J.E. Fenstad, R.O. Gandy, G.E. Sacks (Eds.)
GENERALIZED RECURSION THEORY II

The intrinsic recursion theory on the countable or continuous functionals

J.M.E. Hyland

§0. Introduction. The collection $\mathcal{C} = \{C_\sigma \mid \sigma \text{ is a type symbol}\}$ of countable (or continuous) functionals was first considered in the pioneering papers of Kleene [9] and Kreisel [10]. Already in these papers, an intrinsic notion of recursive countable functional was defined. Since that time smoother approaches to the subject have been made by Ershov [2], [3], Feferman [4] and Hyland [7], [8] and a natural "intrinsic recursion theory" has begun to be studied. The purpose of this paper is to describe the present state of knowledge and the major unsolved problems in this area.

What I call the intrinsic recursion theory on the countable functionals is an attractive area to study for the following reasons.

(i) It arises naturally out of the various definitions (Ershov [2], [3], Hyland [7], [8].) of $\mathcal{C}$; and $\mathcal{C}$ is itself the natural collection of continuous objects of higher type defined over the natural numbers (Hyland [8]).

(ii) It has pleasing applications to questions of constructivity (Kreisel [10].

(iii) As is sensible for recursion on objects given by a countable amount of information, every countable functional is recursive in a function (from natural numbers to natural numbers).

(iv) Elementary arguments about weaker notions of recursion turned out to be arguments about the intrinsic recursion theory (see §2).

(v) It provides an instructive challenge to generalized recursion theory as that subject has been developed over the last ten years (see especially §4); note that this challenge is already implicit in Kreisel [10].

Of course $\mathcal{C}$ is also a suitable domain for Kleene's generalized recursion theory via the schemes S1-S9, and a survey of that area, computability on $\mathcal{C}$, is in Gandy and Hyland [5]. Indeed, despite (iv) above, it has proved easier to obtain results about computability. In particular, since the writing of [5], work on 1-sections (initiated by Wainer) and on 2-envelopes (initiated by Bergstra) has been carried to a very satisfactory conclusion by Norman [11]. This work directly contradicts what seemed plausible at the time I wrote my thesis [7]. I discuss the relation between it and what we know about the intrinsic recursion theory in §3.

Work on computability is indirectly related to the main outstanding question concerning the countable functionals: whether the intrinsic recursion theory can be characterized by one of the approaches of generalized recursion theory (most plausibly by inductive definitions see Feferman [4]). This question enables one to look critically at generalized recursion theory. A negative answer, indicating the limitations of the "generalizations" of ordinary recursion theory, would have considerable philosophical interest. However as Feferman [4] observed, it is hard to imagine how to obtain a proof of this kind of impossibility. In §4, I give a vague outline of what appears to be the only natural possibility of an inductive definability approach to the intrinsic recursion theory on the countable functionals. My view is that if this fails, there can be no sensible positive answer to the fundamental question.

§1. Basic definitions. Various definitions of partial recursive functional on $\mathcal{C}$ have been proposed. That given here is from Hyland [7].

For countable functionals $G_1,\ldots,G_n$ and F, $\{e\}_c(G_1,\ldots,G_n) = F$ iff for all associates $\alpha_1,\ldots,\alpha_n$ for $G_1,\ldots,G_n$ $\lambda x.\{e\}(\alpha_1,\ldots,\alpha_n,x)$ is an associate for F.

The partial functionals $\{e\}_c$, are the <u>partial recursive countable functionals</u>. That this definition is equivalent to others in the literature (Ershov [3], Feferman [4]) follows readily from work in Hyland [8]. (In this definition I have omitted codings for the types of the arguments and prospective value of the functional, but never mind). A clear discussion of the relation of this notion to the notion of <u>partial computable functional</u> (i.e. partial S1-S9 recursive) can be found in Feferman [4].

The partial recursive countable functionals give rise to a notion of <u>countably recursive in</u>: F is countably recursive in G iff there is a partial recursive countable functional mapping G to F (cf. Gandy and Hyland [5] §3.8). This gives rise to the <u>countable degrees</u> as follows. The degree of a countable functional F,

$$\deg(F) = \{G \mid G \text{ is countably recursive in } F\}.$$

These form the countable degrees $\mathcal{D}_1$, ordered by inclusion. A degree is of type n iff it is of form deg(F) with F of type n; $\mathcal{D}_n$ denotes the collection of degrees of type n. Clearly,

$$\mathcal{D} = \bigcup \{\mathcal{D}_n \mid n \geq 1\}.$$

A functional of type n is said to be <u>irreducible</u> (with respect to countable recursion) iff it does not have the same degree as a functional of type less than n.

We define the <u>countable k-section</u> of F to be

$$\text{ct-}k\text{-sc}(F) = \{G \mid G \text{ of type } k \text{ and countably recursive in } F\}.$$

Partial recursive countable functionals with numerical values define a notion of semi-recursive (s.r) or recursively enumerable set. The subsets of C_k <u>s.r. in</u> F are those of the form

$$\{G \mid G \text{ of type } k \text{ and } \{e\}_c\{F,G\} = 0\},$$

for some index e. The <u>countable (k+1)-envelope</u> of F is defined to be

$$\text{ct-}(k+1)\text{-env}(F) = \{A \subseteq C_k \mid A \text{ s.r. in } F\}.$$

§2. The countable degrees. In this section we survey what little is known about the countable degrees. Many open problems remain and we discuss a few of these.

The proofs of some simple results can be easily derived from the literature.

Theorem 2.1. There is a type 1 degree minimal amongst all the countable degrees.

Proof:- By an easy adaptation of Spector [12].

Corollary 2.2. (answering a question in Hinman [6]) $\mathcal{D}$ is not dense in $\mathcal{D}_1$; whence the type 2 Kleene degrees of countable functionals are not dense in $\mathcal{D}_1$ (ordinary degrees).

(2.1) has some content beyond Spector's result in view of the next result.

Theorem 2.3. There are irreducible type 2 objects.

Proof:- It is this result which is really proved in Hinman [6].

Remark (2.3) can be strengthened in various ways:

1) (Folklore?) One can obtain an irreducible F whose Kleene one section is the recursive functions.

2) With difficulty one can obtain an irreducible F with ct-1-sc(F) = the recursive functions and (Harrington) with F not the continuous extension of an effective operation.

3) Since Hinman's proof involves a simple spoiling argument, one can modify elementary degree theory constructions to get irreducible type 2 objects in place of type 1 objects.

A k-section which is not the k-section of any type k object is called topless. Bergstra [1] showed the existence of topless sections for Kleene computations (S1-S9); a different approach to this result is provided by Norman [11]. Bergstra's method adapts with difficulty to countable recursion; we only get a result at the very first level.

Theorem 2.4. There exists F of type 2 such that ct-1-sc(F) is topless.

A basic problem in the theory of the countable degrees would be to determine the relation between the familiar structure $\mathcal{D}_1$ of the ordinary degrees, and the $\mathcal{D}_n$'s and $\mathcal{D}$ which include $\mathcal{D}_1$. The difficulties are indicated by the following long-standing question (which also shows the limitations of 3) of the above remark).

OPEN PROBLEM Are there minimal degrees which are irreducible of type 2?

Though by (2.3), we know that $\mathcal{D}_2$ properly includes $\mathcal{D}_1$, nothing is known of the corresponding result for higher levels of the type structure.

OPEN PROBLEM For $n \geq 2$, does $\mathcal{D}_{n+1}$ properly include $\mathcal{D}_n$?

Remarks 1) A formulation of this question for Kleene's computations on $\mathcal{C}$ was answered (by Bergstra [1]) in the affirmative. For computability, the relation between the degree structures at different types is complicated by the existence of elements (e.g. the fan functional - see Gandy and Hyland [5]) at type 3, which are not 1-obtainable (i.e. computable from a function).

2) It seems most likely that the answer is "yes". For the C_n's become more complicated as n increases in the sense of the

Proposition If $m > n$, then there is no continuous onto map from C_n to C_m.

This is essentially Cantor's Theorem (together with type-changing manipulations). As observed by Norman [11], the proposition can also be obtained from (3.2).

3) Results on countable sections, or extensions of (2.4) might be contributions to the solution of this problem. But neither possibility seems very easy.

§3. Associates and envelopes. The aim of thissection is to determine what are the countable 2-envelopes of countable functionals. The result will be in marked contrast to what is known about Kleene 2-

envelopes (Norman [11]). It is clear from the definitions that we will need to know something about associates.

Write Ass(F) for the set of associates (the details of the definition will be of no significance) for a countable functional F, and set

$$\mathrm{Ass}_n = \bigcup \{\mathrm{Ass}(F) \mid F \in C_n\}.$$

It seems to be a matter of folklore that for $n \geq 2$, Ass_n is a complete Π^1_{n-1} set of functions, but as I do not know a published proof, I sketch the simple basis for this result here.

<u>Lemma 3.1.</u> The sets of the form

$$\{\vec{x} \mid (\forall \alpha)(\alpha \in \mathrm{Ass}_n \longrightarrow P(\vec{x},\alpha))\},$$

where P is Σ^0_1, $\vec{x}$ denotes a sequence of variables of types 0 and 1 and $n \geq 1$, are closed under universal qualification (effectively in an index for P).

<u>Proof</u>:- This follows immediately from the fact that there exist recursive (indeed elementary) maps from Ass_n <u>onto</u> $\mathbb{N}^{\mathbb{N}} \times \mathrm{Ass}_n$. (Observe that for any σ, C_σ is isomorphic to $C_\sigma \times C_\sigma$. So for $n \geq 1$ C_n is isomorphic to $C_n \times C_n$ which easily maps onto $C_1 \times C_n$. The map so constructed will be onto at the level of associates).

<u>Proposition 3.2.</u> For $n \geq 2$, Ass_n is a complete Π^1_{n-1} set.

Proof:- The case $n = 2$, the basis for an induction, is easy. Suppose result true for n. Then an arbitrary Σ^1_n set is of the form

$$\{\vec{x} \mid (\exists \alpha)\ \Phi(\vec{x},\alpha) \in \mathrm{Ass}_n\},$$

for some recursive functional Φ, i.e. of the form

$$\{\vec{x} \mid (\exists \beta)(\beta \in \mathrm{Ass}_n \ \&\ (\exists \alpha)(\beta = \Phi(\vec{x},\alpha)))\}.$$

Thus an arbitraty Π^1_n set A is of the form

$$\{\vec{x} \mid (\forall \alpha)(\forall \beta)(\beta \in \mathrm{Ass}_n \longrightarrow \beta \neq \Phi(\vec{x},\alpha))\},$$

so using (3.1) of the form

$$\{\vec{x} \mid (\forall \beta)(\beta \in \mathrm{Ass}_n \longrightarrow (\exists y)T|e, \overline{\vec{x}(y)}, \overline{\beta(y)})\}$$

for a suitable T-predicate.

Now define Ψ by

$$\Psi(\vec{x})(u) = \begin{cases} 0 \text{ if not } (\exists v \subseteq u) T(e, \overline{\vec{x}(\mathrm{lh}(v)}, v) \\ 1 \quad \text{otherwise.} \end{cases}$$

Then $\vec{x} \in A$ iff $\Psi(\vec{x}) \in \mathrm{Ass}_{n+1}$. Since Ass_{n+1} is clearly Π^1_n, we have completed the induction step.

There are some immediate corollaries of the above result and proof. Let ${}^n0$ be the everywhere zero functional of type n.

Corollary 3.3. For $n \geq 2$, $\mathrm{Ass}({}^n0)$ is a complete Π^1_{n-1} set.

Proof:- By the above argument, $\vec{x} \in A$ iff $\Psi(\vec{x}) \in \mathrm{Ass}({}^n0)$.

Corollary 3.4. For $n \geq 2$, ct-2-env$({}^n0) = \Pi^1_n$.

Proof:- ct-2-env$({}^n0)$ consists of all sets of the form

$$\{\vec{x} \mid (\forall \alpha)(\alpha \in \mathrm{Ass}({}^n0) \longrightarrow P(\vec{x},\alpha))\}, \text{ where } P \text{ is } \Sigma^0_1.$$

So clearly ct-2-env$({}^n0) \subseteq \Pi^1_n$ and is closed under substitution of recursive functionals. It remains to show that ct-2-env$({}^n0)$ contains a complete Π^1_n set. But this follows by the first part of the argument for (3.2). (The existence of a recursive onto map from $\mathrm{Ass}({}^n0)$ to $\mathbb{N}^{\mathbb{N}} \times \mathrm{Ass}({}^n0)$ is much as in the proof of (3.1) - though it lacks the structural motivation of that result).

The generalizations of (3.3) and (3.4) to arbitrary F (of type $n \geq 2$), involves ugly coding problems; my proofs rely on equivalences from Hyland [8] so I do not give them here, but simply state the results. Suppose F is of type n, $n \geq 2$ and let h_F give the value of F on some recursive dense sequence in C_{n-1}.

Theorem 3.5. (a) Ass(F) is a complete $\Pi^1_{n-1}(h_F)$ set

(b) ct-2-env(F) $= \Pi^1_n(h_F)$.

Remarks 1) (3.5)(a) is proved in full detail from a completely different point of view in Norman [11]; (3.5)(b) could also be obtained using his methods.

2) (3.5)(b) should be contrasted with the result of Norman [11], that in the sense of Kleene (S1-S9) recursion, 2-env(F) = $\Pi^1_{n-2}(h_F)$ (F of type 3 or more).

(3) The most significant feature of Norman [11] is his ability to handle 1-sections (see his Theorem 3). At the moment there is nothing corresponding for countable 1-sections (cf. Remark 3 of §2).

§4. Definitions by recursion on the inductive definition of C_2. The outstanding question concerning the countable functionals is whether one can obtain the intrinsic recursion theory by applying the usual ideas of generalized recursion theory. This problem was raised in embryonic form by Kreisel [10]. It was considered in Hyland [7], but the tentatively negative conclusion reached there was based in part on conjectures which have since been disproved. It is discussed in detail in Feferman [4], where a positive answer to the corresponding question for the partial continuous functionals is indicated. At first sight the problem seems to be one of finding the "right structure" to put on $\mathcal{C}$ (cf. the final section of Feferman [4]). However $\mathcal{C}$ doesn't seem to have any structure in the sense of model theory apart from the usual structure on C_0, the natural numbers, and the type structure (essentially, evaluation and λ-abstraction); and this much gives rise to Kleene's computations (S1-S9) on $\mathcal{C}$. One would appear to search in vain for further natural inductive schemeta, while there must exist suitable unnatural ones since the partial recursive countable functionals can clearly be enumerated. This is the impasse reached by Feferman [4]. In this section, I sketch the lines of what seems to be the only plausible way out.

My suggestion is based on two observations.

1) Once we have C_0, C_1 and C_2, the rest of $\mathcal{C}$ is determined by demanding closure under explicit definition (i.e. avoiding ${}^2E$).

2) (A point made to me by Gandy). The natural numbers are inductively defined and thereby carry a good recursion theory; but C_2 is also inductively defined (Brouwer, Konig) and should carry a good

recursion theory in virtue of this fact.

Since there is no problem with the recursion theory on C_0 and C_1, it would seem that the inductive definition of C_2 is the one element of the structure of $\mathcal{C}$ missing from what is described above. If we can add an appropriate process of definition by recursion over the inductive definition of C_2, we ought to get the natural recursion theory on $\mathcal{C}$ from the point of view of computation schemes or inductive definitions. If this does not coincide with the intrinsic recursion theory, one could conclude that the intrinsic recursion theory on the countable functionals falls outside the scope of the main developments of generalized recursion theory.

For F in C_2 and u a sequence (number) define F_u by

$$F_u(\beta) = F(u*\beta),$$

where * denotes concatenation. Both the fan functional and Gandy's functional Γ of [5], can be defined as functionals Δ for appropriate (primitive recursive) $\mathcal{F}$, in the following simple way:

$$\Delta(\lambda\beta.k,\alpha) = \alpha(k)$$

$$\Delta(F,\alpha) = \mathcal{F}(F, \lambda u \neq < >.\Delta(F_u,\alpha),\alpha).$$

But the fact that such a definition uniquely determines Δ, depends on the fact that for the corresponding $\mathcal{F}$,

$$\alpha(k) = \mathcal{F}(\lambda\beta.k,\ \lambda u \neq < >.\alpha(k),\alpha).$$

In other words we must take account of the fact that

(*) is not decidable (countably recursive) whether of not an element of C_2 is a member of the basis for the inductive definition of C_2 (viz. the constant functionals).

It is not hard to give a formulation of a "computation scheme" SI (with which one could augment S1-S9) which would define functionals as above.

If $\{e\}(\lambda\beta.k,\ \lambda u \neq < >.\alpha(k) = \alpha(k)$ then $\{e'\}(\lambda\beta.k,\alpha) = \alpha(k)$.

(SI)

If for all $n \neq < >$, $\{e'\}(F_u,\alpha)$ is defined, then

$$\{e'\}(F,\alpha) = \{e\}(F,\lambda u \neq \langle\ \rangle.\{e'\}(F_u,\alpha)).$$

(Here e' is the new index which codes up e together with other appropriate information). It seems clear that (SI) will not close the gap indicated in §3 between countable and S1-S9 semi-recursion (2 envelopes). However I have been able to obtain no evidence against the following conjecture.

CONJECTURE. S1-S9 + SI suffice to generate the recursive countable functionals of type 3.

(Of course, SI may not be quite right for the job).

The augmentation of Kleene's schemes by SI is rather crude. It would be more satisfactory (both for general reasons and particularly since we are trying to use the inductive definition of C_2) to use the approach of inductive schemata as described in Feferman [4]. However from this point of view it is not at all obvious how to take account of (*) above. One seems to get involved either with non-monotone schemata, or with "partial" schemata, and I have not been able to devise a convincing formulation with either. There seems to be a genuine conceptual problem here:

What inductive schemata encapsulate the idea of definition by recursion on the inductive definition of C_2.

I hope that I have said enough in this section to show that the problem whether or not there is a natural inductive definability approach to the intrinsic recursion theory on the countable functionals is an accessible one. While I am optimistic about the specific conjecture above, I feel that the overall answer is likely to be "no".

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