A SURVEY OF SOME USEFUL PARTIAL ORDER RELATIONS ON TERMS OF THE LAMBDA CALCULUS

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§0 **Introduction.** The equality in models for the λ-calculus gives rise to equality relations on terms of the λ-calculus, where by an equality relation we mean an equivalence relation preserved under context substitution. We focus attention on equality relations as often these are given syntactically and so prior to any model. Of course from a given equality relation one can always define a model (the model of terms factored out by the relation) which gives rise to it.

The most interesting purely semantic models for the λ-calculus, the continuous lattices of Dana Scott, are equipped with a partial order. This gives rise to what we call a partial order relation (p.o.r.) on terms of the λ-calculus, that is a pre-partial-order (i.e. transitive relation) preserved under context substitution. To any p.o.r. there corresponds the equality relation obtained by setting two terms equal iff each is less than or equal to the other. So the p.o.r. induces an ordinary partial order on the equivalence classes.

We take the view (arising out of the theses of Barendregt and Wadsworth) that terms with no head normal form (i.e. terms whose closure is unsolvable) have no computational value and so may sensibly be set equal. Thus we say that a p.o.r. is sensible iff it extends that p.o.r. obtained from β-equality by setting all terms with no head normal form equal, and less than any term; this latter p.o.r. is thus the minimal sensible p.o.r.

Our aim in this paper is to map out some of the main landmarks in the territory of sensible p.o.r.'s. To this end we make use of the $\lambda\Omega$-calculus as described in Wadsworth (1971). This arises by adding a constant Ω to the pure λ-calculus. Ω will be a minimal element in all our p.o.r.'s; that is to say Ω canonically represents the terms without head normal form. Thus the addition of Ω adds nothing to the expressive power of the λ-calculus as Ω can always be replaced by $(\lambda x.xx)(\lambda x.xx)$.

An equality relation is consistent iff it does not set all terms equal; a p.o.r. is consistent iff its induced equality relation is so. Barendregt (1971) shows that the minimal sensible p.o.r. is consistent. Our paper contains many consistent sensible p.o.r.'s, and thereby many alternative proofs of Barendregt's result; the interest of his analysis is that it shows directly the computational irrelevance of terms with no head normal form.

§1 Head normal forms. We define which terms of the λ-calculus are head normal forms (h.n.f.'s) as follows:

(a) all variables are h.n.f.'s;

(b) if $X_1,\ldots,X_k$ are terms, and x is a variable, then $xX_1\ldots X_k$ is an h.n.f.;

(c) if P is an h.n.f. then so is $\lambda x.P$.

A term M has h.n.f. iff there is an h.n.f. N with $M =_\beta N$. Otherwise M has no h.n.f.. An h.n.f. has the form,

$$\lambda x_1\ldots x_i.zX_1\ldots X_j,$$

and z is the <u>head</u> <u>variable</u>. A non-h.n.f. has the form,

$$\lambda x_1 \ldots x_i . (\lambda y.P) X_1 \ldots X_j ;$$

the <u>head</u> <u>redex</u> is $(\lambda y.P)X_1$, and the (possibly infinite) reduction of a term, obtained by always reducing the head redex if any, is the <u>head</u> <u>reduction</u> of that term. By the Standardization Theorem, a term has h.n.f. iff its head reduction terminates; hence the set of terms with no h.n.f. has strong closure properties (Wadsworth (1971)). A term has h.n.f. iff its closure is solvable in the sense of Barendregt (1971).

Let $\lambda x_1 \ldots x_m . z X_1 \ldots X_i$ and $\lambda y_1 \ldots y_n . w Y_1 \ldots Y_j$ be two h.n.f.'s. By α-conversion we may take x_r to be y_r for $r \leq \min(m,n)$, and so we assume the two terms are,

(1) $\lambda x_1 \ldots x_m . z X_1 \ldots X_i$ and $\lambda x_1 \ldots x_n . w Y_1 \ldots Y_j$.

The two h.n.f.'s are

(i) <u>similar</u> iff (when arranged as in (1)) $m = n$, $i = j$ and z is w, and (ii) <u>inseparable</u> iff (when arranged as in (1)) $(m-i) = (n-j)$ and z is w.

<u>Proposition 1.1</u>. Let M be any term and let M β-reduce (respectively $\beta\eta$-reduce) to M_1 and to M_2 both h.n.f.'s. Then M_1 and M_2 are similar (respectively inseparable).

Proof: Immediate by the Church-Rosser Theorem.

The rest of this section presents a technical analysis of the theorem of Böhm (1968), by way of some lemmas which will be important later. Proofs are omitted as the methods are fairly well known, and details appear in Hyland (1975).

Lemma 1.2. (a) Suppose M, N have h.n.f.'s which are not inseparable; then there is a context C[ ] such that $C[M] =_\beta x$,

$$C[N] =_\beta y,$$

where x and y are distinct variables.

(b) Suppose M has no h.n.f. while N has an h.n.f.; then there is a context C[ ] such that C[M] has no h.n.f.,

$$C[N] =_\beta y, \text{ for some variable } y.$$

Proof: See Hyland (1975).

Now we define for $k \geq 1$, (a) the terms M and N have the same k-normal form (henceforth written $M =_k N$), and (b) the set of k-pairs of the pair (M,N). The definition is by induction on k as follows:

Case k = 1. $M =_1 N$ iff either both M and N have no h.n.f. or both M and N have h.n.f.'s, and the h.n.f.'s to which M and N reduce are inseparable. (Proposition 1.1 shows that this last requirement is unambiguous). In the first case, there are no 1-pairs of (M,N). It remains to consider the second case. We may assume that M and N reduce to the h.n.f.'s of (1) above (to fix things just consider β-reduction) where (m-i) = (n-j) and z is w. Suppose without loss of generality that $n \leq m$, and consider,

$$Mx_1 \ldots x_m =_\beta zX_1 \ldots X_i,$$

$$Nx_1 \ldots x_m =_\beta wY_1 \ldots Y_j x_{n+1} \ldots x_m, \text{ which is } zY_1 \ldots Y_i, \text{ say.}$$

Then the 1-pairs of (M,N) are the pairs (X_r, Y_r) for $1 \leq r \leq i$.

Induction step. $M =_{k+1} N$ iff $M =_1 N$ and for any 1-pairs (X,Y) of (M,N) we have $X =_k Y$. The (k+1)-pairs of (M,N) are the k-pairs of the 1-pairs of (M,N).

Lemma 1.3. Given terms M and N, with (X,Y) k-pairs of (M,N), there is a context C[ ] and substitutions (R/x,....) such that,

$$C[M] =_\beta X(R/x,\dots) \text{ a substitution instance of X, and}$$

$$C[N] =_\beta Y(R/x,\dots) \text{ the same substitution instance of Y.}$$

The terms R substituted are of the form $\lambda x_1 \dots x_h . x_h x_1 \dots x_{h-1}$, for h sufficiently large.

Proof: See Hyland (1975).

Remark. The substitutions of (1.3) have the following trivial effect on the similarity type (respectively inseparability type) of X and Y. X and Y β-reduce (respectively $\beta\eta$-reduce) to similar (respectively inseparable) h.n.f.'s iff X(R/x,....) and Y(R/x,....) do so.

Corollary 1.4. (Böhm). If terms M and N have distinct $\beta\eta$-normal forms then there is a context C[ ] such that $C[M] =_\beta x$,

$$C[N] =_\beta y,$$

where x and y are distinct variables.

Proof: By (1.2), (1.3) and the observation that if M and N have distinct $\beta\eta$-normal forms, then there is some k-pair (X,Y) of (M,N) such that X and Y have h.n.f.'s which are not inseparable.

§2 Ω-approximants. We recall that we have introduced a constant Ω into our language to represent the terms with no h.n.f.. The closure properties of the set of terms with no h.n.f. make it sensible to introduce Ω-reductions as follows. Terms of the forms ΩM and $\lambda x.\Omega$ are Ω-redexes and both Ω-reduce to Ω. A term M is in $\beta\Omega$-normal form iff it contains no β-redexes and no Ω-redexes; it is in $\beta\eta\Omega$-normal form iff it also contains no η-redexes.

Attempts to present arbitrary λ-terms as limits of normal forms which approximate them, give rise to the notion of an Ω-approximant. We shall need two such notions (depending on whether or not we are taking η-reduction into account). For a given term M, we define its sets of approximants $\omega(M)$ and $\omega\eta(M)$ as follows:

$\omega(M) = \{L \mid L$ is a $\beta\Omega$-normal form obtained from some N, where $N =_{\beta} M$, by replacing subterms of N by $\Omega\}$;

$\omega\eta(M) = \{L \mid L$ is a $\beta\eta\Omega$-normal form obtained from some N, where $N =_{\beta\eta} M$, by replacing subterms of N by $\Omega\}$.

<u>Proposition 2.1</u>. (a) $C[M]$ β-reduces (respectively $\beta\eta$-reduces) to the β-normal (respectively $\beta\eta$-normal) form N iff for some $L \in \omega(M)$ (respectively $L \in \omega\eta(M)$) $C[L]$ does so.

(b) $C[M]$ β-reduces (respectively $\beta\eta$-reduces) to a h.n.f. of a given similarity type (inseparability type) iff for some $L \in \omega(M)$ (respectively $L \in \omega\eta(M)$) $C[L]$ does so.

Proof: Wadsworth (1971) proves one of the cases in detail by a method which easily extends to the others.

<u>Lemma 2.2</u>. If the $\beta\Omega$-normal form L is not in $\omega(N)$, then for some (X,Y) k-pairs of (L,N) we have,

(i) X β-reduces to a h.n.f. X',

(ii) if Y has h.n.f. then Y β-reduces to a h.n.f. which is not similar to X'.

Proof: The lemma is easily proved for all N by induction on the structure of L.

<u>Theorem 2.3</u>. $\omega(M) \subset \omega(N)$ iff whenever $C[M]$ β-reduces to the h.n.f. M' then $C[N]$ β-reduces to a similar h.n.f.

Proof: That L.H.S. implies R.H.S. is immediate by a couple of applications of (2.1).

Suppose not L.H.S.. Then there is $L \varepsilon \omega(M)$, L not in $\omega(N)$. Now by (2.2) take k-pairs (X,Y) of (L,N) satisfying (i) and (ii) above. By (1.3) there is a context $C[\ ]$such that $C[L]$ and $C[N]$ β-reduce to substitution instances of X and Y. By the remark following (1.3) we can conclude that $C[L]$ has h.n.f., but $C[L]$ and $C[N]$ do not β-reduce to similar h.n.f.'s. Hence by applying (2.1) we have not R.H.S.. This completes the proof of the theorem.

<u>Corollary 2.4</u>. (Independant result of Levy and of Welch) $\omega(M) \subset \omega(N)$ does define a (consistent) p.o.r. on λ-terms.

Proof: The relation on the R.H.S. of (2.3) is clearly preserved under context substitution.

<u>Remark</u>. The relation of (2.3) properly extends the minimal sensible p.o.r. as (for example) it sets all the members of the usual sequence Y_0, Y_1,... of fixed point operators, equal.

<u>Lemma 2.5</u>. If the $\beta\eta\Omega$-normal form L is not in $\omega\eta(N)$, then for some (X,Y) k-pairs of (L,N), we have,

(i) X $\beta\eta$-reduces to the variable x,

(ii) Y does not $\beta\eta$-reduce to x.

Proof: The lemma is easily proved for all N by induction on the structure of L.

<u>Theorem 2.6</u>. $\omega\eta(M) \subset \omega\eta(N)$ iff whenever $C[M]$ $\beta\eta$-reduces to the $\beta\eta$-normal form M' then $C[N]$ $\beta\eta$-reduces to M'.

Proof: That L.H.S. implies R.H.S. is immediate by a couple of applications of (2.1).

Suppose not L.H.S.. Then there is $L \varepsilon \omega\eta(M)$, L not in $\omega\eta(N)$. Things are not so simple now as they were in the proof of (2.3), so we dispose of the easy case first. Suppose there exist k-pairs (X,Y) of (L,N) such that X has h.n.f. but if Y has h.n.f. then it is not inseparable from that of X. Then not R.H.S. follows easily from (1.2), (1.3) and the remark following (1.3).

So henceforth assume that for all k-pairs (X,Y) of (L,N), if X has h.n.f. then Y has h.n.f. inseparable from that of X.

Now by (2.5) take k-pairs (X,Y) of (L,N) satisfying (i) and (ii) of (2.5). Then $X =_{\beta\eta} x$,

and $Y =_{\beta\eta} \lambda y_1 \ldots y_k . xY_1 \ldots Y_k$,

and it follows from our assumption anove that Y has no normal form. Consider the substitution instances X' and Y' of X and Y determined by (1.3). It suffices to show that Y' has no normal form. (This does not follow from the general nature of the substitutions, but from the special form of Y). Note that even if in the substitution instances X' and Y', some R has been substituted for the variable x, there must be (k+1)-pairs (X_i, Y_i) say satisfying (i) and (ii) of (2.5), where X_i is a variable y_i say and nothing is substituted for y_i by the appropriate context determined by (1.3). So we can assume that nothing is substituted for x in X and Y. But then for all r-pairs (A,B) of (X,Y) nothing has been substituted for the head variable of B. By considering normal reductions, since Y has no normal form, neither has Y'. The proof is now completed as for (2.3).

§3 Scott's models. In this section we outline the main results of Hyland (1975). We are concerned with the values of λ-terms in continuous lattice models for the λ-calculus. D denotes some (arbitrary) continuous lattice isomorphic to its function space, which is constructed from a continuous lattice D_0 and the initial maps,

$$\phi_0: D_0 \longrightarrow D_1, \text{ defined by } \phi_0(d_0) = \lambda x.d_0, \text{ and}$$

$$\psi_0: D_1 \longrightarrow D_0, \text{ defined by } \psi_0(d_1) = d_1(\perp).$$

$P\omega$ denotes the Graph Model described in Scott's "Data Types as Lattices". (The Scott Model D is fully described in Scott's "Continuous Lattices"). The value of a term M in these models will be denoted by $[\![M]\!]_D$ and $[\![M]\!]_{P\omega}$ respectively. $\sqsubseteq$ denotes the order relation and $\bigsqcup$ the sup operation in either lattice.

Proofs of all the results of this section appear in Hyland (1975), and we do not include them here. Furthermore, Wadsworth presented his considerable improvement on our original proof of (3.1)(a) and his own proof of (3.2)(a) at a conference in Orleans, 1972. So the basic ideas should be familiar.

Theorem 3.1. (a) $[\![M]\!]_D = \bigsqcup\{ [\![L]\!]_D | L \in \omega(M)\} = \bigsqcup\{ [\![L]\!]_D | L \in \omega\eta(M)\}$.

(b) $[\![M]\!]_{P\omega} = \bigsqcup\{ [\![L]\!]_{P\omega} | L \in \omega(M)\}$.

Next we make some definitions which extend those of §1. We introduce relations $<^s_k$ and $<^g_k$ for $k \geq 1$, by induction on k. The superscripts s and g are to indicate that the relations are important for the Scott and Graph Models respectively.

Definition. $M <^s_1 N$ iff whenever M has h.n.f. then $M =_1 N$. Then by induction, $M <^s_{k+1} N$ iff $M <^s_1 N$ and for any 1-pairs (X,Y) of (M,N) we have $X <^s_k Y$.

The h.n.f. $\lambda x_1 \ldots x_m . zX_1 \ldots X_i$ is <u>more functional than</u> the h.n.f. $\lambda y_1 \ldots y_n . wY_1 \ldots Y_j$ iff the two h.n.f.'s are inseparable and $m \geq n$.

<u>Definition</u>. $M <_1^g N$ iff whenever M has h.n.f., then N β-reduces to a h.n.f. which is more functional than that to which M β-reduces.

Then by induction, $M <_{k+1}^g N$ iff $M <_1^g N$ and for any 1-pairs (X,Y) of (M,N) we have $X <_k^s Y$.

<u>Theorem 3.2</u>. (a) The following are equivalent:

(i) $[\![M]\!]_D \sqsubseteq [\![N]\!]_D$;

(ii) for all $k \geq 1$, $M <_k^s N$;

(iii) whenever $C[M]$ has h.n.f. then so has $C[N]$.

(b) The following are equivalent:

(i) $[\![M]\!]_{P\omega} \sqsubseteq [\![N]\!]_{P\omega}$;

(ii) for all $k \geq 1$, $M <_k^g N$;

(iii) whenever $C[M]$ has h.n.f. then $C[N]$ β-reduces to a h.n.f. more functional than that to which $C[M]$ β-reduces.

The p.o.r. induced by the Scott Model D has a beautiful uniqueness property.

<u>Theorem 3.3</u>. The p.o.r. characterised in (3.2)(a) is the unique maximal consistent sensible p.o.r.

§4 <u>Concluding remarks</u>. We have in (2.3), (2.6) and (3.2) characterised four sensible p.o.r.'s. It is easy to see that they are all distinct (though note that the induced equality of (2.3) and (3.2)(b) is the same). The most significant feature of the results to my mind is this. Each of the four p.o.r.'s has a characterisation in terms of contexts, of a natural form: if a context acting on one term does such and such, then so does the context acting on the other. In other words, each p.o.r. is characterised in terms of its computational significance. This in my view should be a feature of any interesting sensible p.o.r. so at least we know what to look for should we search for more sensible p.o.r.'s.

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