) and (i) implies (ii): since both proofs are rather long, we give the results as separate propositions.

Proposition 2: If S^A exists in Loc, then A is locally compact.

Proof: - A point $p:1 \to S^A$ of S^A corresponds to a locale map $\tilde{p}:A \to S$ and hence is determined by the $\wedge \bigvee$ map $\tilde{p}^*:S \to A$ and so by element $\tilde{p}^*(m)$ of A. Conversely an element a say of A gives rise to a locale map $\tilde{p}_a:A \to S$ where $\tilde{p}_a{}^*(m) = a$ and thence to a point $p_a:1 \to S^A$ of S^A. There is an obvious partial order on the points of S^A, the pointwise order on maps between lattices (it makes no difference whether we consider the $p:1 \to S^A$ or its left adjoint $p^*:S^A \to 1$). We claim first under the above bijection between points of S^A and elements of A, the order on the points corresponds to that on A.

Sublemma 2.1 $p_a \le p_{a'}$ if and only if $a \le a'$.

Proof: - One way round is easy. Given $p:1 \to S^A$, the map $\tilde{p}:A \to S$, must be the composite

$$A \cong 1 \times A \xrightarrow[p \times 1]{} S^A \times A \xrightarrow[ev]{} S$$

where $ev:S^A \times A \to S$ is the "exponential transpose" of $id:S^A \to S^A$. Hence if $p_a \le p_{a'}$, then $\tilde{p}_a \le \tilde{p}_{a'}$ and so $a = \tilde{p}_a{}^*(m) \le \tilde{p}_{a'}^*(m) = a'$. For the converse, suppose $a \le a'$ and consider the element of $S \times A$

$$[\top \wedge a,\ m \wedge a'],$$

the ideal generated by $\top \wedge a$ and $m \wedge a'$. Since $a \le a'$, this consists of just those elements in the lattice product below one or other of $\top \wedge a$ and $m \wedge a'$. Now consider the maps $h:S \times A \to S$ and $t, f:1 \to S$ where $h^*(m) = [\top \wedge a, m \wedge a']$ and $t^*(m) = \top$, $f^*(m) = \bot$. (The reader should be able to identify the latter two as maps of spaces <u>and</u> as the maps <u>true</u> and <u>false</u> in any topos). Now by our characterization of $[\top \wedge a, m \wedge a']$, the composites

$$A \cong 1 \times A \xrightarrow[t \times 1]{} S \times A \xrightarrow[h]{} S$$

and

$$A \cong 1 \times A \xrightarrow[f \times 1]{} S \times A \xrightarrow[h]{} S$$

are $p_{a'}$ and p_a respectively. Taking "exponential transpose" again we see that the composites

$$1 \xrightarrow[t]{} S \xrightarrow[\tilde{h}]{} S^A$$

and
$$1 \xrightarrow[f]{} S \xrightarrow[\tilde{h}]{} S^A$$

are $p_{a'}$ and p_a respectively ($\tilde{h}$ is the "transpose" of h). Since $f \le t$, it follows immediately that $p_a \le p_{a'}$. This completes the proof of (2.1).

It is now natural to identify a and p_a, and when d is an element of S^A write $a \in d$ rather than $p \in d$ for $p_a{}^*(d) = T$.

Sublemma 2.2 For any element d of S^A, $\{a | a \in d\}$ is upwards closed in A.

Proof: - If $a \le a'$ and $a \in d$ then $p_a{}^*(d) = T$ and $p_a \le p_{a'}$ by (2.1) so $p_{a'}{}^*(d) = T$, that is $a' \in d$.

We next show that the sets $\{a | a \in d\}$, d an element of S^A, satisfy Scott's property of inaccessibility to directed joins (so they are Scott open in A).

Sublemma 2.3 Let d be an element of S^A, and I an ideal in A such that $\bigvee I \in d$. Then for some a in I, $a \in d$.

Proof: - (We use $\in$ both to denote ordinary set membership and as above to denote the "membership" of points in elements of locales. The context should prevent confusion). Consider $\uparrow(I)$ the upward closed sets of I under inclusion. This clearly forms a locale and in view of (2.2) we define $f: \uparrow(I) \to S^A$ by setting

$$f^*: S^A \to \uparrow(I);\ d \to \{a \in I | a \in d\}$$

Each element a of I gives rise to a point $q_a: 1 \to \uparrow(I)$ of $\uparrow(I)$ determined by

$$q_a{}^*: \uparrow(I) \to 1;\ U \to \bigvee\{T | a \in U\},$$

(i.e. $q_a{}^*(U)$ is T if and only if $a \in U$). A simple deduction shows that $f_o q_a = p_a$.

Define a point $q: 1 \to \uparrow(I)$ of $\uparrow(I)$ by setting

$$q^*: \uparrow(I) \quad 1;\ U \to \bigvee\{T | \exists a \in I.\ a \in U\},$$

(i.e. $q^*(U)$ is T if and only if U has a number).

Clearly for each a in I, $q_a \le q$ so $p_a \le f_o q$ and so $a \le (\widetilde{f_o q})^*(m)$.

Thus $\bigvee I \le (\widetilde{f_o q})^*(m)$.

Now if $\bigvee I \in d$, then $(f_o g)^*(m) \in d$ whence $(f_o q)^*(d) = T$ that is $q^* f^*(d) = T$; but this implies $\exists a \in I.\ a \in f^*(d)$ so that for some $a \in I$, $a \in d$. This completes the proof of (2.3).

<u>Sublemma 2.4</u> $a = \bigvee\{a^* \mid \exists d.\, d \wedge a^* \le ev^*(m)\ \&\ a \in d\}$ where as above ev is the "exponential transpose" of $id: S^A \to S^A$.

Proof: - As in the proof of (2.1) we have $\tilde{p}_a = ev_o(p_a \times 1)$, and by properties of the product in Loc we have

$$(p_a \times 1)^*(d' \wedge a') = \bigvee\{T \mid a \in d'\} \wedge a' = \bigvee\{a' \mid a \in d'\}.$$

Thus $a = \tilde{p}_a^*(m)$

$$= (p_a \times 1)^* ev^*(m)$$

$$= \bigvee \{(p_a \times 1)^*(d' \wedge a') \mid d' \wedge a' \le ev^*(m)\}$$

$$= \bigvee\{a^* \mid \exists d.\, d \wedge a^* \le ev^*(m)\ \&\ a \in d\}.$$

This proves (2.4).

We now complete the proof of Proposition 2 by showing

$$a = \bigvee\{a' \mid a' \ll a\},$$

for any a in A.

In (2.4) we see that

$$d \wedge a^* \le ev^*(m) \text{ and } a \in d \text{ implies } a^* \le a \quad (\dagger).$$

We claim that in fact such an a* is way below a. Let I be an ideal such $\bigvee I \ge a$; since $a \in d$, by (2.2) $\bigvee I \in d$; hence by (2.3) $\exists a' \in I.\ a'' \in d$; now since $d \wedge a^* \le ev^*(m)$, we can apply (†) to a" to obtain $a^* \le a'$; thus $a^* \in I$.

This shows that

$$d \wedge a^* \le ev^*(m) \text{ and } a \in d \text{ implies } a^* \ll a,$$

and so by (2.4) we have

$$a = \bigvee\{a' \mid a' \ll a\},$$

and A is locally compact.

To show that (i) implies (ii), we exhibit a presentation of the locale B^A for A locally compact and B arbitrary. As in the proof of Proposition 2, the points of B^A correspond to locale maps A to B, that is to $\wedge\bigvee$-maps from B to A. The presentation of B^A which we give is the theory of such $\wedge\bigvee$-maps.

Proposition 3: If A is a locally compact locale, then A has function spaces in Loc.

Proof: - Consider a geometric theory based on propositions "$a \ll f^*(b)$" for $a \in A$, $b \in B$; of course "$a \ll f^*(b)$" is regarded as a single syntactic entity; the notation indicates how the points of the locale we are constructing give rise to $\wedge\bigvee$-maps from B to A. The theory is given by the axioms:

$$a \ll f^*(b) \vdash a' \ll f^*(b') \qquad (a' \leq a, b \leq b');$$

$$\vdash \bot \ll f^*(b); \; a \ll f^*(b),\; a' \ll f^*(b) \vdash a \vee a' \ll f^*(b);$$

$$\vdash a' \ll f^*(T)\,(a' \ll T); \; a \ll f^*(b),\; a \ll f^*(b') \vdash a' \ll f^*(b \wedge b')\,(a' \ll a);$$

$$a \ll f^*(b) \vdash \bigvee_{a \ll a'} a' \ll f^*(b);$$

$$a \ll f^*(b) \vdash \bigvee_{\substack{\text{finite covers} \\ \{a_\alpha | \alpha \in J\} \text{ of } a \\ \text{for which } J \leq I}} \left(\bigwedge_J a_\alpha \ll f^*(b_\alpha)\right) \qquad (\{b_\alpha | \alpha \in I\} \text{ covers } b).$$

The locale B^A is the propositional (Lindenbaum) algebra associated to this theory. Concretely, $\vdash$ is a preorder on the geometric formulae (it is sufficient to consider arbitrary disjunctions of finite conjunctions of the $a \ll f^*(b)$ and B^A is obtained by factoring out by the corresponding equivalence relation to give a partial order. Thus the elements $[a \ll f^*(b)]$ (equivalence class of $a \ll f^*(b)$) are subbasic - any element is an arbitrary join of finite meets of such - so an $\wedge\bigvee$-map from B^A will be determined by its value on the $[a \ll f^*(b)]$'s.

In this proof, for locale maps F, G ... we let F^*, G^* denote the inverse image map which is an $\wedge\bigvee$-map in the opposite direction (the "logical" direction). First suppose we are given $G: C \times A \to B$; we define $\tilde{G}: C \to B^A$ by setting

$$\tilde{G}^*([a \ll f^*(b)]) = \bigvee \{c \mid \exists a' \gg a,\; c \wedge a' \leq G^*(b)\}$$

To check that $\tilde{G}^*$ extends to a (unique) $\wedge\bigvee$-map B^A C, it suffices to check that $\tilde{G}^*$ preserves the axioms for B^A. These are all trivial except the last, for which we need some Lemmas.

<u>Sublemma 3.1</u> Let A be locally compact and $\{x_\alpha | \alpha \in I\}$ a directed set of elements (C-ideals) in $C \times A$: then

$$\bigvee\{x_\alpha | \alpha \in I\} = \{c \wedge a | \forall a' \ll a.\ c \text{ is a union of } c' \text{ such that } c' \wedge a' \text{ is in some } x_\alpha\}.$$

Proof: - It suffices to show that { } is closed under the axioms for $C \times A$ (i.e. is a C-ideal).

Suppose $c_\delta \wedge a \in \{\ \}$; then clearly $\bigvee c_\delta \wedge a \in \{\ \}$.

Suppose $c \wedge a_\delta \in \{\ \}$; to show $c \wedge \bigvee a_\delta \in \{\ \}$, pick $a' \ll a = \bigvee a_\delta$; we find a finite set F of δ's and $a'_\delta \ll a_\delta$ $\delta \in F$ with $\bigvee_F a'_\delta = a'$; now for each $\delta \in F$,

$$c = \bigvee\{c' \leq c | c' \wedge a'_\delta \text{ is in some } x_\delta\},$$

whence taking finite intersections we get

$$c = \bigvee\{c' \leq c | c' \wedge a'_\delta \text{ is in some } x_\delta \text{ for each } \delta \in F\}$$
$$= \bigvee\{c' \leq c | c' \wedge a' \text{ is in some } x_\delta\} \text{ using the fact that } \{x_\delta | \delta \in I\} \text{ is directed;}$$

this shows $c \wedge \bigvee a_\delta \in \{\ \}$ as required.

<u>Sublemma 3.2</u> Let A be locally compact and $\{x_\alpha | \alpha \in I\}$ a finite set of elements of $C \times A$: then

$$\bigvee\{x_\alpha | \alpha \in I\} = \{c \wedge a | \forall a' \ll a.\ c \text{ is a union of } c' \text{ such that there is } a' = \bigvee\{a'_\alpha | \alpha \in I\} \text{ with } c' \wedge a'_\alpha \in x_\alpha,\ \alpha \in I\}.$$

Proof: - Again it suffices to show that { } is closed under the axioms for $C \times A$.

Suppose $c_\delta \wedge a \in \{\ \}$; then clearly $\bigvee c_\delta \wedge a \in \{\ \}$.

Suppose $c \wedge a_\delta \in \{\ \}$; to show $c \wedge \bigvee a_\delta \in \{\ \}$, pick $a' \ll a = \bigvee a_\delta$; find a finite set F of δ's and $a'_\delta \ll a_\delta$ $\delta \in F$ with $\bigvee\{a'_\delta | \delta \in F\} = a'$; now for each $\delta \in F$,

$$c = \bigvee\{c' \leq c | \text{ there is } a'_\delta = \bigvee\{a'_{\delta,\alpha} | \alpha \in I\} \text{ with } c' \wedge a'_{\delta,\alpha} \in x_\alpha\},$$

so taking finite intersections we have

$$c = \bigvee\{c' \le c \mid \text{for each } \delta \in F, \text{ there is } a'_\delta = \bigvee\{a'_{\delta,\alpha} \mid \alpha \in I\}$$
$$\text{with } c' \wedge a'_{\delta,\alpha} \in x_\alpha\},$$

whence setting $a'_\alpha = \bigvee\{a'_{\delta,\alpha} \mid \alpha \in F\}$ we see that

$$c = \bigvee\{c' \le c \mid \text{ there is } a' = \bigvee\{a'_\alpha \mid \alpha \in I\} \text{ with}$$
$$c' \wedge a'_\alpha \in x_\alpha\};$$

this shows $c \wedge \bigvee a_\delta \in \{\ \}$ as required.

Sublemma 3.3 Let A be locally compact and $\{x_\alpha \mid \alpha \in I\}$ a set of elements of $C \times A$: then

$$\bigvee\{x_\alpha \mid \alpha \in I\} = \{c \wedge a \mid \forall a' \ll a.\ c = \bigvee\{c' \le c \mid \exists \text{ finite}$$
$$J \subseteq I \text{ and } a' = \ \{a'_\alpha \mid \alpha \in J\},\ c' \wedge a'_\alpha \in x_\alpha,\ \alpha \in J\}\}.$$

Proof: - Sublemmas 3.1 and 3.2 give

$c \wedge a \in \bigvee\{x_\alpha \mid \alpha \in I\}$ iff $\forall a' \ll a \quad c = \bigvee\{c' \le c \mid \exists$ finite $J \subseteq I$.

$$\forall a'' \ll a'.\ \ c' = \bigvee\{c'' \mid \exists a'' = \bigvee\{a''_\alpha \mid \alpha \in J\} \ \&\ c'' \wedge a''_\alpha \in x_\alpha,$$
$$\alpha \in J\}\}.$$

Clearly $c \wedge a \in \{\ \}$ of (3.3) implies the condition on the right. Conversely if $c \wedge a \in \bigvee\{x_\alpha \mid \alpha \in I\}$, then for any $a' \ll a$ pick $a' \ll a'' \ll a$ & we get $c = \bigvee\{c' \le c \mid \exists \text{fin. } J \le I\ \&$

$c' = \bigvee\{c'' \le c' \mid \exists a' = \bigvee\{a'_\alpha \mid \alpha \in J\}\ \&\ c'' \wedge a'_\alpha \in x_\alpha\}\}$,

whence $c = \bigvee\{c' \le c \mid \exists \text{fin. } J \subseteq I\ \&\ a' = \bigvee\{a'_\alpha \mid \alpha \in J\}\ \&\ c' \wedge a'_\alpha \in x_\alpha\}$;

this shows $c \wedge a \in \{\ \}$ of (3.3).

[Alternatively, one can give a direct proof along the lines of (3.1) & (3.2)].

Now suppose c is such that $\exists a' \gg a\ \ c \wedge a' \le G^*(b)$ where $b = \bigvee\{b_\alpha \mid \alpha \in I\}$: then pick $a''.\ a \ll a'' \ll a'$; by (3.3),

$$c = \bigvee\{c' \le c \mid \exists \text{finite } J \subseteq I \text{ and } a'' = \bigvee\{a''_\alpha \mid \alpha \in J\}$$
$$\text{with } c' \wedge a''_\alpha \in G^*(b_\alpha),\ a \in J\};$$

but we can find $a_\alpha \ll a''_\alpha \quad \alpha \in J$ with $a = \bigvee\{a_\alpha \mid \alpha \in J\}$ for any $a'' = \bigvee\{a''_\alpha \mid \alpha \in J\}$; so

$$c = \bigvee\{c' \le c \mid \text{ for some finite } J \subseteq I \text{ and } a = \bigvee\{a_\alpha \mid \alpha \in J\}$$
$$\text{there is } a_\alpha \ll a''_\alpha \text{ and } c' \wedge a''_\alpha \in G^*(b_\alpha),\ \alpha \in J\};$$

but this shows that

$$\tilde{G}^*([a \ll f^*(b)]) \leq \bigvee_{\substack{\text{finite } J \subseteq I \\ \& \ a = \{a_\alpha | \alpha \in J\}}} \bigwedge_J \tilde{G}^*([a_\alpha \ f^*(b_\alpha)]).$$

<u>Remark</u> The last axiom is equivalent to the following two:

$$a \ll f^*(b) \vdash \bigvee_{a \in I} a \ll f^*(b_\alpha) \quad \left(\begin{array}{l}\{b_\alpha | \alpha \in I\} \text{ a directed set with} \\ \bigvee\{b_\alpha | \alpha \in I\} = b\end{array}\right)$$

$$a \ll f^*(b) \vdash \bigvee_{\substack{\text{finite covers} \\ \{a_\alpha | \alpha \in I\} \text{ of } a}} \bigwedge_I (a_\alpha \ll f^*(b_\alpha) \quad \left(\begin{array}{l}\{b_\alpha | \alpha \in I\} \text{ a finite set} \\ \text{with } \{b_\alpha | \alpha \in I\} = b\end{array}\right).$$

Sublemma 3.1 shows that the first of these is preserved and Sublemma 3.2 does the second.

Now suppose that we are given $F: C \to B^A$; we define $\bar{F}: C \times A \to B$ by setting

$$\bar{F}^*(b) = [\{c \wedge a | c \leq F^*(a \ll f^*(b))\}] \text{ (i.e. C-ideal generated by \{ \})}$$

$$= \bigvee\{c \wedge a | c \leq F^*(a \ll f^*(b))\}.$$

We check that $\bar{F}^*$ is an $\wedge\bigvee$ map. We need a Lemma identifying $\bar{F}^*(b)$.

<u>Sublemma 3.4</u> $\bar{F}^*(b) = \{c \ \ a | \forall a' \ll a \quad c \leq F(a' \ll f^*(b))\}$.

Proof: - Again as { } is generated by the $c \wedge a | c \leq F^*(a \ll f^*(b))$, it suffices to show that { } is closed under the axioms for $C \times A$.

If $c_\delta \wedge a \in \{\ \}$, clearly $\bigvee c_\delta \wedge a \in \{\ \}$.

If $c \wedge a_\delta \in \{\ \}$; to show $c \wedge \bigvee a_\delta \in \{\ \}$ pick $a' \ll a = \bigvee a_\delta$; find a finite set F of δ's and $a'_\delta \ll a_\delta$ with $\bigvee\{a'_\delta | \delta \in F\} = a'$; so for each $\delta \in F$, $c \leq F^*(a'_\alpha \ll f^*(b))$; so by one of the axioms for B^A,

$$c \leq F^*(a' \ll f^*(b));$$

but this shows $c \wedge \bigvee a_\delta \in \{\ \}$ as required.

Let $\{b_i | i \in I\}$ be a finite set; we wish to show

$$\bigwedge\{\bar{F}^*(b_i) | i \in I\} \leq \bar{F}^*(\bigwedge\{b_i | i \in I\});$$

but $\bigwedge\{\bar{F}^*(b_i) | i \in I\} = \{c \wedge a \,|\, \text{for each } i \in I \quad c \leq F^*(a' \ll f^*(b_i))$ each $a' \ll a\}$;

but if $a' \ll a$ we can pick $a' \ll a'' \ll a$, deduce from

$$c \leq F^*(a'' \ll g^*(b_i)) \text{ each } i \in I$$

that $c \leq F^*(a' \ll f^*(\ b_i))$; so

$$\bigwedge \{\bar{F}^*(b_i) \mid i \in I\} \leq \{c \wedge a \mid \forall a' \ll a \quad c \leq F^*(a' \ll f^*(\bigwedge b_i))\} = \bar{F}^*(\bigwedge b_i).$$

Let $\{b_\alpha \mid \alpha \in I\}$ be any set in B; we wish to show

$$\bigvee \{\bar{F}^*(b_\alpha) \mid \alpha \in I\} \geq \bar{F}^*(\bigvee \{b_\alpha \mid \alpha \in I\});$$

but $F^*(\bigvee b_\alpha) = \bigvee \{c \wedge a \mid c \leq F^*(a \ll f^*(\bigvee b_\alpha))\}$ and

$$\text{if } c \leq F^*(a \ll f^*(\bigvee b_\alpha)) \leq \bigvee_{\substack{J \subseteq I \text{ finite} \\ \& \ \{a_\alpha \mid \alpha \in J\} = a}} \bigwedge F^*(a_\alpha \ll f^*(b_\alpha)),$$

$$c = \bigvee \{c' \leq c \mid \text{there is finite } J \leq I \ \& \ a = \bigvee \{a_\alpha \mid \alpha \in J\} \text{ with } c' \quad a_\alpha \leq \bar{F}^*(b_\alpha)\}$$

so that $c \wedge a \leq \bigvee \{\bar{F}^*(b_\alpha) \mid \alpha \in I\}$ by (3.3).

If we start with $G: C \times A \to B$, we have (in view of (3.4))

$$\bar{\bar{G}}^*(b) = \{c \wedge a \mid \forall a' \ll a \quad c \leq \bigvee \{c' \mid \exists a'' \gg a'.\ c' \wedge a'' \leq G^*(b)\}$$

$$= \bigvee \{c \wedge a \mid c \leq \bigvee \{c' \mid \exists a'' \gg a.\ c' \wedge a'' \quad G^*(b)\}.$$

Clearly if $c \wedge a \leq G^*(b)$ then $c \wedge a \leq \bar{\bar{G}}^*(b)$ by top line: so $G^* \leq \bar{\bar{G}}^*$.

Conversely using bottom line, we wish to show that

if $c \wedge a$ is such that ( ) $c \leq \bigvee \{c' \mid \exists a'' \gg a. c' \wedge a'' \leq G^*(b)\}$,

then $c \wedge a \leq G^*(b)$;

but ( ) implies $c \leq \bigvee \{c' \mid c' \wedge a \leq G^*(b)\}$ whence $c \wedge a \leq G^*(b)$.

This shows $\bar{\bar{G}}^* \leq G^*$ so we have $G = \bar{\bar{G}}$.

On the other hand if $F: C \to B^A$, we have (in view of (3.4))

$$\bar{\bar{F}}^*([a \ll f^*(b)]) = \bigvee \{c \mid \exists a' \gg a.\ \forall a'' \ll a'.\ c \leq F^*([a'' \ll f^*(b)])\}.$$

Clearly if $c \ \in \{ \ \}$ of RHS then $c \leq F^*([a \ll f^*(b)])$: so $\bar{\bar{F}}^* \leq F^*$.

Conversely since $[a \ll f^*(b)] = \bigvee_{a \ll a'} [a' \ll f^*(b)]$,

$$F^*([a \ll f^*(b)]) = \bigvee_{a \ll a'} F^*(a' \ll f^*(b)).$$

but if $a \ll a'$ then $c = F^*(a' \ll f^*(b))$ is such that

$\exists a' \gg a$ s.t. $\forall a'' \ll a' \quad c \leq F([a'' \ll f^*(b)])$,

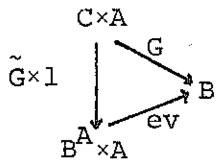
whence $c \leq \bar{\bar{F}}^*(a \ll f^*(b))$.

This shows $F^* \leq \bar{\bar{F}}^*$ so we have $F = \bar{\bar{F}}$.

It remains to check naturality of the isomorphism

$$\underset{\sim}{\text{Loc}}\ (C \times A, B) \cong \underset{\sim}{\text{Loc}}\ (C, B^A).$$

It suffices to show that for any $G: C \times A \to B$,



commutes,

where $ev : B^A \times A \to B$, the evaluation map, is the exponential transpose of the identity so that

$$ev^*(b) = \bigvee_a \{[a \ll f^*(b)] \wedge a\}.$$

Then $(\tilde{G}^* \times 1^*)ev^*(b) = \bigvee_a \{\bigvee\{c \mid \exists a' \gg a.\ c \wedge a' \le G^*(b)\} \wedge a\},$

$$= \bigvee\{c \wedge a \mid \exists a' \gg a.\ c \wedge a' \le G^*(b)\}.$$

Thus clearly $(\tilde{G}^* \times 1^*)ev^*(b) \le G^*(b)$.

Conversely, if $c \wedge a \le G^*(b)$, take any $a^* \ll a$: we have $c \wedge a^* \le (\tilde{G}^* \times 1^*)ev^*(b)$; but $c \wedge a = \bigvee\{c \wedge a^* \mid a^* \ll a\}$, so we have $c \wedge a \le (G^* \times 1^*)ev(b)$. This shows $G^*(b) \le \tilde{G}^* \times 1^* ev^*(b)$, so indeed

$$ev_o(\tilde{G} \times 1) = G.$$

This completes the proof of Proposition 3.

Propositions 2 and 3 immediately give our main result which we restate.

<u>Theorem 1.</u> The following are equivalent for a locale A;

(i) A is locally compact;

(ii) A has function spaces in Loc;

(iii) The function space S^A exists in Loc.

Let us compare our result with those obtained by earlier authors. Their results were that in the category of T_o-spaces (and in certain reflective subcategories) a space X has function spaces if and only if it is locally compact. First let us derive the constructive version of the characterization for the category of sober spaces or equivalently the category $\underset{\sim}{\text{Sp}}$ of spatial locales (locales with enough points).

Corollary 4. The following are equivalent for a spatial locale A;

(i) A is locally compact;

(ii) A has function spaces in Sp;

(iii) S^A exists in Sp.

Proof: - (ii) implies (iii) is trivial.

For (iii) implies (i), note that S is spatial and the whole argument for Proposition 2 takes place in Sp if A is in Sp. (Since S is locally compact, $S \times A$ is spatial by the general and constructive result (cf. Johnstone [6])

"C,A spatial and A locally compact implies $C \times A$ (in Loc) spatial, and so equal to the product in Sp".

For (i) implies (ii), use Theorem 1 to deduce that A has function spaces in Loc. Now let B, C be in Sp. Then we have natural isomorphisms indicated by

$C \times A \to B$ in Sp	as $C \times A$ is the same in Loc as in Sp as A locally compact
$C \times A \to B$ in Loc	
$C \to B^A$ in Loc	(by Theorem 1)
$C \to \text{pts}\ (B^A)$ in Sp	where pts is the right adjoint to the inclusion of sober spaces in locales

This identifies pts (B^A) as the function space in Sp.

We leave it to the reader to deduce in a similar fashion, the characterization for (T_0) topological spaces.

Corollary 5. The following are equivalent for a (T_0) space X;

(i) $\Omega(X)$ is locally compact ($\Omega(X)$, the locale of opens of X);

(ii) X has function spaces in Top (T_0- spaces);

(iii) S^X exists in Top (T_0-spaces).

Note that classically,

1 if a locale is locally compact, then it is spatial, and

2 if a locale is locally compact, then its space of points is locally

compact in the sense natural for topological spaces that every point has a neighbourhood base consisting of compact sets.

So by 1, our main theorem says that the function spaces in **Loc** and **Sp** coincide, while by 2, Corollaries 4 and 5 identify function spaces in the appropriate category as the locally compact spaces. Of course 1 is wildly non-constructive, and the formulations of this paper are needed for constructive results: for example, constructively, it is not the case that A locally compact implies pts (A) has function spaces in **Sp**. Less obviously, the usual proof of 2 is non-constructive, and I do not know if "X is locally compact" can be constructively substituted for "$\Omega(X)$ locally compact" in Corollary 5.

Finally in this paper we discuss some examples of function spaces. We can usually simplify the construction in the proof of Proposition 3. In particular we often make do with an axiomatization in terms of bases for A and B. First we consider the function space S^A which played a major role in Theorem 1. Curiously, though Proposition 2 was proved solely by reference to the points of S^A, we did not need to know that S^A was spatial. But it is.

Proposition 6: If A is locally compact, then S^A (which classifies the open sets in / elements of the locale A) is spatial (has enough points).

Proof: - By the construction in Proposition 3, we see that S^A is given by the following theory on propositions $a \ll 0$ (for a way below the generic open) [$a \ll 0$ is the same as $a \ll f^*(m)$, where m is the distinguished middle element of the lattice S]:

$$a \ll 0 \vdash a' \ll 0 \qquad (a' \leq a);$$

$$\vdash \bot \ll 0; \; a \ll 0,\; a' \ll 0 \vdash a \vee a' \ll 0,$$

$$a \ll 0 \vdash \bigvee_{a' \gg a} a' \ll 0.$$

From this we get a $\wedge$-semi-lattice $\cong A^{op}$ with coverages (in the sense of Johnstone [6]).

$$\{a' \ll O | a' \gg a\} \text{ covers } a \ll O.$$

Now a point a' is in $a \ll O$ if and only if $a \ll a'$. Let $K = \{(a_\alpha \ll O)\}$ be downward closed (or an ideal) in the $\wedge$-semi-lattice such that for all points a' in $(a \ll O)$, a' is in some $(a_\alpha \ll O)$ in K; then

$$\{a' \ll O \; a' \gg a\} \leq K,$$

and K covers $(a \ll O)$. Thus if K covers all points in $(a \ll O)$, it covers $(a \ll O)$ in the sense of the theory. Thus S^A has enough points.

Next we consider some properties of discrete spaces and their function spaces. Obviously any discrete space is locally compact. For each natural number n let $\underline{n}$ denote the discrete space or locale with just n points. Let $\mathbb{N}$ be the discrete space or locale of all natural numbers.

Our first result is very simple.

<u>Proposition 7:</u> For any natural number n and locale B,

$$B^{\underline{n}} \cong B \; \ldots \times B \text{ (n times)}.$$

Proof: - By adjointness, it is sufficient to show that

$$C \times \underline{n} \cong C + \ldots + C \text{ (n times)}.$$

We indicate the argument for $n = 2$.

The locale $\underline{2}$ is P(2) and has a basis (& classically is) consisting of $T = \{0,1\}$, $\{0\}$, $\{1\}$, $\perp = \emptyset$. Then a typical element of $C \times \underline{2}$ is the ideal

$$[\{(c_0 \; c_1) \wedge T, \; c_0 \wedge \{0\}, \; c_1 \wedge \{1\}, \; T \wedge \}]$$

Which is completely determined by the pair c_0, c_1.

This sets up the isomorphism

$$C \times \underline{2} \cong C + C,$$

as $C + C$ is simply the <u>lattice</u> product of C with itself.

Our next observation concerns the familiar Cantor space.

<u>Proposition 8:</u> The function space $\underline{2}^{\mathbb{N}}$ is compact and locally compact. (In fact, it is compact regular which is stronger).

Proof: - An axiomatization, equivalent to that obtained from Proposition 3, is based on propositions "u" where u is finite binary sequence (u is the initial segment of a map from $\mathbb{N}$ to $\underline{2}$):

$$\vdash \langle\rangle, \quad u \vdash v \quad \text{(if u extends v)},$$

$$u \vdash u*0 \vee u*1 \quad \text{(where * denotes concatenation)},$$

$$u, v \vdash \bot \quad (\text{if } u_i \neq v_i \text{ some i}).$$

It is easy to see that an extension-closed collection $\{v_\alpha\}$ covers u if and only if for some finite k all length k extensions of u are in $\{v_\alpha\}$. But this shows that $[u] \ll [u]$ in $\underline{2}^{\mathbb{N}}$ which gives the result. (Special case $u = \langle\rangle$ gives compactness).

The statement that $\underline{2}^{\mathbb{N}}$ is spatial is equivalent to the intuitionists' Fan Theorem (König's Lemma) - see Fourman & Hyland [3]. The function space $\mathbb{N}^{\mathbb{N}}$ has a similar axiomatization to that for $\underline{2}^{\mathbb{N}}$ above. It is not locally compact and the statement that it is spatial is equivalent to the intuitionists' Monotonic Bar Induction. We close with a pair of results which account for the well-known implication

"Monotonic Bar Induction implies Fan Theorem".

<u>Proposition 9:</u> Let A be locally compact and have at least one point. Then for any B,

$$B^A \text{ spatial implies B spatial.}$$

Proof: - There are maps $B \to B^A$ (constant map - i.e. exponential transpose of the projection $B \times A \to B$) and $B^A \to B$ (evaluate at a point of A), making B into a retract of B^A. The result is immediate.

<u>Proposition 10:</u>

$$\mathbb{N}^{\mathbb{N}} \cong (\underline{2}^{\mathbb{N}})^{(\underline{2}^{\mathbb{N}})}$$

Proof: - By use of adjointness, it is clearly enough to show that

$$\underline{2}^{(\underline{2}^{\mathbb{N}})} \cong \mathbb{N}.$$

This falls into three parts: showing that $\underline{2}^{(\underline{2}^{\mathbb{N}})}$ has enough points, showing that the points are discrete and showing that the points are enumerable. By the construction of Proposition 3 and our description of $\underline{2}^{\mathbb{N}}$ in Proposition 8, $\underline{2}^{(\underline{2}^{\mathbb{N}})}$ can be given by the following theory on propositions $u \in P_0$, $u \in P_1$ (implicitly $u \ll f^*(\{0\})$ and $u \ll f^*(\{1\})$) where u runs over binary sequences:

$$u \in P_i \vdash v \in P_i \quad (v \text{ extends } u \text{ i.e. } [v] \leq [u] \text{ in } 2^{\mathbb{N}})$$

$$u{*}0 \in P_i,\ u{*}1 \in P_i \vdash u \in P_i$$

$$u \in P_0,\ u \in P_1 \vdash \bot$$

$$\vdash \bigvee_{\substack{\text{finite sets } F_0, F_1 \\ \text{of binary sequences} \\ \text{with } F_0 \cup F_1 \text{ covers} \\ \langle\rangle \text{ in } 2^{\mathbb{N}}}} \left(\bigwedge_{v \in F_0} v \in P_0\right) \wedge \left(\bigwedge_{v \in F_1} v \in P_1\right).$$

Thus we get a $\wedge$-semi-lattice consisting (more or less) of pairs (F_0, F_1) of finite sets of binary sequences such that (i) if $u \in F_0$ and $v \in F_1$ then u and v are incompatible ($\exists i. u_i \neq v_i$), and (ii) if $u{*}0$ and $u{*}1$ are in F_i then u is in F_i. The order is given by

$(F_0, F_1) \leq (G_0, G_1)$ if and only if everything in G_i extends something in F_i.

(or course this is really only a pre-order).
Then $\underline{2}^{(\underline{2}^{N})}$ is obtained (cf. Johnstone [6]) by using the coverages:

(*) $\{(F_0, F_1) \mid (F_0, F_1) \leq (G_0, G_1)$ and $F_0 \cup F_1$ covers $\langle\rangle$ in $2^{N}\}$ covers (G_0, G_1).

Now from the axioms a point of the locale constructed is given by a pair (P_0, P_1) of disjoint, extension-closed sets of binary sequences such that $u{*}0 \in P_i$ and $u{*}1 \in P_i$ implies $\in P_i$ and whose union $P_0 \cup P_1$ covers $\langle\rangle$ in $\underline{2}^{\mathbb{N}}$. By the compactness of $\underline{2}^{\mathbb{N}}$, for any point (P_0, P_1) there exists finite $F_0 \leq P_0$, $F_1 \leq P_1$ with $F_0 \cup F_1$ covering $\langle\rangle$ in $\underline{2}^{\mathbb{N}}$ and (F_0, F_1) in our $\wedge$-semi-lattice. But then P_i is the extension-closure of F_i so that $[(F_0, F_1)]$ is the lattice element consisting just of the point (P_0, P_1) and then (*) above says that any $[(G_0, G_1)]$ is covered by the points in it. Thus $2^{(2^{\mathbb{N}})}$ has enough points and is discrete. To enumerate the points of $2^{(2^{\mathbb{N}})}$, note that given (F_o, F_1) as just above we can uniquely compute the extension-least elements of (F_o, F_1) obtaining $(\bar{F}_o, \bar{F}_1)$ independently of the choice of (F_o, F_1). But it is simple to enumerate the $(\bar{F}_o, \bar{F}_1)$'s. (If the reader is not worried about constructivity he could prove this result more simply by taking $2^{\mathbb{N}}$ to be a compact space).

With the above curious connection between Cantor and Baire space we end this paper.

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