CONTINUITY IN SPATIAL TOPOSES

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0. INTRODUCTION

This paper is concerned with spaces of continuous functions in categories of sheaves over topological spaces. As such it is a contribution to the underdeveloped area of higher-order constructions in topos theory. Recent work on this includes external representations of sober spaces Fourman and Scott [2]) and Banach spaces (Hofmann [4] , Mulvey [10]) and work in general topology (Grayson [3]) . Here we consider representations of objects (filter spaces) which are more general (and so have "less structure") than topological spaces. For these we can investigate a significant higher order construction, the formation of function spaces. At the moment however we have to pay for this generality; we can establish our most significant results only for filter spaces which "come from the real world" i.e. are represented as sets by sections of a projection $T \times X \longrightarrow T$ over T . Indeed we restrict presentation throughout to such objects. Occasional glimpses of the more general view occur (in sections 3 and 6), and the reader of Fourman and Scott [2] will readily see how to generalize our definitions and represent a wide class of internal filter spaces. But since I cannot i) characterize this wider class in the internal logic nor ii) prove much about it, it seemed best to avoid the unnecessary generalities.

The paper divides into two parts. Sections 1 to 4 deal with quite general questions of continuity, convergence and function spaces in sheaves over a topological space T . The material is constructive. Sections 5 and 6 discuss situations where all functions are continuous (a phenomenon first investigated by Scott [11]) . The treatment I give is not detailed as I do not believe it can be in its final form. In particular it is not constructive. It seems likely that some relation with the Cauchy approximations of Mulvey will emerge in a more constructive treatment.

In the evolution of the ideas of this paper sections 5 and 6 came first. They arose out of a suggestion of Scott's that the continuous functionals might appear as higher types in sheaf models (see Corollary 7 of §6). I am grateful to him for this and also for valuable advice on continuous lattices. At the time that the strategy of the first part of the paper was crystallizing in my mind, I had useful discussions on topology with Michael Fourman and Robin Grayson.

The material in this paper suggests further lines for investigation:

i) function spaces in Grothendieck toposes (for example the Johnstone topos [8], that suggested by Lawvere [9], $SETS^{N^{op}}$ where N is Ershov's category of enumerated sets);

ii) other kinds of higher-order structure;

iii) some general theory of the interpretation of higher-order constructs (here Lemma 1 of §6 is most suggestive).

I can only hope that the inadequacies of this paper do not discourage people from pursuing such topics.

1. CONTINUOUS LATTICES AND THEIR REPRESENTATION IN Sh(T)

The basic theory of continuous lattices is set out in Scott [12]. This theory can be readily developed in the logic of toposes. In particular the following hold constructively:

i) continuous lattices can be considered either as special kinds of complete lattices, or as topological spaces under Scott's induced topology (henceforth the <u>Scott topology</u>);

ii) for continuous lattices topological continuity is identical with lattice continuity (defined by the preservation of directed sups);

iii) the category of continuous lattices is cartesian closed.
(<u>Hint for iii)</u> If $d \in D$ and $d' \in D'$, then $[d;d'] : D \longrightarrow D'$ should be defined by $[d;d'](x) = \bigvee \{ d' \mid d \ll x \}$, where we use $\ll$ for the "strict less than" relation.)

A certain amount of rewriting of [12] is necessary for a constructive treatment, as some classical results about general complete lattices appear to be essentially non-constructive. The category of complete lattices with maps preserving directed sups is cartesian closed; however, the injection of this category into that of topological spaces by taking the Scott topology does not appear constructively to be full. (Indeed it is not clear constructively that the Scott topology in a complete lattice is even T_0 in a positive sense.)

We have a general way of representing an external topological space X in Sh(T) (see Fourman and Scott [2]) as a space X_T, whose (partial) elements are continuous sections of the projection $T \times X \longrightarrow T$ and whose topology is derived from the product topology on $T \times X$ in the obvious way. We will show that if X is a continuous lattice externally then X_T is one internally. Recall that the relevant order on a topological space X is defined by

$$x \le y \quad \text{iff} \quad (\forall \text{ open } O)(x \in O \rightarrow y \in O) \quad .$$

LEMMA 1. $Ea \cap Eb \cap [\![a \le b]\!] = \text{In}\{ t \mid a(t) \le b(t) \}$

Proof. Note that the opens of any topological space X form a (possibly non-topological) filter space $\mathcal{O}(X)$, which is represented internally (at least as a set) by $\mathcal{O}(X_T)$, the sheaf of continuous $\mathcal{O}(X)$-valued functions. Thus it is sufficient to observe that the formula defining $x \le y$ is strictly local ($\vdots$ a geometric sequent) (see Hyland [5]).

LEMMA 2. Let D be a complete lattice; let D_T be the representation in $Sh(T)$ of the space obtained by giving D the Scott topology; let $\le$ be defined on D_T by $[\![x \le y]\!] = \text{In}\{ t \mid x(t) \le y(t) \}$ (here $\le$ is strict). Then $(D_T, \le)$ is a complete lattice in $Sh(T)$.

Proof. The finitary $\vee$ is continuous on any complete lattice, so we can define $\vee$ on D_T by

$$(x \vee y)(t) = x(t) \vee y(t) \quad , \quad \text{for } t \in Ex \cap Ey \quad .$$

It is simple to check directly that this definition gives the internal join. (The formula expressing this is strictly local in the extended sense of Hyland [5] , so there is a general reason). It remains to show completeness of D_T . Take a collection $\{x_i\}_{i \in I}$ of partial elements of D_T , corresponding to a (global) subsheaf, and define $\bigvee x_i$ by

$$\bigvee x_i\,(t) = \bigvee \{ x_i(t) \mid t \in Ex_i \}$$

We must show that $\bigvee x_i$ is in D_T (i.e. that it is continuous). Suppose $\bigvee x_i\,(t) \in O$, with O Scott open in D ; then for some finite $J \subseteq I$, $\bigvee\{ x_i(t) \mid i \in J \} \in O$; but the finitary join $x = \bigvee\{ x_i \mid i \in J \}$ is in D_T and $t \in x^{-1}(O)$ open in T ; so

$$\bigvee x_i\,(t) \in \bigvee x_i\,(x^{-1}(O)) \subseteq O$$

which shows that $\bigvee x_i$ is continuous. Now it is easy to check that $\bigvee x_i$ is internally the sup of the subsheaf corresponding to $\{x_i\}_{i \in I}$. We have only to localize the above to conclude that D_T is complete.

REMARK By Lemma 1 , the complete lattice D_T of Lemma 2 satisfies

$$(\forall xy)(x \le y \leftrightarrow (\forall \text{ Scott open } O)(x \in O \rightarrow y \in O)) \quad . \qquad (*)$$

This holds for any complete lattice classically, but the proof is non-constructive; however, I do not know of an example of a complete lattice in a topos which does not satisfy (*) . (In Lemma 2 , $\le$ is strict; as defined before Lemma 1 , it is not strict; either form could be defined in terms of the other and they are equivalent when quantified as in (*) . This kind of distinction is of no significance in this paper and will henceforth be ignored.)

We now wish to show that if D in Lemma 2 is a continuous lattice then so is $(D_T, \leq)$ as defined there. So we must consider the "strict less than" $\ll$. This relation is not so simple to characterize as $\leq$, but the information in Lemma 3 is all we need.

In general if we represent a space X by the sheaf X_T of continuous X-valued functions, we denote by x_U the element of X_T with extend U open in T and constant value $x \in X$. When X is topological, O open in X , U open in T , then O_U denotes the (basic) open set of X_T satisfying

$$[\![a \in O_U]\!] = U \cap a^{-1}(O) .$$

Given a continuous lattice D , we let

$$O_a = \{ d \mid a \ll d \} = \mathrm{In} \{ d \mid a \leq d \}$$

be the (Scott) open set in D determined by $a \in D$.

LEMMA 3. Let D be a continuous lattice and D_T be the represented complete lattice (as in Lemma 2).

i) Given x in D_T ; we define a subsheaf S_x of D_T of extent Ex by giving its set of partial elements:

$$S_x = \{ a_U \mid (\forall t \in U)(a \ll x(t)) \} .$$

Then in D_T ,

$$x = \bigvee S_x .$$

ii) If $a \in D$, $y \in D_T$ and $U \subseteq Ey$, then

$$U \cap [\![a_U \ll y]\!] = U \cap y^{-1}(O_a) .$$

Proof.

<u>For i)</u> : If $a \ll x(t)$, then $x(t) \in O$. Let $U = x^{-1}(O_a)$. Then a_U is in S_x . Since $a \ll x(t)$ was arbitrary,

$$x(t) = \bigvee S_x(t) .$$

<u>For ii)</u> : Recall $a_U \ll y$ iff $(\forall \text{directed } S)(\bigvee S \geq y \rightarrow (\exists s \in S)\ s \geq a_U)$. Suppose $t \in U \cap y^{-1}(O_a)$ and $t \in [\![S \text{ is directed} \wedge \bigvee S \geq y]\!]$. Since an internally directed set is locally directed when looked at from the outside (by Lemma 1) , we can conclude from

$$a \ll y(t) \leq \bigvee S\ (t)$$

that for some s in S with $t \in Es$, $a \ll s(t)$. But then clearly $t \in [\![s \geq a_U]\!]$ using Lemma 1 . So we have $t \in [\![a_U \ll y]\!]$. Conversely suppose

$t \in U \cap [\![a_U \ll y]\!]$; then by i) there is s in S_y with $t \in [\![s \geq a_U]\!]$, so for $t' \in U \cap [\![s \geq a_U]\!]$,

$$y(t') \gg s(t') \geq a \quad ,$$

whence $t \in y^{-1}(O_a)$. This completes the proof.

THEOREM 4. Suppose D is a continuous lattice, D_T is the representation of D as a topological space in Sh(T) and $\leq$ is as in Lemmas 1 and 2 . Then $(D_T,\leq)$ is a continuous lattice and the internal Scott topology on D_T is the same as that given by the topological representation.

Proof. $(D_T,\leq)$ is a complete lattice by Lemma 2 . By Lemma 3(ii) the elements a_U of S_x for $x \in D_T$ satisfy

$$[\![a_U \ll x]\!] \geq U \quad ,$$

so by Lemma 3(i) for any x in D_T ,

$$x = \bigvee \{ y \mid y \ll x \} .$$

By the general theory of continuous lattices, the sets $(O_a)_U$ (and their restrictions) form a basis for the topology, and by Lemmas 1 and 3 , these are Scott open internally. On the other hand by Lemma 3(i) every Scott open set is a union (locally) of sets O_{a_U} (which is the restriction of $(O_a)_U$ to U and hence is open in the topological representation). Thus the two topologies coincide.

COROLLARY 5. Suppose D, D' are continuous lattices and D_T , D'_T as above. Then $[D_T,D'_T]$ the sheaf of continuous maps from D_T to D'_T is the same whether we take lattice or topological continuity.

Proof. By Theorem 4 , D_T and D'_T are internal continuous lattices, so the result holds by ii) of the first paragraph of this section.

THEOREM 6. Let D and D' be continuous lattices. Then the following (internal) continuous lattices are isomorphic:

i) $[D_T,D'_T]$ the internal function space with the topology of pointwise convergence (product topology) and usual order;

ii) $[D,D']_T$ the representation of the external function space $[D,D']$.

Proof. The topological representation (Fourman and Scott [2]) identifies sections over U of $[D_T,D'_T]$ with continuous maps $U \times D \longrightarrow D'$; by adjointness these can be identified with continuous maps $U \longrightarrow [D,D']$. This gives the bijection between $[D_T,D'_T]$ and $[D,D']_T$ as sheaves (i.e. internal sets). To show isomorphism as continuous lattices we consider the partial order $\leq$. In

$[D_T, D'_T]$, (the strict version of) this can be defined by

$$f \le g \quad \text{iff} \quad (\forall d)\ f(d) \le g(d) \wedge Ef \wedge Eg \quad .$$

But since $\le$ on D'_T is strictly local, so is $\le$ on D_T and hence (by Hyland [5])

$$[\![f \le g]\!] = \mathrm{In}\{ t \mid f(t) \le g(t) \} \quad .$$

But this is just how $\le$ is defined on $[D,D']_T$ so the two continuous lattices are isomorphic in lattice structure (and cf. Theorem 4 topological structure).

REMARK One can show constructively that continuous lattices are sober spaces (the necessary information is in Scott [12]) . Hence it is not surprising that one can generalize the material of this section to give an account of the behaviour of the category of all continuous lattices in Sh(T) . We leave this to the reader; we already have all the information we will be able to apply.

2. FILTER SPACES AND CONTINUOUS LATTICES IN Sh(T)

We first describe the main abstract theorem in Hyland [7] . It gives a (constructive) way of representing filter spaces as structures on continuous lattices.

A convergence structure $\{ A_x \mid x \in X \}$ on a continuous lattice $(D,\le)$ is a collection of subsets A_x of D such that

i) each A_x has a maximal element p_x ;

ii) if $d \le e \le p_x$ and $d \in A_x$ then $e \in A_x$.

If $\{ A_x \mid x \in X \}$ on D and $\{ B_y \mid y \in Y \}$ on D' are two convergence structures, we take as continuous maps, functions $f : X \longrightarrow Y$ on the index sets for which there exists a continuous $\bar{f} : D \longrightarrow D'$ such that $\bar{f}(A_x) \subseteq B_{f(x)}$. Convergence structures with continuous maps as morphisms form the category CSCL of convergence structures on continuous lattices.

As in Hyland [7] , we adopt the following notion of filter space. A filter space (X,F) is a set X with a notion of filter convergence F which associates with each $x \in X$ a collection $F(x)$ of filters on X such that

i) $[x]$ the principal filter generated by x is in $F(x)$;

ii) if $\Phi \supseteq \Psi \in F(x)$ then $\Phi \in F(x)$;

iii) for any Φ , $\Phi \in F(x)$ iff $\Phi \cap [x] \in F(x)$.

As usual, the continuous maps from a filter space (X,F) to a filter space (Y,G) are those $f : X \longrightarrow Y$ which carry convergent filters to convergent filters (i.e. the "image" under f of $\Phi \in F(x)$ is a filter base which generates a filter

$f(\Phi) \in G(f(x))$). Filter spaces with continuous maps as morphisms form the category FIL .

Given a convergence structure $\{ A_x \mid x \in X \}$ on D , we define a filter structure on X as follows. For O open in D , define $U_O \subseteq X$ by

$$U_O = \{ x \mid p_x \in O \} \quad .$$

Clearly $U_O \cap U_{O'} = U_{O \cap O'}$, so for $d \in D$ we may define a (possibly degenerate) filter Φ_d as that generated by the filter base

$$\{ U_O \mid d \in O \} \quad .$$

Now define a filter structure F on X by setting

$$F(x) = \{ \Phi \mid \Phi \supseteq \Phi_d \text{ for some } d \in A_x \} \quad .$$

Conversely, given a filter space (X,F) , we take the continuous (in fact algebraic) lattice $Fil(X)$ of all filters (including the degenerate ones) on X . Define

$$F'(x) = F(x) \cap \{ \Phi \mid \Phi \subseteq [x] \} \quad .$$

Then $\{ F'(x) \mid x \in X \}$ is a convergence structure on $Fil(X)$.

The main abstract result of Hyland [7] can now be stated as follows:

THEOREM 1. The above constructions (of objects of FIL from objects of CSCL and vice versa) preserve the continuous maps in the categories so that we have full and faithful functors from CSCL to FIL and from FIL to CSCL . These functors give rise to an equivalence of categories whereby FIL is mapped to a reflective subcategory of CSCL .

In this paper we exploit the fact that Theorem 1 is constructive.

We now consider how to represent a convergence structure $\{ A_x \mid x \in X \}$ on a continuous lattice D , in $Sh(T)$. The representation D_T of D was given in §1 . We represent X as follows: give X the filter structure corresponding to the convergence structure $\{ A_x \mid x \in X \}$; represent X by X_T , the sheaf of X-valued functions on T which are continuous with respect to this filter structure. Finally, we must give a value to $[\![p \in A_a]\!]$ where p is in D_T and a in X_T . We let

$$[\![p \in A_a]\!] = \text{In} \{ t \mid p(t) \in A_{a(t)} \} \quad .$$

This defines $A_a \subseteq D_T$ for a in X_T . Then we have the following:

LEMMA 2. $\{ A_a \mid a \in X_T \}$ on D_T is a convergence structure on a continuous lattice in $Sh(T)$.

Proof. The proof is straightforward, but note that the maximal element p_a of A_a is defined by

$$p_a(t) \;=\; \bigvee\{\, p(t) \mid t \in [\![\, p \in A_a \,]\!] \,\} \quad .$$

In particular, $p_a(t)$ is <u>not</u> equal to $p_{a(t)}$; this latter is not necessarily continuous in t .

REMARK In the (important) case where the A_x's are disjoint (i.e. are given as the equivalence classes of $A = \bigcup\{ A_x \mid x \in X \}$ under an equivalence relation $\sim$), we can get at the above representation in the following simple manner. We let

$$A_T \;=\; \{\, p \mid (\forall\, t \in Ep)\quad p(t) \in A \,\}$$

and define for p and q in A_T ,

$$[\![\, p \sim q \,]\!] \;=\; \mathrm{In}\,\{\, t \mid p(t) \sim q(t) \,\} \quad .$$

This defines an internal equivalence relation $\sim$ on A_T , and when we quotient out, we get X_T ; then the equivalence classes are just the A_a's with a in X_T as defined above.

(We sketch a proof of this. The quotient is

$$A_T/\sim \;=\; \{\, [p] \mid p \in A_T \,\}$$

where

$$[p] \;=\; \{\, q \mid [\![\, Eq \to p \sim q \,]\!] = T \,\} \qquad \text{and} \qquad E[p] = Ep \quad ;$$

we have the obvious restrictions; and do not need to sheafify as the presheaf $A_T/\sim$ is already a sheaf. The equivalence class $[p]$ corresponds to the filter continuous map

$$Ep \longrightarrow X \quad ; \quad t \mapsto [p(t)]$$

(where $[p(t)]$ is the equivalence class of $p(t)$ in A). Conversely, any continuous map $a : V \longrightarrow X$ gives rise to an element of $A_T/\sim$, defined by setting

$$q_a(t) \;=\; a(\nu(t)) \quad ,$$

(where $\nu(t)$ is the neighbourhood filter at t) , and taking $[q_a]$.)

It is a trivial but useful fact that our representation provides an injection of the category CSCL into the category of convergence structures on continuous lattices in $Sh(T)$ with (global) continuous maps as morphisms.

LEMMA 3. Let $\{ A_x \mid x \in X \}$ and $\{ B_y \mid y \in Y \}$ be convergence structures on D and D' respectively. Let $z : X \longrightarrow Y$ be a continuous map. Then the map $\hat{z} : X_T \longrightarrow Y_T$ defined by

$$\hat{z}(a)\,(t) \;=\; z(a(t)) \quad ,$$

is a continuous map from $\{ A_a \mid a \in X_T \}$ on D_T to $\{ B_b \mid b \in Y_T \}$ on D'_T .

Proof. There is a continuous map $f : D \longrightarrow D'$ such that

$f(A_x) \subseteq B_{z(x)}$.

We define the internal continuous map $\hat{f} : D_T \longrightarrow D'_T$ by

$$(\hat{f}(p))\ (t) = f(p(t)) \quad .$$

"$f(A_x) \subseteq B_{z(x)}$" is strictly local in the sense of Hyland [5] , so

$$E_a \cap [\![\hat{f}(A_a) \subseteq B_{\hat{z}(a)}]\!] = \mathrm{In}\{ t \mid f(A_{a(t)} \subseteq B_{z(a(t))} \} = Ea \quad .$$

Thus $\hat{z}$ is a continuous map.

COROLLARY 4. If $\{ A_x \mid x \in X \}$ and $\{ A'_x \mid x \in X \}$ are convergence structures on continuous lattices D and D' which determine the same filter structure on X , then $\{ A_a \mid a \in X_T \}$ and $\{ A'_a \mid a \in X_T \}$ on D_T and D'_T determine the same internal filter structure on X_T .

Proof. This is immediate by Lemma 3 and the constructivity of Theorem 1 .

In virtue of Corollary 4 , our representation of convergence structures on continuous lattices gives rise to a well-defined representation of external filter spaces in Sh(T) . Suppose (X,F) is a filter space. Associated with it is the canonical convergence structure

$$\{ F'(x) \mid x \in X \} \quad \text{on} \quad \mathrm{Fil}(X) \quad ,$$

and this is represented internally by

$$\{ F'(a) \mid a \in X_T \} \quad \text{on} \quad (\mathrm{Fil}(X))_T \quad .$$

Thus we can characterize our internal representation (X_T, F_T) of (X,F) by

$$[\![\Phi \in F_T(a)]\!] = [\![(\exists p \in F'(a))\ \ \Phi \supseteq \Phi_p]\!] \quad , \qquad (*)$$

where Φ is in the internal $\mathrm{Fil}(X_T)$ and a is in X_T . The main aim of the rest of this section is to determine what (*) means in concrete terms.

First we analyse what Φ_p is. Given $A \subseteq X$ we define $A_T \subseteq X_T$ by

$$[\![a \in A_T]\!] = \mathrm{In}\ \{ t \mid a(t) \in A \} \quad .$$

Now given p in $(\mathrm{Fil}(X))_T$ we define a filter $\phi(p)$ on X_T by stipulating that

$$[\![A_T \in \phi(p)]\!] = \{ t \mid A \in p(t) \} = p^{-1}(\{ \Psi \mid A \in \Psi \})$$

and that for an arbitrary subsheaf S of S_T ,

$$[\![S \in \phi(p)]\!] = [\![(\exists A_T \in \phi(p))\ \ A_T \subseteq S]\!] = \bigvee \{ [\![A_T \in \phi(p) \wedge A_T \subseteq S]\!] \mid A \subseteq X \} \quad .$$

Note that $\phi(p)$ is generated by the (restrictions of the) A_T's in $\phi(p)$. It is an internal filter on X_T with extent Ep .

LEMMA 5. i) For $A \subseteq X$, let $O(A)$ ($= \{ \Psi \mid A \in \Psi \}_T$ in the notation of section 1) be the internal set in $(\mathrm{Fil}(X))_T$ determined by

$$[\![p \in O(A)]\!] = \{ t \mid A \in p(t) \} .$$

Then $A_T = U_{O(A)}$.

ii) $\phi(p) = \Phi_p$.

Proof. For i), first note that

$$[\![a \in U_{O(A)}]\!] = [\![p_a \in O(A)]\!] = \{ t \mid A \in p_a(t) \} ,$$

and $\{ t \mid A \in p_a(t) \} \subseteq \mathrm{In}\{ t \mid a(t) \in A \} = [\![a \in A_T]\!]$ as $p_a(t) \leq p_{a(t)}$.
On the other hand if $A \in a(t)$ for all t in V open in T, then

$$p : V \longrightarrow \mathrm{Fil}(X) \quad ; \quad t \mapsto a(\nu(t))$$

(where $\nu(t)$ is the neighbourhood filter on t) is a continuous map such that for all $t \in V$ $p(t) \in F'(a(t))$, and $A \in p(t)$.
Hence $V \subseteq \{ t \mid (\exists p) \; t \in [\![p \in F'(a)]\!] \text{ and } A \in p(t) \} = \{ t \mid A \in p_a(t) \}$.
This completes the proof of i) (but cf. Lemma 8).

For ii) note that since the sets $\{ \Psi \mid A \in \Psi \}$ form a basis for the topology on $\mathrm{Fil}(X)$, Φ_p is generated by the (restrictions of the) $U_{O(A)}$'s in it. Now by the remark that the same goes for $\phi(p)$ and the A_T's , by i) it remains to check that

$$[\![U_{O(A)} \in \Phi_p]\!] = [\![p \in O(A)]\!] = p^{-1}(\{ \Psi \mid A \in \Psi \}) = \{ t \mid A \in p(t) \}$$
$$= [\![A_T \in \phi(p)]\!] .$$

Next it will be useful to determine the relation between the two internal continuous lattices $\mathrm{Fil}(X_T)$ and $(\mathrm{Fil}(X))_T$. (That there is something significant here is indicated by the fact that $[\![p \in O(A)]\!] = [\![A_T \in \phi(p)]\!]$.) We consider the map

$$\phi : p \mapsto \phi(p) .$$

Clearly it is a sheaf map from $(\mathrm{Fil}(X))_T$ to $\mathrm{Fil}(X_T)$. Also

$$\phi(p) = \phi(q) \quad \text{iff} \quad p^{-1}(\{ \Psi \mid A \in \Psi \}) = q^{-1}(\{ \Psi \mid A \in \Psi \}) \quad \text{all}$$
$A \subseteq X$. iff $p = q$;

so it is clear that ϕ is injective (externally and internally). Furthermore if $\{p_\alpha\}$ is an internal directed set in $(\mathrm{Fil}(X))_T$ (i.e. externally it is "locally directed"), then

$$[\![A_T \in \phi(\bigvee p_\alpha)]\!] = \{ t \mid A \in \bigvee p_\alpha(t) \} = \bigcup_\alpha \{ t \mid A \in p_\alpha(t) \}$$
$$= \bigvee_\alpha [\![A_T \in \phi(p_\alpha)]\!] .$$

Thus $\phi : p \mapsto \phi(p)$ is an injective continuous map from $(\mathrm{Fil}(X))_T$ to $\mathrm{Fil}(X_T)$. Clearly ϕ has a one-sided inverse which we can define as follows. Given Φ in $\mathrm{Fil}(X_T)$ we define $f_\Phi : E\Phi \longrightarrow \mathrm{Fil}(X)$ by

$$f_\Phi(t) = \{ A \mid t \in [\![A_T \in \Phi]\!] \} \quad .$$

Since $f_\Phi^{-1}(\{ \Psi \mid A \in \Psi \}) = [\![A_T \in \Phi]\!]$, f_Φ is continuous and so clearly the map $\Phi \mapsto f_\Phi$ is a sheaf map from $\mathrm{Fil}(X_T)$ to $(\mathrm{Fil}(X))_T$. Furthermore $\Phi \mapsto f_\Phi$ is continuous as if $\{\Phi_\alpha\}$ is an internally directed set in $\mathrm{Fil}(X_T)$, then

$$\begin{aligned} f_{\bigvee\Phi_\alpha}(t) &= \{ A \mid t \in [\![A_T \in \textstyle\bigvee \Phi_\alpha]\!] \} \\ &= \{ A \mid (\exists\, \Phi_\alpha) \quad t \in [\![A_T \in \Phi_\alpha]\!] \\ &= \textstyle\bigvee f_{\Phi_\alpha}(t) \quad . \end{aligned}$$

We can now state the precise relationship between $\mathrm{Fil}(X_T)$ and $(\mathrm{Fil}(X))_T$.

LEMMA 6. $(\mathrm{Fil}(X))_T$ is a projection (in the sense of Scott [12]) of $\mathrm{Fil}(X_T)$ via the maps

$$\mathrm{Fil}(X_T) \longrightarrow (\mathrm{Fil}(X))_T \quad ; \quad \Phi \mapsto f_\Phi \quad ,$$

and

$$(\mathrm{Fil}(X))_T \longrightarrow \mathrm{Fil}(X_T) \quad ; \quad p \mapsto \phi(p) \quad .$$

Proof. After the above discussion, it remains to check

i) $f_{\phi(p)}(t) = \{ A \mid t \in [\![A_T \in \phi(p)]\!] \} = p(t)$, so $p = f_{\phi(p)}$,

and ii) $[\![A_T \in \phi(f_\Phi)]\!] = [\![A_T \in \Phi]\!]$, so $\phi(f_\Phi) \subseteq \Phi$ (or if you prefer, $[\![\phi(f_\Phi) \subseteq \Phi]\!] = T$) .

The next Lemma relates the maximal element p_a of $F'(a)$ to other concepts which we have introduced.

LEMMA 7. Let $[a]$ be the principal filter generated by a in $\mathrm{Fil}(X_T)$ and let $\nu(t)$ represent the neighbourhood filter at t . Then

$$p_a(t) = f_{[a]}(t) = a(\nu(t)) \quad .$$

Proof.
$$\begin{aligned} f_{[a]}(t) &= \{ A \mid t \in [\![A_T \in [a]]\!] \} \\ &= \{ A \mid t \in \mathrm{In}\{ s \mid a(s) \in A \} \} = a\nu(t) \quad ; \end{aligned}$$

whence also throughout the extent of a , $f_{[a]}(t) \in F'(a(t))$. Since clearly if $p(t) \in F'(a(t))$ throughout an open set V , then in V , $\phi(p) \subseteq [a]$, using Lemma 6 , we see that in V , $p \subseteq f_{[a]}$; thus $f_{[a]}$ is p_a the maximal element in $F'(a)$.

We may define $F'_T(a)$ by

$$[\![\Phi \in F_T(a)]\!] = [\![\Phi \in F_T(a) \wedge \Phi \subseteq [a]]\!] ,$$

and $f^*F'(a)$ by

$$[\![\Phi \in f^*F'(a)]\!] = [\![f_\Phi \in F'(a)]\!] .$$

It is easy to see that

$$[\![\Phi \in f^*F'(a)]\!] = [\![\Phi \in F_T(a) \wedge f_\Phi \leq p_a]\!] ,$$

whence we see that

$$[\![\Phi \in F'_T(a)]\!] \subseteq [\![\Phi \in f^*F'(a)]\!]$$

while we would not expect the converse to hold. We now give a theorem which characterizes F_T.

THEOREM 8. i) $[\![\Phi \in F_T(a)]\!] = [\![(\exists \Psi)(\Psi \subseteq \Phi \wedge \Psi \in F'_T(a))]\!]$

$= [\![(\exists \Psi)(\Psi \subseteq \Phi \wedge \Psi \in f^*F'(a))]\!]$

$= \bigcup \{ \operatorname{In} \{ t \mid f_\Phi(t) \supseteq p(t) \in F'(a(t)) \} \mid p \in (Fil(X))_T \}$.

ii) If (X,F) is a limit space (i.e. for each $x \in X$, $F(x)$ is a filter) then

$$[\![\Phi \in F_T(a)]\!] = \operatorname{In} \{ t \mid f_\Phi(t) \in F(a(t)) \} .$$

Proof. i) is straightforward with the help of Lemmas 6 and 7. For ii) it is obvious that if $t \in \Phi \in F_T(a)$, then for some p, $t \in \operatorname{In} \{ s \mid p(s) \in F'(a(a)) \}$ and $t \in \operatorname{In} \{ s \mid f_\Phi(s) \supseteq p(s) \}$, whence $t \in \operatorname{In} \{ s \mid f_\Phi(s) \in F(a(s)) \}$. To get the converse we use the fact that if each $F(x)$ is a filter, then if $t \in \operatorname{In} \{ s \mid f_\Phi(s) \in F(a(s)) \}$, then $t \in \operatorname{In} \{ s \mid f_\Phi(s) \cap p_a(s) \in F'(a(s)) \}$ and as $s \to f_\Phi(s) \in p_a(s)$ is continuous, this is sufficient to show that $t \in [\![\Phi \in F_T(s)]\!]$.

REMARK It appears that in general, the pleasant characterization as in ii) holds only for the Φ in $f^*F'(a)$.

3. THE INDUCED TOPOLOGY ON FILTER SPACES

There is an injection of TOP (the category of topological spaces and continuous maps) into FIL : associate to points x of a topological space $(X,\mathcal{O}(X))$ those filters which include the neighbourhood filter $\nu(x)$ at x ; maps are topologically continuous iff they are continuous with respect to the corresponding filter structures. The induced topology provides a left adjoint to this injection. Given a filter space (X,F) we say that $O \subseteq X$ is open (in the induced topology) iff whenever $x \in O$ and $\Phi \in F(x)$, then $O \in \Phi$.

Filter continuous maps are continuous with respect to the induced topologies. It is fairly easy to see that the fact that the induced topology provides a left adjoint to the injection of TOP in FIL , is constructive. We need this fact together with information about the induced topology on our represented filter spaces in $Sh(T)$, in order to prove the most general form of our results about function spaces.

THEOREM Let (X,F) be a filter space. Suppose $O \subseteq T \times X$ is open in the induced topology on the product $T \times X$ (in FIL), and $O_T \subseteq X_T$ is defined by

$$[\![a \in O_T]\!] = \{ t \mid (t,a(t)) \in O \} .$$

Then O_T is open in the induced topology on (X_T,F_T) , and every such open arises in this way.

Proof. First suppose O open in $T \times X$ and V open in T are such that

$$V \subseteq [\![a \in O_T]\!] \cap [\![\Phi \in F_T'(a)]\!] ,$$

for some a in X_T and Φ in $Fil(X_T)$. We wish to show that $V \subseteq [\![O_T \in \Phi]\!]$. Let $p = f_\Phi$, so that $V \subseteq [\![p \in F'(a)]\!]$. Thus for any $t \in V$, $\nu(t) \times p(t)$ converges to $(t,a(t))$ in $T \times X$. Hence $O \in \nu(t) \times p(t)$; so there is open U containing t , and $A \in p(t)$ with $U \times A \subseteq O$. Let $U' = U \cap p^{-1}(\{\Psi \mid A \in \Psi\})$. Then clearly $t \in U'$, $U' \subseteq [\![A_T \in \phi(p)]\!]$ and $U' \subseteq [\![A_T \subseteq O_T]\!]$. Hence $t \in [\![O_T \in \Phi]\!]$ and since t was an arbitrary point of V , $V \subseteq [\![O_T \in \Phi]\!]$ as required.

Now suppose W is open in the induced topology on X_T . We first prove the following:

SUBLEMMA Suppose a and a' are in X_T and $a(t) = a'(t)$; then

$$t \in [\![a \in W]\!] \quad \text{iff} \quad t \in [\![a' \in W]\!] .$$

Proof of sublemma. It is sufficient to show that if a is in X_T and $a(t) = x$, then

$$t \in [\![a \in W]\!] \quad \text{iff} \quad t \in [\![x_T \in W]\!] .$$

If $t \in [\![a \in W]\!]$, take p ($: t \mapsto a\nu(t)$, say) such that $t \in [\![p \in F'(a)]\!]$. Now $t \in [\![W \in \phi(p)]\!]$, and hence for some $A \subseteq X$,

$$t \in [\![A_T \subseteq W]\!] \cap \{ s \mid A \in p(s) \} .$$

Since $A \in p(t)$, $x \in A$ so $[\![x_T \in A_T]\!] = T$; thus

$$t \in [\![x_T \in A_T \wedge A_T \subseteq W]\!] \subseteq [\![x_T \in W]\!] .$$

Conversely, if $t \in [\![x_T \in W]\!]$, define $p(s) = a\nu(s)$ and let q be $(p(t))_T$. Since $p(t) \in F'(x)$, $[\![q \in F(x_T)]\!] = T$. Now $t \in [\![W \in \phi(q)]\!]$, so that there is open V containing t and $A \in q(t)$ such that $V \subseteq [\![A_T \subseteq W]\!]$. But $q(t) = p(t) = a\nu(t)$, so we can find open V' containing t such that $a(V') \subseteq A$. Then $t \in V \cap V' \subseteq [\![a \in A_T \wedge A_T \subseteq W]\!] \subseteq [\![a \in W]\!]$. This completes the proof of the sublemma.

Now we define $O \subseteq T \times X$ by

$$O = \{ (t,x) \mid t \in [\![x_T \in W]\!] \} \quad .$$

Take $(t,x) \in O$ and an arbitrary $\Phi \in F'(x)$. Since $[\![\Phi_T \in F'(x_T)]\!] = T$, $t \in [\![x_T \in W]\!] \subseteq [\![W \in \phi(\Phi_T)]\!]$. Hence there is open V containing t and $A \in \Phi$ such that

$$V \subseteq [\![A_T \subseteq W]\!] \quad .$$

Clearly $V \times A \subseteq O$. Hence $O \in \nu(t) \times \Phi$. Since $\Phi \in F'(x)$ was arbitrary this shows that O is open in the induced topology on $T \times X$.

Now we observe that

$$\begin{aligned}
[\![a \in O_T]\!] &= \{ t \mid (t,a(t)) \in O \} \\
&= \{ t \mid t \in [\![a(t)_T \in W]\!] \} \\
&= \{ t \mid t \in [\![a \in W]\!] \} \quad \text{by the sublemma} \\
&= [\![a \in W]\!] \quad .
\end{aligned}$$

Hence $O_T = W$, and this completes the proof.

4. FUNCTION SPACES IN Sh(T)

We first determine what is the sheaf of continuous maps between represented convergence structures in $Sh(T)$. As is the case with topological spaces, we need a condition on the range space to ensure a good representation of the continuous maps. We state this in terms of the <u>induced topology</u> on filter spaces (see e.g. Hyland [6] and §3).

LEMMA 1 Let $\{ A_x \mid x \in X \}$ and $\{ B_y \mid y \in Y \}$ be convergence structures on D and D' respectively; let their representations in $Sh(T)$ be $\{ A_a \mid a \in X_T \}$ on D_T and $\{ B_b \mid b \in Y_T \}$ on D'_T . Let (X,F) and (Y,G) be the filter spaces corresponding to the convergence structures and let their function space in FIL be $([X,Y],H)$. Then

i) an element $c : U \to [X,Y]$ of $[X,Y]_T$ gives rise internally to a continuous map $\hat{c}$ (of extent U) from $\{ A_a \mid a \in X_T \}$ on D_T to $\{ B_b \mid b \in Y_T \}$ on D'_T by the equation

$$(\hat{c}(a))(t) = (c(t))(a(t)) \quad ;$$

ii) if the induced topology on (Y,G) is T_0 , then every internal continuous map arises in this way, so that the sheaf of continuous maps between the convergence structures can be identified with $[X,Y]_T$.

Proof. We treat the case when the convergence structures are $\{ F'(x) \mid x \in X \}$ on $Fil(X)$ and $\{ G'(y) \mid y \in Y \}$ on $Fil(Y)$. The general case follows in view of the constructivity of Theorem 1 of §2 .

For i) , let $c : U \longrightarrow [X,Y]$ be continuous. Define a continuous map $r : U \longrightarrow [Fil(X),Fil(Y)]$ by

$$(r(t))(\Phi) = (c(\nu(t)))(\Phi) ,$$

where $\nu(t)$ is the neighbourhood filter at t . We identify r with the corresponding element of $[(Fil(X))_T,(Fil(Y))_T]$ by Theorem 6 of §1 . As the condition " $r(F'(a)) \subseteq G'(c(a))$ " is strictly local in the sense of Hyland [5] ,

$$U \cap Ea \cap [\![r(F'(a)) \subseteq G'(\hat{c}(a))]\!] = In \{ t \mid r(t)(F'(a(t)) \subseteq G'(c(t)(a(t))) \}$$

$$= U \cap Ea .$$

Thus r witnesses the fact that $\hat{c}$ is a continuous map from $\{ F'(a) \mid a \in X_T \}$ on $(Fil(X))_T$ to $\{ G'(b) \mid b \in Y_T \}$ on $(Fil(Y))_T$ (defined over U) .

For ii) , let z be in $[X_T,Y_T]$ with $Ez = V$. First we must show that if $a(t) = a'(t)$, then $z(a)(t) = z(a')(t)$. Now since z is an internal continuous map from (X_T,F_T) to (Y_T,G_T) , it is a continuous map between the corresponding (internal) induced topological spaces. Let P be open in the induced topology on Y and suppose $z(a)(t)$ P ; let $U = z(a)^{-1}(P)$. Then,

$$t \in U \subseteq [\![z(a) \in P_T]\!] = \{ s \mid z(a)(s) \in P \} .$$

Thus, $t \in U \subseteq [\![a \in z^{-1}(P_T)]\!]$. Now $z^{-1}(P_T)$ is open in the induced topology on X_T , so by the sublemma in Theorem of § 3 .

$$t \in [\![a' \in z^{-1}(P_T)]\!] .$$

Thus $z(a')(t) \in P$.

This shows that for all open P in Y , $z(a)(t) \in P$ iff $z(a')(t) \in P$, whence since the induced topology on Y is T_0 , $z(a)(t) = z(a')(t)$. Now it is easy to see that the map $c : V \longrightarrow Y^X$, $t \mapsto z(x_T)(t)$, satisfies

$$z(a)(t) = c(t)(a(t)) .$$

Since z is in $[X_T,Y_T]$ we can find continuous maps $r_i : V_i \longrightarrow [Fil(X),Fil(Y)]$, where the V_i cover V and

$$Ea \cap V_i \subseteq [\![r_i(F'(a)) \subseteq G'(z(a))]\!] ,$$

for any a in X_T . It simplifies matters to take sups and find an r in

$[Fil(X), Fil(Y)]_T$ with $Er = V$ such that

$$Ea \cap V \subseteq [\![r(F'(a)) \subseteq G'(z(a))]\!]$$

for any a in X_T. Now using the strictly local form of " $r(F'(a)) \subseteq G'(z(a))$ ", we can deduce that

$$Ea \cap V \subseteq \{ t \mid r(t)(F'(a(t)) \subseteq G'(c(t)(a(t))) \} .$$

In the first place this shows that for t in V,

$$r(t)F'(x) \subseteq G'(c(t)(x)) ,$$

so that c maps V to $[X,Y]$. Furthermore by taking the obvious projection of $[Fil(X), Fil(Y)]$ to $Fil([X,Y])$ we can find a continuous $r' : V \longrightarrow Fil(X,Y)$, such that for t in V and Φ in $Fil(X)$,

$$r'(t)(\Phi) = r(t)(\Phi) ,$$

(where on the left-hand side we have the usual application of filters). But it is easy to see that for t in V we have $r'(t) \subseteq c\nu(t)$, so we can deduce

$$c\nu(t)(F'(x)) \subseteq G'(c(t)(x)) ,$$

whence c is continuous from V to $[X,Y]$, i.e. c is in $[X,Y]_T$. This completes the proof.

We can now immediately deduce our main theorem on function spaces. Note that it is an analogue for filter spaces of Theorem 6 of §1 for continuous lattices.

THEOREM 2. Let X and Y be filter spaces and suppose the induced topology on Y is T_0. Then the following (internal) filter spaces are isomorphic:

i) $[X_T, Y_T]$ the internal function space of X_T and Y_T (with the usual filter structure determined by continuous convergence) ;

ii) $[X,Y]_T$ the representation of the external function space $[X,Y]$.

Proof. In view of Lemma 1 and §2 it only remains to check that the filter structures coincide. If H is the filter structure on $[X,Y]$, then for (ii), H_T is determined by $H'(c)$, $c \in [X,Y]_T$, where

$$[\![r \in H'(c)]\!] = In \{ t \mid r(t) \in H'(c(t)) \} .$$

For (i) the internal filter structure is determined in just the same way from $H^*(c)$ say where

$$[\![r \in H^*(c)]\!] = [\![(\forall a)(\forall p \in F'(a))[r(p) \in G'(c(a))]]\!]$$

(F, G being the filter structures on X and Y). But the condition determining H^* is strictly local and so we deduce that H' and H^* coincide whence the filter structures coincide.

REMARK I simply do not know for what filter spaces in Sh(T) , a generalization of Theorem 2 , will go through.

5. WHEN ALL FUNCTIONS ARE CONTINUOUS

For the classical mathematician, the most bizarre feature of Brouwer's view of mathematics is his theorem that all functions from reals to reals are continuous (indeed uniformly continuous on compact intervals). However Scott [11] showed that this theorem holds for the Dedekind reals in sheaves over Baire space. Scott's argument can be generalized in a variety of ways. We present some of these generalized "all functions are continuous" results in the next two sections. (Note that the uniformity of the continuity is automatic in spatial topoi - see Hyland [5] , Fourman and Hyland [1] .)

In order to avoid repetition in proofs, we sketch the strategy of the argument in Scott [11] . We are given external spaces X , Y and we wish to show that

$$Sh(T) \models \text{ all maps from } X_T \text{ to } Y_T \text{ are continuous} .$$

1) Given a map z from X_T to Y_T with $Ez = U$, we show that for all t in U , and a , a' in X_T ,

$$\text{if } a(t) = a'(t) \text{ then } z(a)(t) = z(a')(t) .$$

This enables us to define a map $\hat{c} : U \times X \longrightarrow Y$ such that for t in U and a in X_T ,

$$z(a)(t) = \hat{c}(t, a(t)) .$$

2) We show that $\hat{c}$ is (in a suitable sense) continuous. Thus we can identify $\hat{c}$ with a continuous map $c : U \longrightarrow [X,Y]$, where $[X,Y]$ is a suitable space of continuous functions from X to Y .

3) We deduce from 2) that *it holds in Sh(T)* that z is in a suitable sense continuous.

In this section we will be concerned with parts 1) and 2) of this argument; part 3) will be discussed in §6 .

Much of the ensuing argument is in terms of sequential convergence. The reader should be aware of the cartesian closed category of sequential spaces (a coreflective subcategory of topological spaces) and its relation with L-spaces and filter spaces as in Hyland [6] . Johnstone [8] gives much information on sequential convergence in a general setting; he has introduced the name "subsequential space" for the L-spaces of [6] , and we shall adopt this terminology. We continue to write "$x_n \downarrow x$" for "(x_n) converges to x" ; " $\rightarrow$ " is reserved for implication and to indicate mappings.

For our first result, we let T be Hausdorff, first countable, zero-dimensional (i.e. with a basis of clopen sets) and without isolated points. Suppose that X and Y are sequential spaces and that Y is Hausdorff: suppose further that Y satisfies the following condition:

(*) if $y_{nm} \to y_n$ but $y_n \not\to y$, then there is a strictly increasing $m(n)$ with $y_{nm(n)} \not\to y$.

PROPOSITION 1 Given T , X , Y as above, U open in T , a continuous map $c : U \longrightarrow [X,Y]$ (where $[X,Y]$ is the sequential function space) gives rise to a partial sheaf map $\bar{c} : X_T \longrightarrow Y_T$ with extent U , defined by

$$\bar{c}(a)(t) = c(t)(a(t)) ,$$

and every partial sheaf map arises in this way. Thus the sheaf of all functions from X_T to Y_T is isomorphic to $[X,Y]_T$.

Proof. The only problem is to show that any sheaf map $z : X_T \longrightarrow Y_T$ with $Ez = U$ arises as indicated.

For part 1) of the general strategy, let $a(t) = a'(t)$. From the conditions on T we can easily find disjoint open sets V , V' included in $Ea \cap Ea' \cap U$ such that $W = V \cup V' \cup \{t\}$ is open and $t \in C\ell V \cap C\ell V'$. Define $\bar{a}$ with $E\bar{a} = W$, by $\bar{a}\ V = a\ V$, $\bar{a}\ V' = a'\ V'$, $\bar{a}(t) = a(t) = a'(t)$. By construction $\bar{a}$ is in X_T , and

$$t \in C\ell(\mathrm{In}\ \{ s \mid a(s) = \bar{a}(s) \}) \cap C\ell(\mathrm{In}\ \{ s \mid a'(s) = \bar{a}(s) \}) .$$

Now since Y is Hausdorff, we have for general b , b' in X_T ,

$$C\ell(\mathrm{In}\ \{ s \mid b(s) = b'(s) \}) \subseteq \{ s \mid z(b)(s) = z(b')(s) \} ,$$

by the extensionality of z . Thus

$$z(a)(t) = a(\bar{a})(t) = z(a')(t) .$$

Thus we can define $\hat{c} : U \times X \longrightarrow Y$ such that

$$z(a)(t) = \hat{c}(t,a(t)) .$$

For part 2) of the general strategy, suppose there is $x_n \to x$ in X and $t_n \to t$ in T such that $\hat{c}(t_n,x_n) \not\to \hat{c}(t,x)$ in Y . First we show that we may assume that the t_n's are distinct from t . We may assume that the t_n's and x_n's are chosen so that no subsequence of $\hat{c}(t_n,x_n)$ converges to $\hat{c}(t,x)$. If infinitely many t_n's are distinct from t , we get what we want by taking subsequences. Otherwise we may assume the t_n's are all t , i.e. that $\hat{c}(t,x_n) \not\to \hat{c}(t,x)$ while $x_n \to x$. As T has no isolated points we may pick $s_n \to t$ with s_n distinct from t . Now for each n , $\hat{c}(s_m,x_n) \to \hat{c}(t,x_n)$, so we may apply (*) and find

$m(n)$ such that $\hat{c}(s_{m(n)},x_n) \not\to \hat{c}(t,x)$. Setting $t_n = s_{m(n)}$, we have what we want. Now we may take $x_n \to x$, $t_n \to t$, with the t_n's all distinct from t and from each other and with $\hat{c}(t_n,x_n) \not\to \hat{c}(t,x)$. Now (taking subsequences if necessary) we may find disjoint clopen sets U_n included in U , with $t_n \in U_n$ and $t \in U \setminus \bigcup \{U_n | n \in \mathbb{N}\}$. Consider $a : U \longrightarrow X$ defined by

$$a(s) = \begin{cases} x_n & \text{if } s \in U_n \\ x & \text{if } s \in U \setminus \bigcup\{U_n | n \in \mathbb{N}\} \end{cases} .$$

Clearly a is continuous, i.e. a is in X_T ; and $a(t) = x$. Also $z(a)$ is in Y_T , so $z(a)(t_n) \to z(a)(t)$ i.e. $\hat{c}(t_n,x_n) \to \hat{c}(t,x)$, which is a contradiction. Thus $\hat{c}$ is continuous. By adjointness $\hat{c}$ gives rise to a continuous $c : U \longrightarrow [X,Y]$, and clearly z is in $\overline{c}$ as defined in the statement of the Proposition. This completes the proof.

REMARK The condition (*) used in the above proof may seem artificial. But it is satisfied by most interesting examples, and it is inherited by function spaces (if a subsequential space Y satisfies (*) then so does $[X,Y]$ for any subsequential X). We state some particular corollaries of Proposition 1 .

COROLLARY 2 If T is as above, the sheaf model for the finite types over the natural numbers are the sheaves of sequentially continuous "continuous functional" valued functions of the appropriate type. (The continuous functionals are discussed in detail in Hyland [6] .)

COROLLARY 3 If T is as above, (& by Prop. 4 if T is a manifold), the sheaf model for $\mathbb{R}^{\mathbb{R}}$ ($\mathbb{R}$ the Dedekind reals) is the sheaf of continuous $[\mathbb{R},\mathbb{R}]$-valued functions on T . (Here $[\mathbb{R},\mathbb{R}]$ is the space of continuous maps from $\mathbb{R}$ to $\mathbb{R}$ with the compact-open topology which is the sequential space topology.)

For our next result, we suppose that T is (locally) of the form $\mathbb{R} \times S$ where S is some first countable, normal space.

PROPOSITION 4 Given T as above, U open in T , a continuous map $c : U \longrightarrow [R,R]$ gives rise to a partial sheaf map $\overline{c} : R \longrightarrow R$ with extent U , defined by

$$\overline{c}(a)(t) = c(t)(a(t)) ,$$

and every such map arises in this way. Thus the sheaf of all functions from $\mathbb{R}$ to $\mathbb{R}$ is isomorphic to $[\mathbb{R},\mathbb{R}]_T$.

Proof. Again the problem is to show that any partial sheaf map $z : R \longrightarrow R$ with $Ez = U$, arises as indicated.

For part 1) of the general strategy, let $a(t) = a'(t)$. The case when t is in the interior of $\{ t' \mid a(t') = a'(t') \}$ is trivial and need concern us no further. Otherwise, t is in the closure of one of $V = \{t' \mid a(t') > a'(t')\}$ and $V' = \{t' \mid a(t') < a'(t')\}$. Were $t \in C\ell V \cap C\ell V'$, we would be home as we could define

$$\bar{a}(t') = \begin{cases} a(t') & \text{if } a(t') > a'(t') \\ a'(t') & \text{if } a(t') < a'(t') \\ a(t') = a'(t') & \text{otherwise} , \end{cases}$$

and the argument would be as in the proof of Proposition 1 . Now write conventionally $t = (r,s)$, $t' = (r',s')$, $r,r' \in \mathbb{R}$ and $s,s' \in S$, and assume $t \in C\ell V$. Then t must be in the closure of one of $\{t' \mid t' \in V \text{ and } r' < r\}$ and $\{t' \mid t' \in V \text{ and } r' > r\}$. Assume without loss of generality that it is in the former. Define

$$a''(r',s') = \begin{cases} a'(r',s') & \text{if } r' \leq s \\ a'(r',s') + a(r',s) - a'(r',s) + (r'-r) & \text{if } r' \geq r \end{cases} .$$

Then (a) $t \in C\ell \operatorname{In} \{t' \mid a'(t') = a''(t')\}$, so that $z(a')(t) = z(a'')(t)$, and (b) $t \in C\ell \{t' \mid a(t') > a''(t')\} \cap C\ell \{t' \mid a(t') < a''(t')\}$, so that by an argument above $z(a')(t) = z(a'')(t)$. Hence $z(a)(t) = z(a')(t)$. This completes the argument for all cases, so we can define $c : U \times R \longrightarrow R$ such that

$$z(a)(t) = c(t,a(t)) .$$

For part 2) of the general strategy, the argument proceeds as in the proof of Proposition 1 , until one comes to extend a continuous map on a convergent sequence. Then we make use of the Tietze extension theorem. The proof is then completed as Proposition 1 .

REMARK Recently Grayson has made a civilized generalization of Proposition 4 to first countable completely regular spaces.

6. ALL FUNCTIONS ARE CONTINUOUS (INTERNALLY)

In this final section we investigate cases when our sheaf models satisfy "all functions are continuous". We concentrate on the case which we considered first in §5 , that is the case when T is Hausdorff, first countable, zero-dimensional and without isolated points.

A number of questions immediately arise from consideration of Proposition 1 of §5 :

i) does it hold that every map is sequentially continuous ?

ii) does it hold that every map is topologically continuous ?

iii) is either the sequential convergence or topology on $[X,Y]_T$ internally definable from the (internal) topological spaces X_T and Y_T ?

iv) what connection (if any) is there between the material of the final two sections of this paper and the earlier sections ?

First let us see what we can say about the relation $\downarrow$ of sequential convergence on an internal topological space of the form X_T : we have

$$a_n \downarrow a \quad \text{iff} \quad (\forall \text{ open } O)(a \in O \rightarrow (\exists m)(\forall n \geq m) \; a_n \in O) \; .$$

LEMMA 1 Let $U \subseteq T$ be an open set included in the extent of $a \in X_T$ and $a_n \in (X_T)^{\mathbb{N}}$. Let $[a_n]$ be the Fréchet filter determined by the a_n's restricted to U and for $t \in U$ let $\nu(t)$ be the neighbourhood filter of t within U . Then,

$$U \subseteq [\![a_n \downarrow a]\!] \quad \text{iff} \quad \text{for all } t \in U , \; [a_n](\nu(t)) \text{ converges to } a(t) \; .$$

(More sloppily and memorably, $[\![a_n \downarrow a]\!] = \text{In} \{ t' \mid [a_n](\nu(t)) \downarrow a(t) \}$.)

Proof. First suppose $U \subseteq [\![a_n \downarrow a]\!]$ and $t \in U$, and consider an open set O containing $a(t)$. Then $t \in [\![a \in O_T]\!]$, so we have,

$$t \in U \cap a^{-1}(O) \subseteq [\![(\exists n)(\forall m \geq n) \; a_m \in O_T]\!] \quad .$$

Thus we can pick an n so that

$$t \in [\![(\forall m \geq n) \; a_m \in O_T]\!] = W \text{ , say } .$$

Then clearly,

$$\{ a_m \mid m \geq n \} (W) \subseteq O \; ,$$

i.e. $O \in [a_n](\nu(t))$ as required.

Conversely, suppose $[a_n](\nu(t))$ converges to $a(t)$ for $t \in U$. If O open containing $a(t)$, then there is n and open $W \subseteq U$ with $t \in W$, such that $\{ a_m \mid m \geq n \} (W) \subseteq O$. Thus

$$t \in W \subseteq [\![(\forall m \geq n) \; a_m \in O_T]\!] \; .$$

But since (the restrictions of) the O_T's form a basis for the topology on X_T , this is enough to show $U \subseteq [\![a_n \downarrow a]\!]$.

COROLLARY 2 If T is first countable, then $U \subseteq [\![a_n \downarrow a]\!]$ iff whenever $t_n \downarrow t$ in T , then $a_n(t_n) \downarrow a(t)$.

PROPOSITION 3 Under the conditions of Proposition 1 of §5 , $\models$ every map from X_T to Y_T is sequentially continuous.

Proof. This follows directly from Lemma 1 and Proposition 1 of §5 .

It is a routine matter to check that the basic theory of sequential and subsequential spaces (definitions as in Johnstone [8]) is constructive. In particular, given sequential continuity, topological continuity of the maps from X_T to Y_T will follow if X_T is a sequential space. However this is not generally the case. We recall an example from §7 of Hyland [6] . The space C_1 is Baire space and C_2 the space of continuous functions from C_1 to the natural numbers with the discrete topology. There is a set $O \subseteq C_1 \times C_2$ such that O is clopen in the sequential topology on $C_1 \times C_2$ but includes no non-empty $O_1 \times O_2$ with O_1 open in C_1 and O_2 open in C_2 (see (7.3) and (11.1) of Hyland [6]). Take T to be C_1 , X to be C_2 and Y to be the two point discrete space. Then this example provides a map from X_T to Y_T which by Proposition 3 is sequentially continuous but by Fourman and Scott is not topologically continuous. This example also provides sequentially open subsets of X_T which are not open (take the inverse image of an element of Y_T).

The following rather weak result does give some cases when it does hold that every map is topologically continuous.

PROPOSITION 4 Suppose the conditions of Proposition 1 of §5 are satisfied and furthermore X is first countable. Then, $\models$ every map from X_T to Y_T is topologically continuous.

Proof. The partial maps are represented by sequentially continuous maps $U \times X \longrightarrow Y$. But under the given conditions $U \times X$ is sequential, so the maps will all be topologically continuous. The result now follows from Fourman and Scott [2].

Given the facts sketched above, it is implausible that the topology on $[X,Y]_T$ could be internally defined from that on X_T and Y_T . For the relation of sequential convergence, there is the following result, whose proof is simple, except for a step much like the proof of Proposition 1 of §5 , and which is therefore omitted.

PROPOSITION 5 In the circumstances of Proposition 1 of §5 , the relation of sequential convergence on $[X,Y]_T$ can be characterized by

$$[\![c_n \downarrow c]\!] = [\![(\forall x_n \downarrow x) \;\; c_n(x_n) \downarrow c(x)]\!] .$$

(Here we have identified $c \in [X,Y]_T$ with the corresponding sheaf map $\overline{c}$.)

A connection between the work on sequential convergence and that on filter spaces is given by §9 of Hyland [6] . The material there is constructive once axioms of countable and dependent choices are given. Thus we have the following result :

PROPOSITION 6 Suppose that T is (locally) second countable but that otherwise we are in the situation of Proposition 1 of §5 ; we may give X and Y the countably-generated filter structures described in §9 of Hyland [6] ; X_T and Y_T are then the underlying sheaves of internal filter spaces as defined in §2 ; and then

$\models$ all maps from X_T to Y_T are filter continuous

COROLLARY 7 For T as in Proposition 6 , the finite types over the natural numbers are represented by the sheaves of continuous "continuous functional"-valued functions of appropriate type (see Hyland [6]) ; and

$\models$ all maps between these types are sequentially and filter continuous.

REMARK A result similar to Corollary 7 holds for finite types over the Dedekind reals. The argument from Proposition 4 to internal continuity is given in essence in Scott [11] . So we state,

PROPOSITION 8 If T is as in Proposition 4 of §5 , then

$\models$ all maps from $\mathbb{R}$ to $\mathbb{R}$ are continuous, (in any of the many equivalent senses) and in fact uniformly continuous on closed bounded intervals.

It remains to state the following:

CHALLENGE Constructivize the results of §§5 and 6 .

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