First Steps in Synthetic Domain Theory

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1 Introduction

1.1 Aims of synthetic domain theory

Domain theory is the study of various concrete categories **C** typically of (directed) complete partial orders in which constructions fundamental to the analysis of computing can be performed.

- One can take fixed points of endofunctions $X \to X$, so as to give meaning to functions defined by general recursion.
- One can find fixed points for various operations $\mathbf{C} \to \mathbf{C}$ in order to provide interpretations for recursively defined types.

Domain theory is a well developed body of mathematics, in which the stress is on limits of increasing sequences. Continuity serves in the theory as a substitute for effectivity.

This paper is concerned with an approach to programming semantics of a different flavour. The motivating slogan is "domains are sets". (More exactly but less memorably, "domains are certain kinds of constructive sets".) An investigation of the general kind presented here was proposed by Dana Scott in a talk at a meeting of the Peripatetic Seminar on Sheaves and Logic in Sussex in 1980. He had in mind the example of Synthetic Differential Geometry where generalized manifolds are treated as (special kinds of) sets with the result that the development of the basic theory becomes highly intuitive: and he asked for a treatment of domain theory in a similar spirit. Initial progress was slow, but it now appears that the major conceptual advance was made by Scott's student Rosolini in his thesis [10]. There he took the recursively enumerable subobject classifier Σ as the lynch-pin of the theory. Rosolini also described axioms in the internal logic which unified features of the effective and recursive toposes, and took the first steps towards formulating a synthetic theory of computation. (Mention should also be made in this context of Lawvere's student Mulry who had identified the recursively enumerable subobject classifier, in the course of his study of the recursive topos, see [6].) In a theory of computation, effectivity must be analyzed directly. The category theoretic approach is to consider intrinsic structure of objects within a category (see the discussion in [7]); and as envisaged by Scott and Rosolini, this exploits the analogy with continuity. The result is the development of a kind of coding-free recursion theory, and it is this that is stressed in this paper.

1.2 Scope of the paper

The study of Synthetic Differential Geometry has two complementary aspects: a synthetic (or axiomatic) aspect and a semantic aspect (involving categorical models), and so it is also with Synthetic Domain Theory. In this paper the emphasis is on the first of these two aspects. The approach taken is to attempt to axiomatize effectivity in a categorical framework by means of properties of a classifier Σ of 'semi-decidable' subsets. (One might regard Σ as playing the same central role in Synthetic Domain Theory as the synthetic line R does in Synthetic Differential Geometry.)

The reader should perhaps be aware of limitations on the scope of the enquiry. We only reach the first aspect of domain theory: the existence of fixed points of endofunctions. Furthermore the word domain is used only in the general sense associated with Scott (though *not* as synonomous with what are usually called Scott domains) as distinct from the sense associated with Berry. This has a number of consequences. Computations may run in parallel and so the 'semi-decidable subsets' are closed under unions (Axiom 7). Also, for good objects the intrinsic order on function spaces is pointwise. Properties of Scott's topological notion of domain are reflected in all of the paper from section 3 onwards, though some aspects of the later sections are more widely applicable. Perhaps this is a drawback of the presentation; what we have is far from being an axiomatization of domains in the general sense. It remains a challenging problem to produce a synthetic theory of (the various flavours of) stable domains.

1.3 Background assumptions

We shall work within a *non-trivial* category of sets S, and need to state what we assume about this category. Of course one can assume that it is a topos, but the full structure of a topos will not play any part in our axiomatization. Hence it seems important to give some more precise indication of the kind of situation which one should have in mind. We assume that we work in a category with properties enjoyed by (amongst others) the category of modest sets within the effective topos.

In addition to the tacit assumption of non-triviality, the assumptions which we have in mind are as follows:

1. S is a locally cartesian closed subcategory of a topos $\mathcal{E}$ (in the internal sense);
2. S has a natural number object N;
3. S is the category of separated objects for a topology j on a subpretopos of the topos;
4. S is a small category contained within and complete relative to the category of j-separated objects of the topos.

A discussion of (1) for the particular case of the modest sets in the effective topos can be found in Hyland [2]. It seems best in view of some applications not to assume in (2) that N is the natural number object in the ambient topos. But we have the usual structure

$$1 \xrightarrow{0} \mathsf{N} \xrightarrow{s} \mathsf{N}$$

satisfying Lawvere's 'initial algebra' universal property. And in view of (3) we may assume Freyd's formulation of a natural number object in terms of a coproduct $1+\mathsf{N} \cong \mathsf{N}$ and coequalizer $\mathsf{N} \rightrightarrows \mathsf{N} \to 1$. As regards (3), Carboni and Mantovani have characterized the categories which arise as j-separated objects for a topology j on a pretopos. Such a category

- is regular (in the logical sense - it has finite limits and stable (regular epi, mono) factorization);
- has stable (epi, regular mono) factorization;
- has finite stable coproducts;
- has all 'quotients' of equivalence relations and these are quasi-effective (that is, the natural map from an equivalence relation to the kernel pair of its quotient is an epimorphism).

There is some analysis of (4) in Hyland, Robinson, Rosolini [3] and a fuller discussion in Robinson [9]. The full force of this assumption does not seem necessary for the theory developed here; and it does not hold in some of our models. It is important however if we wish to model strongly polymorphic type systems.

We give some examples of toposes in which we can find at least some of our background assumptions satisfied, and which support models for much of the synthetic domain theory which we shall describe.

1. Suitable realizability toposes. It seems that these may not all model the full theory which will be described below, but many do. For example
   - The effective topos. (Historically this was the motivating example.)
   - Toposes based on domain theoretic models for the lambda calculus.
2. Some toposes based on other notions of functional interpretation (modified realizability, Dialectica Interpretation). I have only made "back of an envelope" calculations in these cases, but they should be a good source of counter examples.
3. Topological toposes. These include
   - Johnstone's topological topos see [4].
   - Scott toposes in the general sense indicated in [10].
4. Recursive versions of topological toposes. The recursive topos of Mulry. (This was the first topos in which an r.e. subobject classifier was identified.) (The natural example is analogous to the simplest "well-adapted" models for synthetic differential geometry.

1.4 Conventions and notation

We shall use the usual set-theoretic language appropriate within (pre-)toposes, and refer to subobjects as subsets and to regular epis as surjections. A pretopos also provides good notions of (finite) intersections and unions of subobjects; and we suppose enough completeness to give us small (internal) intersections and unions. Note however that the existence of two factorization systems already adds a nuance to the set theory which reflects the j-modality. The regular monomorphisms are j-closed subobjects, while the epimorphisms are j-dense maps.

We mention a few standard category theoretic conventions. If A is an object of a category (with terminal object), then we shall write $A: A \to 1$ for the unique map to the terminal object 1. The identity map on A is denoted by $1_A: A \to A$, and the subscript will be dropped wherever possible. Also we use some non-standard notation. If

$a: 1 \to A$ is a point or element of A, then I shall write $k_a: X \to A$ for the map of the form $X \xrightarrow{X} 1 \xrightarrow{a} A$. Finally if $u: A \to C$ and $v: B \to C$ are maps, we write $[u, v]: A + B \to C$ for the induced map from the coproduct.

1.5 Acknowledgements

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2 Basic theory of partial maps

2.1 Semi-decidable or Σ-subsets

We follow Rosolini in making the notion of a Σ-subobject of an object the cornerstone of the theory. These Σ-subobjects should be thought of as recursively enumerable or (better) semi-decidable subsets, and we shall refer to them as Σ-subsets. Our first assumption is that this notion is classified.

Assumption 1 *We assume that we have an object Σ equipped with a subobject $t: 1 \to \Sigma$. The pullbacks of $t: 1 \to \Sigma$ are called Σ-subsets, and we assume that $t: 1 \to \Sigma$ is a generic Σ-subset in the sense that any Σ-subset $A \to X$ appears in a pullback diagram*

$$\begin{tikzcd} X \arrow[r, "a"] & \Sigma \\ A \arrow[u, tail] \arrow[r] & 1 \arrow[u, tail, "t"'] \end{tikzcd}$$

for a unique map $a: X \to \Sigma$. We shall write $A \subseteq_\Sigma X$ for A is a Σ-subset of X.

This assumption has the obvious consequences.

Proposition 2.1.1 *The collection of Σ-subsets satisfies the basic closure properies:*

(i) *For any object X, $X \subseteq_\Sigma X$, that is the maximal subset $1: X \to X$ is a Σ-subset.*

(ii) *The pullback of a Σ-subset is a Σ-subset.*

Note that the object Σ^X internalises the notion of a Σ-subset of X; its global sections correspond to the Σ-subsets.

2.2 The lift functor

Now we can exploit the background assumption that S is locally cartesian closed to obtain from the classifying map $1 \to \Sigma$ a further piece of structure, namely a Σ-partial map classifier. In the usual way, we let $\perp(X) \to \Sigma$ be $\Pi_t(X \to 1)$ and obtain a pullback

$$\begin{tikzcd} \perp(X) \arrow[r] & \Sigma \\ X \arrow[u, "\eta_X"] \arrow[r] & 1 \arrow[u, "t"'] \end{tikzcd}$$

The map $\eta = \eta_X : X \to \perp(X)$ classifies partial maps whose domain is a Σ-subset. We refer to these as *Σ-partial maps*; we let $X \rightharpoonup_\Sigma Y$ denote a Σ-partial map from X to Y. If $A \subseteq_\Sigma X$ and $u: A \to Y$ is a Σ-partial map $X \rightharpoonup_\Sigma Y$, then there is a unique map $\bar{u}: X \to \perp(Y)$ such that

$$\begin{tikzcd} X \arrow[r, "\bar{u}"] & \perp(Y) \\ A \arrow[u] \arrow[r] & Y \arrow[u, "\eta_Y"'] \end{tikzcd}$$

is a pullback. In particular, $\perp(1) \cong \Sigma$ and $\eta: 1 \to \perp(1)$ corresponds to $t: 1 \to \Sigma$. The object $\perp(X)$ is called *the lift* of X and is sometimes written $X_\perp$.

$\perp$ extends to a functor in a natural way. If $u: X \to Y$ then $\perp(u) = u_\perp$ classifies the Σ-partial map $X_\perp \rightharpoonup_\Sigma Y$ defined on $X \subseteq_\Sigma X_\perp$ as u. Furthermore it is easy to see that $\perp$ is an S-functor. Finally note that the lift functor has familiar preservation properties.

Proposition 2.2.1 *The lift functor $\perp$ preserves connected limits in both the external and internal senses.*

We can of course iterate the lift functor, and so in particular can obtain a sequence of objects Σ_n defined by

$$\Sigma_0 = 1; \quad \Sigma_{n+1} = \perp(\Sigma_n).$$

Clearly the sequences of maps

$$1 \xrightarrow{\eta} \Sigma \xrightarrow{\eta} \Sigma_2 \ldots \Sigma_{n-1} \xrightarrow{\eta} \Sigma_n$$

classify sequences of subobjects

$$A_0 \subseteq_\Sigma A_1 \subseteq_\Sigma A_2 \ldots A_{n-1} \subseteq_\Sigma X$$

of an object X where each A_i is a Σ-subset of A_{i+1}.

2.3 The lift monad

Our first axiom states the closure of (representatives for) Σ-subsets under composition.

Axiom 2 *If A is a Σ-subset of B and B is a Σ-subset of X, then A is a Σ-subset of X.*

There is an immediate consequence.

Proposition 2.3.1 *The collection of Σ-subsets of an object is closed under (finite) intersection.*

Corollary 2.3.2 *If $A \subseteq_\Sigma B$, then Σ^A is a retract of Σ^B.*

In the language of Rosolini [10], Axiom 2 says that the Σ-subsets form a dominance classified by $t: 1 \to \Sigma$. Axiom 2 has a useful alternative formulation in terms of the lift.

Proposition 2.3.3 *Given the classifying map $t: 1 \to \Sigma$ for Σ-subsets, the following are equivalent*

(i) *Σ-subsets form a dominance, that is Axiom 2 holds;*

(ii) *the composite $(1 \xrightarrow{t} \Sigma \xrightarrow{\eta} \Sigma_\perp)$ is a Σ-subset;*

(iii) *there is a (necessarily unique) natural transformation $\mu: (\perp)^2 \to \perp$ such that $(\perp, \eta, \mu)$ is a monad.*

Externally we have the usual 'subset ordering' or *inclusion ordering* on the Σ-subsets of X. Suppose that $a: X \to \Sigma$ and $b: X \to \Sigma$ classify $A \subseteq_\Sigma X$ and $B \subseteq_\Sigma X$ respectively. The inclusion relation induces a relation on classifying maps: if $A \subseteq B$, then we write $a \subseteq_\Sigma b$.

It is a consequence of Axiom 2 that we can identify in **S** a monic representing the subset order on Σ. We denote such a subobject of Σ^2 (which automatically exists in an ambient topos) by

$$\supseteq \rightarrow \Sigma^2 \quad \text{or} \quad \subseteq \rightarrow \Sigma^2$$

as appropriate. (It is convenient notationally to distinguish between the subobjects $\subseteq$ and $\supseteq$ of Σ^2 which are isomorphic via the twist map.) We have two maps $\perp(\Sigma): \Sigma_\perp \to \Sigma$ and $\mu_1: \Sigma_\perp \to \Sigma$ and these induce a map $(\mu_1, \perp(\Sigma)): \Sigma_\perp \to \Sigma \times \Sigma$. It is a further consequence of Axom 2 that intersection is also represented in **S**. We let $\cap: \Sigma \times \Sigma \to \Sigma$ be the classifier of the Σ-subset $(t, t): 1 \to \Sigma \times \Sigma$.

Proposition 2.3.4 (i) *A map $(a, b): X \to \Sigma^2$ factors through $(\mu_1, \perp(\Sigma))$ if and only if $a \subseteq_\Sigma b$;*

(ii) *$(\mu_1, \perp(\Sigma)): \Sigma_\perp \to \Sigma^2$ represents the 'subset order' $\subseteq$ on Σ;*

(iii) *$(\mu_1, \perp(\Sigma)): \Sigma_\perp \to \Sigma^2$ is the equalizer of the maps $\mathrm{fst}, \cap: \Sigma^2 \to \Sigma$*

(iv) *$(\Sigma, \top, \subseteq, \cap)$ forms a (meet) semilattice.*

Finally it is easy to see that the monad $(\perp, \eta, \mu)$ is strong. Hence the second part of 2.3.5 follows from the first.

Proposition 2.3.5 (i) *$(\Sigma_\perp \xrightarrow{\mu} \Sigma)$ is the free $\perp$-algebra on 1.*

(ii) *Each Σ^X has the structure of a $\perp$-algebra.*

2.4 The intrinsic order

The subset ordering on $\Sigma^{(\Sigma^X)}$ gives rise to a preorder on any object X of **S**.

Definition 1 The *intrinsic pre-order* on an object X is the relation $\leq$ which appears in the pullback

$$\begin{tikzcd} X^2 \arrow[r] & (\Sigma^2)^{(\Sigma^X)} \\ \leq \arrow[u] \arrow[r] & \subseteq^{\Sigma^X} \arrow[u] \end{tikzcd}$$

Using the internal logic, the definition says that

$$S \models (x \le y) \iff (\forall R \in \Sigma^X . x \in R \Rightarrow y \in R).$$

Generally the pointwise preorder on function spaces need not coincide with the intrinsic preorder (see the discussion in [8]). We define the *pointwise preorder* $\preceq$ on a function space B^A in the internal logic by stipulating that

$$S \models (f \preceq g) \iff (\forall a \in A . f(a) \le g(a)).$$

The intrinsic preorder on an object is analogous to the topological specialization order used in algebraic geometry. (Indeed the latter is a special case of the former.) Properties of the intrinsic preorder are most rapidly established using the internal logic. We now state a number of properties which we shall need and whose proofs are quite trivial in these terms. (I cannot resist remarking however that the intrinsic order seems to be a distraction. I would rather avoid reference to it where at all possible, and am distressed at my failure to do so more successfully.)

We shall eventually have an axiom ensuring that the intrinsic preorder on an object of the form Σ^A coincides with the inclusion ordering. One entailment is however both automatic and easy.

Proposition 2.4.1 *$R \le S$ entails $R \subseteq S$ in Σ^A.*

Proposition 2.4.2 *Internally any function preserves the intrinsic preorder:*

$$S \models \forall f \in B^A \, \forall a, c \in A (a \le c \Longrightarrow f(a) \le f(c)).$$

(And hence externally functions preserve the intrinsic order.)

Proof Essentially clear as f induces a map $\subseteq^{\Sigma^X} \rightarrow \subseteq^{\Sigma^Y}$. □

Corollary 2.4.3 *The intrinsic preorder on a product is the product of the intrinsic preorders.*

Note that the pointwise preorder $\preceq$ has a simple alternative characterization.

Proposition 2.4.4 *We have the following in the internal logic:*

$$S \models \forall f, g \in B^A (f \preceq g \iff (\forall S \in \Sigma^B) f^{-1}(S) \subseteq g^{-1}(S)).$$

Preorders are rather a bore. It is convenient to work with objects for which the intrinsic preorder is in fact an order.

Definition 2 For any object X there is a natural map $X \rightarrow \Sigma^{\Sigma^X}$. An object X is a Σ-*space* just when this map is a monic. X is *extensional* just when it is a regular monic.

Proposition 2.4.5 *If X is a Σ-space then the intrinsic preorder is an order.*

In what follows we shall often tacitly assume that we are working with Σ-spaces.

2.5 Objects with bottom

There are clearly a number of distinct notions of an object with a least or bottom element in the intrinsic order. In conformity with the 'algebraic spirit' of the categorical axiomatization, it seems best to work with a rather strong notion of an object equipped with a bottom element. As a conceit I introduce the definition before Axioms 3 and 6 which ensure that there really are bottom elements in these objects.

Definition 3 An object X *(equipped) with bottom* is an algebra $\perp(X) \to X$ for the lift monad. A *strict map* between such objects is a map of algebras.

Note that as we have presented the definition, having a bottom element is an additional piece of structure. However if we restrict attention to Σ-spaces, there is at most one such stucture on any X, so we may regard having a bottom element as a property. Since we may tacitly assume that the objects with which we are dealing are Σ-spaces, we shall not make any fuss over this.

There is considerable interest in the free algebras for the lift monad, that is, in objects which are themselves lifts of some object. We say that such an object $A_\perp$ is *a lift*. We already know that maps into such an object classify Σ-partial maps. But as $\perp^2(A) \xrightarrow{\mu} \perp(A)$ is a free $\perp$-algebra, we also know about strict maps from such an object into objects with bottom. In principle we want to regard these maps as maps preserving $\perp$, but clearly one has to say this in a positive way as one cannot yet identify the bottom elements! One particular case of interest is that of maps to an object of the form Σ^X.

Lemma 2.5.1 *Suppose that $W \subseteq_\Sigma X \times \perp(A)$. Then the diagram*

$$\begin{tikzcd} X \times \perp^2(A) \arrow[r, "1 \times \mu_A"] & X \times \perp(A) \\ W \arrow[u, tail] \arrow[r, equal] & W \arrow[u, tail] \end{tikzcd}$$

is a pullback if and only if $W \subseteq X \times A$.

Proposition 2.5.2 *Take a map $u: \perp(A) \to \Sigma^X$ and let $U \subseteq_\Sigma X \times \perp(A)$ be the subobject classified by the transpose $\bar{u}: X \times \perp(A) \to \Sigma$ of u. Then u is strict if and only if $U \subseteq X \times A$.*

2.6 The empty set

As things stand it is possible that $\Sigma = 1$ and that the collection of Σ-subsets is trivial. The next axiom ensures that this is not the case.

Axiom 3 *For any object X, the empty set is a Σ-subset of X.*

Proposition 2.6.1 *The following are equivalent:*

(i) *For any object X, the empty set is a Σ-subset of X.*

(ii) *The empty set is a Σ-subset of 1.*

(iii) *There is a map $f: 1 \to \Sigma$ distinct from $t: 1 \to \Sigma$ in the sense that the diagram*

$$\begin{tikzcd} 1 \arrow[r, "t"] & \Sigma \\ 0 \arrow[u] \arrow[r] & 1 \arrow[u, "f"'] \end{tikzcd}$$

is a pullback.

One simple consequence of Axiom 3 is that decidable subsets are semi-decidable. As usual a *decidable* subset of an object X is a subobject $A \subseteq X$ which is complemented: there is an isomorphism $A + B \cong X$ inducing $A \subseteq X$.

Proposition 2.6.2 *Decidable subsets are Σ-subsets.*

Proof Decidable subsets are classified by maps into $2 = 1+1$. As we have the pullback

$$\begin{tikzcd} 2 \arrow[r, "{[t, f]}"] & \Sigma \\ 1 \arrow[u] \arrow[r, equal] & 1 \arrow[u] \end{tikzcd}$$

they are also classified by maps into Σ. □

On the basis of Axiom 3 we start to get some recognizable structure. Note first that $\perp(0) = 1$ and the map $f: 1 \to \Sigma$ is equal to $\perp(0 \to 1)$. The diagram

$$\begin{tikzcd} \Sigma \arrow[r, "f_\perp"] & \Sigma_\perp \\ 1 \arrow[u, "t"] \arrow[r] & \Sigma \arrow[u, "\eta"'] \end{tikzcd}$$

is automatically a pullback. Thus we can identify three distinct points of $\Sigma_\perp$:

- the point $(\eta_\Sigma) \cdot t$ of $\Sigma_\perp$ which is the classifying map for $1 \subseteq_\Sigma 1 \subseteq_\Sigma 1$;
- the point $\eta \cdot f = (f_\perp) \cdot t$ of $\Sigma_\perp$ which is the classifying map for $0 \subseteq_\Sigma 1 \subseteq_\Sigma 1$;
- the point $\perp(0 \to \Sigma)$ of $\Sigma_\perp$ which is the classifying map for $0 \subseteq_\Sigma 0 \subseteq_\Sigma 1$.

The two maps $\perp(\Sigma): \Sigma_\perp \to \Sigma$ and $\mu_1: \Sigma_\perp \to \Sigma$ are clearly distinct as they classify $\Sigma \subseteq_\Sigma \Sigma_\perp$ and $1 \subseteq_\Sigma \Sigma_\perp$ respectively. What's more, we can calculate the pullbacks of f along these maps. We get

$$\begin{tikzcd} \Sigma_\perp \arrow[r, "\perp(\Sigma)"] & \Sigma \\ 1 \arrow[u] \arrow[r, equal] & 1 \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Sigma_\perp \arrow[r, "\mu_1"] & \Sigma \\ \Sigma \arrow[u] \arrow[r] & 1 \arrow[u] \end{tikzcd}$$

where $1 \to \Sigma_\perp$ is $\perp(0 \to \Sigma)$ and $\Sigma \to \Sigma_\perp$ is $\perp^2(0 \to 1)$. The left hand diagram is a pullback because $\perp$ preserves pullbacks. For the right hand one, note that $X \xrightarrow{u} \Sigma_\perp \xrightarrow{\mu} \Sigma$ factors through f if and only if u classifies subobjects $0 \subseteq_\Sigma U \subseteq_\Sigma X$; and that this happens if and only if $u = (\perp f) \cdot (\perp \Sigma) \cdot u$.

The reader will now readily see that the collection of objects $\Sigma_0, \Sigma_1, \ldots.$ and maps between them constructed from the maps t, f, μ by pullbacks and composition mirrors exactly the simplicial category of all nonempty finite ordinals and order-preserving maps.

3 Some dual structure

3.1 Co-Σ-subsets and the co-lift functor

Further structure of Σ can be conveniently described in terms of a notion of co-Σ-subset dual to that of Σ-subset. We assume that $f : 1 \to \Sigma$ classifies these subsets which we think of as the complements of semi-decidable subsets.

Assumption 4 *Call the pullbacks of $f : 1 \to \Sigma$ co-Σ-subsets. We assume that $f : 1 \to \Sigma$ is a generic co-Σ-subset in the sense that any co-Σ-subset A of X appears in a pullback diagram*

$$\begin{tikzcd} X \arrow[r, "a"] & \Sigma \\ A \arrow[u] \arrow[r] & 1 \arrow[u, "f"'] \end{tikzcd}$$

for a unique $a : X \to \Sigma$. We shall write $A \subseteq_{\bar{\Sigma}} X$ for A is a co-Σ-subset of X.

Note that now Σ^X also internalises the notion of co-Σ-subset. In fact more is true.

Proposition 3.1.1 *There is a bijective correspondence between Σ-subsets and co-Σ-subsets, where a co-Σ-subset of X corresponds to the Σ-subset with the same classifying map $X \to \Sigma$. This correspondence reverses the inclusion orders on co-Σ-subsets and Σ-subsets.*

Note that Axiom 3 enables us to 'see' the bottom element in a lift $A_\perp$: it is the map $k_\perp : 1 \to A_\perp$ lying in the unique pullback of form

$$\begin{tikzcd} 1 \arrow[r] & A_\perp \\ 0 \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

As a consequence of Proposition 3.1.1 one can deduce a more intuitive form of Proposition 2.5.2.

Proposition 3.1.2 *A map $u : \perp(A) \to \Sigma^X$ is strict if and only if $u \cdot k_\perp : 1 \to \Sigma^X$ names $\emptyset \subseteq_\Sigma X$. (That is, u is strict if and only if it preserves $\perp$.)*

From the classifying map $f: 1 \to \Sigma$ we can obtain structure dual to that of 2.2. Thus there is a co-Σ-partial map classifier. This consists of a map $\zeta: X \to \mathsf{T}(X)$ such that if $A \subseteq_\Sigma X$ and $u: A \to Y$, then there is a unique map $\bar{u}: X \to \mathsf{T}(Y)$ such that

$$\begin{tikzcd} X \arrow[r, "\bar{u}"] & \mathsf{T}(Y) \\ A \arrow[u] \arrow[r, "u"'] & Y \arrow[u, "\zeta"'] \end{tikzcd}$$

is a pullback. The object $\mathsf{T}(X)$ is called *the co-lift* of X and is sometimes written X_{T}. The co-lift T extends to an S-functor; and T enjoys the standard preservation property.

Proposition 3.1.3 *The co-lift functor T preserves connected limits in both the external and internal senses.*

As things stand, there is some small interaction between the lift and co-lift. We have $\mathsf{T}(1) \cong \Sigma \cong \perp(1)$. However if we start iterating T there is no reason yet to suppose that we shall obtain a sequence of objects (and maps) isomorphic to the Σ_ns (and the face and degeneracy maps) of 2.6.

3.2 The co-lift monad

Next we introduce an axiom dual to Axiom 2.

Axiom 5 *If A is a co-Σ-subset of B and B is a co-Σ-subset of X, then A is a co-Σ-subset of X.*

As in the case of Axiom 2 there is an immediate consequence.

Proposition 3.2.1 *The collection of co-Σ-subsets of an object is closed under (finite) intersection.*

Corollary 3.2.2 *If $A \subseteq_\Sigma B$, then Σ^A is a retract of Σ^B.*

Furthermore as co-Σ-subsets form a dominance classified by $f: 1 \to \Sigma$, the duals of Propositions 2.3.3, 2.3.4 and 2.3.5 hold.

Proposition 3.2.3 *Given that $f: 1 \to \Sigma$ classifies co-Σ-subsets, the following are equivalent*

(i) *co-Σ-subsets form a dominance, that is, Axiom 5 holds;*

(ii) *the composite $(1 \xrightarrow{f} \Sigma \xrightarrow{\zeta} \Sigma_{\mathsf{T}})$ is a co-Σ-subset;*

(iii) *there is a (necessarily unique) natural transformation $\nu: (\mathsf{T})^2 \to \mathsf{T}$ such that (T, ζ, ν) is a monad.*

We have already seen in 3.1.1 that the bijective correspondence between Σ-subsets and co-Σ-subsets reverses the (external) inclusion orderings. Let $a: X \to \Sigma$ and $b: X \to \Sigma$ classify $A \subseteq_\Sigma X$ and $B \subseteq_\Sigma X$ respectively; then we have $A \supseteq B$ if and only if $a \subseteq_\Sigma b$. Thus there is essentially just one inclusion ordering on Σ. Using Axiom 5 we can find a representation of the subset order on Σ dual to that of Proposition 2.3.4. We have two maps $\mathsf{T}(\Sigma): \Sigma_{\mathsf{T}} \to \Sigma$ and $\nu_1: \Sigma_{\mathsf{T}} \to \Sigma$ and these induce a map $(\mathsf{T}(\Sigma), \nu_1): \Sigma_{\mathsf{T}} \to \Sigma \times \Sigma$. It is a further consequence of Axiom 5 that intersection of co-Σ-subsets is also represented in S. However this does not necessarily correspond to *union* of Σ-subsets. Hence we write $\vee: \Sigma \times \Sigma \to \Sigma$ for the classifier of the co-Σ-subset $(f, f): 1 \to \Sigma \times \Sigma$.

Proposition 3.2.4 (i) *A map* $(a,b)\colon X \to \Sigma^2$ *factors through* $(\top(\Sigma),\nu_1)$ *if and only if* $a \subseteq_\Sigma b$;

(ii) $(\top(\Sigma),\nu_1)\colon \Sigma_\top \to \Sigma^2$ *represents the 'subset order'* $\subseteq$ *on* Σ;

(iii) $(\top(\Sigma),\nu_1)\colon \Sigma_\top \to \Sigma^2$ *is the equalizer of the maps* $\mathrm{snd}, \vee\colon \Sigma^2 \to \Sigma$;

(iv) $(\Sigma,\bot,\subseteq,\vee)$ *forms a (join) semilattice.*

For completeness note that the monad $(\top,\zeta,\nu)$ is strong, and we have the dual of 2.3.5.

Proposition 3.2.5 (i) $(\Sigma_\top \xrightarrow{\nu} \Sigma)$ *is the free* $\top$*-algebra on* 1.

(ii) *Each* Σ^X *has the structure of a* $\top$*-algebra.*

As a consequence of 3.2.4 we have an isomorphism $\top(\Sigma) \cong \bot(\Sigma)$. Indeed now if we start with 1 and iterate $\top$ we do get a sequence of objects (and maps) isomorphic to those of 2.6. A complete collection of identifications between the presentations in terms of $\bot$ and $\top$ results. As an example note that

- $\top(t)\colon \top(1) \to \top(\Sigma)$ corresponds to $\eta_\Sigma\colon \Sigma \to \bot(\Sigma)$.

Proposition 3.2.6 (i) *The co-lift functor* $\top$ *preserves* Σ*-subsets.*

(ii) *The lift functor* $\bot$ *preserves co-*Σ*-subsets.*

Proof Suppose that $u\colon X \to \Sigma$ classifies $U \subseteq_\Sigma X$. Consider the diagram

$$\begin{tikzcd}
\top(X) \arrow[r, "\top(u)"] & \top(\Sigma) \cong \bot(\Sigma) \arrow[r] & \Sigma \\
\top(U) \arrow[u] \arrow[r] & \top(1) \cong \Sigma \arrow[u, "\top(t)"', "\eta_\Sigma"] \arrow[r] & 1 \arrow[u, "t"']
\end{tikzcd}$$

where we use some of the identifications just mentioned. The left hand square is a pullback since $\top$ preserves pullbacks; and the right hand square is the classifying pullback diagram for $\Sigma \subseteq_\Sigma \bot(\Sigma)$. Hence the composite

$$\top(X) \xrightarrow{\top(u)} \top(\Sigma) \cong \Sigma_2 \longrightarrow \Sigma$$

classifies $\top(U) \subseteq_\Sigma \top(X)$. This proves (1), and (2) is just the dual. □

Finally note that we have the dual definitions of an object *equipped with a top element* (that is, the algebras for the co-lift monad), and of the *co-strict* maps between them. Objects of the form $A_\top$ (that is, the free algebras) are *co-lifts*. The duals of the result of 2.5 hold.

4 Finitary domain theory

4.1 A higher order axiom

After all the above axioms it still remains possible that t and f are the two distinct maps $1 \to 2$, so that the Σ-subsets, the co-Σ-subsets and the decidable subsets all coincide. This degeneracy is avoided by the main finitary axiom which genuinely exploits the higher order structure in our (locally) cartesian closed category **S**. By 2.6 there is a (monic) map $[t,f]\colon 2 \to \Sigma$.

Axiom 6 *The map*

$$1^{[t,f]}\colon \Sigma^\Sigma \to \Sigma^2$$

represents the inclusion order on Σ.

Let us spell out what this means in concrete terms. Write $e_\top$ and $e_\bot$ for the evaluation maps $1^t\colon \Sigma^\Sigma \to \Sigma$ and $1^f\colon \Sigma^\Sigma \to \Sigma$ respectively. Then $e_\bot \subseteq e_\top$ and there is a commutative diagram

$$\begin{tikzcd} & \Sigma^\Sigma \arrow[dl, "e_\bot"'] \arrow[d] \arrow[dr, "e_\top"] & \\ \Sigma & \Sigma^2 \arrow[l, "\mathrm{fst}"] \arrow[r, "\mathrm{snd}"'] & \Sigma \end{tikzcd}$$

with the following universal property. Suppose that $a \subseteq b\colon X \to \Sigma$; then there is a unique map $(a; b)\colon X \to \Sigma^\Sigma$ such that

$$\begin{tikzcd} & X \arrow[dl, "a"'] \arrow[d] \arrow[dr, "b"] & \\ \Sigma & \Sigma^\Sigma \arrow[l, "e_\bot"] \arrow[r, "e_\top"'] & \Sigma \end{tikzcd}$$

commutes.

Amongst many automatic pullbacks

$$\begin{tikzcd} \Sigma^\Sigma \arrow[r, "e_\top"] & \Sigma \\ \Sigma \arrow[u, "(1; k_\top)"] \arrow[r] & 1 \arrow[u, "t"'] \end{tikzcd}$$

should be noted. This exhibits Σ as a Σ-subset of Σ^Σ.

As an application of the universal property, take $\mu \subseteq \bot(\Sigma)\colon \Sigma_2 \to \Sigma$; thus there is a map $(\mu; \bot(\Sigma))\colon \Sigma_2 \to \Sigma^\Sigma$. It is easy to check that this map is inverse to the map $\Sigma^\Sigma \to \Sigma_\bot$ which classifies $1 \subseteq_\Sigma \Sigma \subseteq_\Sigma \Sigma^\Sigma$. Thus the content of Axiom 6 is given by isomorphisms

$$\Sigma^\Sigma \cong \Sigma_\bot \quad \text{and dually} \quad \Sigma^\Sigma \cong \Sigma_\top.$$

Modulo these isomorphism we can identify many maps. We give some examples:

- $\eta_\Sigma \circ f\colon \Sigma \to \Sigma_\bot$ corresponds to $(k_\bot; k_\top)\colon \Sigma \to \Sigma^\Sigma$;
- $\eta_\Sigma\colon \Sigma \to \Sigma_\bot$ corresponds to $(1; k_\top)\colon \Sigma \to \Sigma^\Sigma$;
- $\bot(f)\colon \Sigma \to \Sigma_\bot$ corresponds to $(k_\bot; 1)\colon \Sigma \to \Sigma^\Sigma$;
- $\mu\colon \Sigma_\bot \to \Sigma$ corresponds to $e_\bot\colon \Sigma^\Sigma \to \Sigma$;
- $\bot(\Sigma)\colon \Sigma_\bot \to \Sigma$ corresponds to $e_\top\colon \Sigma^\Sigma \to \Sigma$.

4.2 Properties of the intrinsic order

One consequence of Axiom 6 is that it identifies the intrinsic order and the inclusion order.

Theorem 4.2.1 *The intrinsic order coincides with the inclusion order on all objects of the form Σ^X.*

Proof The diagram

$$\begin{tikzcd}
(\Sigma^A)^2 \arrow[r] & (\Sigma^2)^{(\Sigma^{(\Sigma^A)})} \\
(\Sigma^A)^\Sigma \arrow[r] \arrow[u] & \left(\Sigma^{(\Sigma^{(\Sigma^A)})}\right)^2 \arrow[u]
\end{tikzcd}$$

clearly commutes. But by Axiom 6 $(\Sigma^A)^\Sigma$ is the subset order on Σ^A. Hence it factors through the pullback of $(\Sigma^{(\Sigma^{(\Sigma^A)})})^2$ which is the intrinsic order. Thus the subset order entails the intrinsic order. But we already have the converse which was Proposition 2.4.1. □

Corollary 4.2.2 *$\perp$ is the least element in $A_\perp$.*

Proof In Σ, $\perp$ is the least element in the inclusion order and hence is least in the intrinsic order. Consider the map $\Sigma \times \perp(A) \to \perp(A)$ appearing in the unique pullback of form

$$\begin{tikzcd}
\Sigma \times \perp(A) \arrow[r] & \perp(A) \\
A \arrow[r, equal] \arrow[u] & A \arrow[u]
\end{tikzcd}$$

We have $f \leq t$ in Σ and so $(f \times 1) \leq (t \times 1): \perp(A) \to \Sigma \times \perp(A)$. Hence by composition $k_\perp \leq 1: \perp(A) \to \perp(A)$. □

4.3 Σ-subsets of lifts and co-lifts

We return to analyze the general behaviour of maps $\perp(A) \to \Sigma^X$. In fact it is simpler to consider the case of a map $\top(A) \to \Sigma^X$. We know by the dual of 4.2.2 that $\top$ is the greatest element of $\top(A)$. But maps preserve the intrinsic order; hence if we let $s \in \Sigma^X$ be the image of $\top$ in Σ^X, s is the greatest element in the image in the intrinsic and hence in the subset order. It follows that if $s: X \to \Sigma$ classifies $S \subseteq_\Sigma X$, then $\top(A) \to \Sigma^X$ factors through the standard split monomorphism $\Sigma^S \to \Sigma^X$. Clearly the resulting map $\top(A) \to \Sigma^S$ is co-strict. Thus we have bijective correspondences:

$$\frac{\top(A) \to \Sigma^X}{S \subseteq_\Sigma X \quad \text{and a co-strict map} \quad \top(A) \to \Sigma^S}$$

$$\begin{array}{c}\hline S \subseteq_\Sigma X \quad \text{and} \quad U \subseteq_\Sigma S \times A.\end{array}$$

As a consequence we derive the following connections between the lift and co-lift.

Proposition 4.3.1 *$\Sigma^{\top(A)} \cong \bot(\Sigma^A)$ and dually $\Sigma^{\bot(A)} \cong \top(\Sigma^A)$, both these isomorphisms being natural in A.*

Proof The first isomorphism is a consequence of the bijective correspondences:

$$\begin{array}{c}
X \to \Sigma^{\top(A)} \\ \hline
\top(A) \to \Sigma^X \\ \hline
S \subseteq_\Sigma X \quad \text{and} \quad U \subseteq_\Sigma S \times A \\ \hline
S \subseteq_\Sigma X \quad \text{and} \quad S \times A \to \Sigma \\ \hline
S \subseteq_\Sigma X \quad \text{and} \quad S \to \Sigma^A \\ \hline
X \to \bot(\Sigma^A)
\end{array}$$

all natural in A. The second isomorphism is just the dual of the first. $\square$

The attentive reader will have noticed that the argument from 4.2.2 onwards goes through simply under the assumption that the subset and intrinsic orders coincide. But Axiom 6 is a special case of Proposition 4.3.1; and so in fact it is equivalent to the coincidence of the two orders.

4.4 Rice's Theorem

One can regard Axiom 6 as a weak version of the undecidability of the halting problem. Hence we can now derive a little bit of non-trivial recursion theory.

Proposition 4.4.1 *For any objects A, B, and C, we have*

$$(B + C)^{\bot(A)} \cong B^{\bot(A)} + C^{\bot(A)}.$$

Proof Let $b\colon B + C \to \Sigma$ classify $B \subseteq_\Sigma B + C$. Consider the diagram

$$\begin{tikzcd}
& (B+C)^{\perp(A)} \arrow[rr] \arrow[dd] & & B+C \arrow[dd] \\
B^{\perp(A)} \arrow[ur] \arrow[rr] \arrow[dd] & & B \arrow[ur] \arrow[dd] & \\
& \Sigma^{\perp(A)} \arrow[rr] & & \Sigma \\
1 \arrow[ur] \arrow[rr] & & 1 \arrow[ur] &
\end{tikzcd}$$

where the right hand face is the pullback exhibiting $B \subseteq_\Sigma B + C$, the left hand face is that pullback raised to the power $\perp(A)$ (and so is a pullback), and the horizontal maps are induced by $\perp\colon 1 \to \perp(A)$. By the naturality in 4.3.1, the map $\Sigma^{\perp(A)} \to \Sigma$ corresponds to $\top(\Sigma^A \to 1)$. Hence as $\top$ preserves pullbacks, the bottom face is a pullback. It follows at once that the top face is a pullback. The same argument works with C in place of B. So we have pullbacks

$$\begin{tikzcd}
(B+C)^{\perp(A)} \arrow[r] & B+C \\
B^{\perp(A)} \arrow[u] \arrow[r] & B \arrow[u]
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
(B+C)^{\perp(A)} \arrow[r] & B+C \\
C^{\perp(A)} \arrow[u] \arrow[r] & C \arrow[u]
\end{tikzcd}$$

But coproducts are stable, so

$$(B+C)^{\perp(A)} \cong B^{\perp(A)} + C^{\perp(A)}.$$

□

Corollary 4.4.2 *If X is an object with bottom, then for any B and C*

$$(B+C)^X \cong B^X + C^X.$$

Proof Since X is a retract of $A = \perp(X)$, this is obvious. □

An immediate consequence is an abstract version of Rice's Theorem. Let us write $P = (\mathsf{N}_\perp)^{\mathsf{N}})$ for the object of Σ-partial functions from N to N.

Corollary 4.4.3 $2^P \cong 2$.

Proof P is a power of a lift and so is an object with bottom; and $2 = 1 + 1$. □

4.5 Further results

In this section we collect together some further consequences of Axiom 6, which we shall need.

From 4.3.1 we can derive an intuitively plausible characterization of the intrinsic order on a lift.

Proposition 4.5.1 *The intrinsic order on $\perp(B)$ is given in the internal logic by*

$$\mathsf{S} \models c \leq_{\perp(B)} d \iff (c \in B \Rightarrow d \in B \wedge c \leq_B d).$$

Corollary 4.5.2 *The functor $\perp$ is externally order-preserving. If $f \preceq g: A \to B$ then $f_\perp \preceq g_\perp: A_\perp \to B_\perp$.*

The following proposition is related to the material of 5.3.

Proposition 4.5.3 *If A has a top, then Σ^A is a lift (and dually if A has a bottom, then Σ^A is a co-lift).*

Proof (Sketch.) Let $\top$ be the top element of A, and set $D = \{R \in \Sigma^A | \top \in R\} \subseteq_\Sigma \Sigma^A$. We obtain a map $\Sigma^A \to \perp(D)$ classifying the partial map $\Sigma^A \rightharpoonup_\Sigma D$ given by $1: D \to D$. Further we obtain a map $\perp(D) \to \Sigma^A$ as follows. Let $E \subseteq_\Sigma A \times D$ be classified by the obvious composite $A \times D \to A \times \Sigma^A \to \Sigma$. As $A \times D \subseteq_\Sigma A \times \Sigma^A$, we get $E \subseteq_\Sigma A \times \perp(D)$. Then we take the transpose of the corresponding classifying map $A \times \perp(D) \to \Sigma$. One checks that these two maps are inverses of one another. □

4.6 Finite unions

So far all our finitary domain theory is the consequence of just one axiom. But there is a further aspect of recursion theory which seems to require a further assumption. We note that while $\cap$ on Σ represents the usual notion of intersection of subobjects, we do not know that $\vee$ represents union. We make this our next axiom.

Axiom 7 *The class of all Σ-subsets is closed under finite unions.*

In recognition of this axiom we shall now write $\cup$ for $\vee$.

Axiom 7 can be expressed quite simply in diagrammatic terms, but we do not go into that here.

With Axiom 7 the misleading symmetry of our treatment is broken. We do not include its dual amongst our axioms, and indeed the dual is false in some of our standard models (in the effective topos for example). (This corresponds to the fact that while the collection of complements of recursively enumerable sets—the Π^0_1 sets—is closed under union, the logical equivalence on which this fact is based

$$(\forall x \phi(x) \vee \forall y \psi(y)) \iff (\forall x \forall y (\phi(x) \vee \psi(y)))$$

is not constructively valid.) Note that the failure of the dual axiom in the effective topos shows that Axiom 7 is independent of the earlier axioms.

The main consequence of Axiom 7 is the abstract version of the familiar fact that sets which are both semi-decidable and co-semi-decidable are decidable. First we exhibit 2 as a regular subobject of $\Sigma \times \Sigma$. It is natural to consider the two maps $(\cup, \cap), (k_\top, k_\perp): \Sigma^2 \to \Sigma^2$.

Proposition 4.6.1 *The above two maps give rise to an equalizer diagram*

$$2 \longrightarrow \Sigma \times \Sigma \rightrightarrows \Sigma \times \Sigma$$

Corollary 4.6.2 *If both $A \subseteq_\Sigma X$ and $A \subseteq_{\bar{\Sigma}} X$, then $A \subseteq X$ is decidable.*

Proof If $a: X \to \Sigma$ classifies $A \subseteq_\Sigma X$ and $b: X \to \Sigma$ classifies $A \subseteq_{\bar{\Sigma}} X$, then it is easy to see that $(a, b): X \to \Sigma^2$ equalizes the maps $\Sigma^2 \rightrightarrows \Sigma^2$. Hence $A \subseteq X$ is decidable. □

5 Infinitary structure

5.1 A notion of ω-chain

Recall that we assumed that we have a natural number object N in our category S. Thus one can consider N-indexed sequences of Σ-subobjects of an object X: these are the maps $\mathsf{N} \to \Sigma^X$ or by transposition the Σ-subsets of $\mathsf{N} \times X$ or by further transposition the maps $X \to \Sigma^{\mathsf{N}}$. We are mainly interested in increasing (N-indexed) sequences of Σ-subobjects of an object X; these are maps $R: \mathsf{N} \to \Sigma^X$ such that $\forall n \in \mathsf{N}. R_n \subseteq R_{n+1}$ holds in S. Clearly we can represent the object of increasing sequences in Σ as the equalizer of maps $\Sigma^{\mathsf{N}} \rightrightarrows \Sigma^{\mathsf{N}}$, one being the identity, and the other the map which takes $\lambda n.\sigma_n \in \Sigma^{\mathsf{N}}$ to $\lambda n.\sigma_n \wedge \sigma_{n+1}$.

First let us show that there is an object ω which classifies increasing sequences in Σ in the sense that we can take Σ^ω to be the object of increasing sequences. (Morally ω corresponds to the object—usually called ω in domain theory—which consists of an infinite increasing sequence of points.) In essence ω is the colimit of the internal diagram

$$0 \to 1 \xrightarrow{f} \Sigma \xrightarrow{\perp(f)} \Sigma_2 \ldots \Sigma_n \xrightarrow{\perp^n(f)} \Sigma_{n+1} \cdots$$

and ω will be the initial algebra for the lift *functor*. We sketch a construction in category theoretic terms (avoiding completeness assumptions on S).

First we need some notation for indexed families. When we consider I-indexed families $X \to I$, it is often the case that we have a natural notation for the implicit fibres $X(i)$ but no notation for the corresponding object X. In these circumstances we shall write $(X(i)) \to I$ or more simply $(X(i))$ for the corresponding object of S/I. We shall use some specific families indexed over N. Let us adopt Martin-Löf's notation $(\mathsf{N}(n))$ for the N-indexed family of finite sets. From it we obtain an N-indexed family $(\Sigma^n) \to \mathsf{N}$. As a subobject of this one can construct an N-indexed family $(\Sigma_n) \to \mathsf{N}$ which internalizes the sequence Σ_n. It seems best to take the subobject

$$\Sigma_n = \{(p_0, \ldots, p_{n-1}) | \forall i (p_i \geq p_{i+1})\} \subset \Sigma^n$$

which is given by a simple equalizer in S/N.

Proposition 5.1.1 *The indexed family $(\Sigma_n) \to \mathsf{N}$ satisfies the recursion equations:*

$$\Sigma_0 \cong 1 \quad (\Sigma_{n+1}) \cong (\perp(\Sigma_n))$$

To obtain an N-indexed diagram, we need to describe the (indexed family) of maps $\perp^n(f)$; but these are induced by the maps $(1, f): \Sigma^n \to \Sigma^n \times \Sigma \cong \Sigma^{n+1}$. Now the colimit ω of the internal diagram will appear in a coequalizer diagram $\amalg_n(\Sigma_n) \rightrightarrows \amalg_n(\Sigma_n) \to \omega$ which lies over the standard coequalizer $\mathsf{N} \rightrightarrows \mathsf{N} \to 1$. (Note that the object ω is both the initial algebra for the lift functor and for the lift monad; but of course the structures are different!)

Recall from 2.6 that Σ_n has $n+1$ distinct global sections. We can internalize these as a family of maps $(\mathsf{N}(n+1)) \to (\Sigma_n)$, and this induces in the colimit, the standard enumeration of $\mathsf{N} \to \omega$. Hence in particular the global sections of ω include analogues of the finite ordinals.

Proposition 5.1.2 *The object Σ^ω is the object of increasing sequences in Σ: the enumeration $\mathsf{N} \to \omega$ induces an equalizer diagram $\Sigma^\omega \to \Sigma^{\mathsf{N}} \rightrightarrows \Sigma^{\mathsf{N}}$ where the two maps $\Sigma^{\mathsf{N}} \to \Sigma^{\mathsf{N}}$ are as described above.*

Proof (Sketch) The covariant functor $\Sigma^{()}$ transforms internal colimits into internal limits; the resulting equalizer diagram is not quite what we want, but we easily derive from it the fact that $\Sigma^\omega \to \Sigma^{\mathsf{N}} \rightrightarrows \Sigma^{\mathsf{N}}$ is an equalizer. □

Definition 4 For any object X of S the object X^ω is *the object of ω-chains in X.*

In general the ω-chains in X will not coincide with the sequences increasing in the intrinsic order. However it is an easy corollary of 5.1.2 that they do coincide in the case of retracts of powers of Σ. In fact they coincide for all linked Σ-spaces (see Phoa [8]).

5.2 Suprema of ω-chains

Suprema of ω-chains (in the traditional sense) play a major role in domain theory; so the next axiom is no surprise.

Axiom 8 *The collection of Σ-subsets of an object is closed under suprema of increasing (and N-indexed) sequences.*

Since ω classifies increasing sequences in Σ we can readily express our axiom in terms of Σ^ω.

Proposition 5.2.1 *The following are equivalent*

(i) *The class of Σ-subsets is closed under N-directed suprema.*

(ii) *There is a map $\bigvee : \Sigma^\omega \to \Sigma$ which is left adjoint to the constant map $\Sigma \to \Sigma^\omega$.*

We can identify ω with the subobject

$$\{(p_n)_{n\in\mathsf{N}} | \forall n(p_n \geq p_{n+1}) \wedge \exists n(p_n = \bot)\} \subseteq \Sigma^{\mathsf{N}}.$$

Perhaps it is worth stressing that this need not be a regular subobject! Clearly $\omega \subseteq \Sigma^{\mathsf{N}}$ is not closed under increasing sequences in Σ^{N}; we identify its closure as

$$\bar{\omega} = \{(p_n)_{n\in\mathsf{N}} | \forall n(p_n \geq p_{n+1})\} \subseteq \Sigma^{\mathsf{N}}.$$

For $\bar{\omega}$ is a retract of Σ^{N} via the map which sends $(p_n)_{n\in\mathsf{N}}$ to $(\wedge_{m\leq n}(p_m))_{n\in\mathsf{N}}$. Hence $\bar{\omega}$ is closed under limits of ω-chains in Σ^{N}: the supremum map $(\Sigma^{\mathsf{N}})^\omega \to \Sigma^{\mathsf{N}}$ restricts to a supremum map $\bar{\omega}^\omega \to \bar{\omega}$. Furthermore the pointwise intersection $(\Sigma^{\mathsf{N}}) \times \Sigma^{\mathsf{N}} \to \Sigma^{\mathsf{N}}$ restricts to a map $\omega \times \bar{\omega} \to \omega$; and its exponential transpose is a map $\bar{\omega} \to \omega^\omega$ which associates to every element of $\bar{\omega}$ a canonical increasing sequence in ω tending to it. In particular the composite $\bar{\omega} \to \omega^\omega \to \bar{\omega}$ is the identity. Other constructions of $\bar{\omega}$ are possible: for example we could set $\bar{\omega}$ to be the internal limit of the standard diagram of form

$$(1 \leftarrow \Sigma \leftarrow \bot(\Sigma)...).$$

This would be given externally as an equalizer.

5.3 The main infinitary axiom

We think of $\bar{\omega}$ as the object ω with a limit to the ω-chain added. So morally it should represent increasing sequences with limit point—at least in sufficiently good objects. In particular therefore we expect ω and $\bar{\omega}$ to have the same Σ-subsets, and we take this as our next axiom.

Axiom 9 *The map $\omega \to \bar{\omega}$ induces an isomorphism $\Sigma^{\bar{\omega}} \to \Sigma^{\omega}$.*

To understand the force of this axiom let us look closely at the object $\Sigma^{\bar{\omega}}$ of Σ-subsets of $\bar{\omega}$. Since $\bar{\omega}$ is a retract of Σ^{N}, it has a top; we write $\infty: 1 \to \bar{\omega}$ for this element. Write $e_\infty: \Sigma^{\bar{\omega}} \to \Sigma$ for the 'evaluation at top' map. From e_∞ we obtain the pullback

$$\begin{tikzcd} \Sigma^{\bar{\omega}} \arrow[r, "e_\infty"] & \Sigma \\ E \arrow[u] \arrow[r] & 1 \arrow[u, "t"'] \end{tikzcd}$$

exhibiting $\Sigma^{\bar{\omega}} \cong \bot(E)$. (The reader should refer to 4.5.3.) Of course $\Sigma^{\bar{\omega}}$ is also an object with top (in fact a co-lift); indeed E has a top. Let us consider this situation in the abstract.

Proposition 5.3.1 *Suppose E has a top element $\uparrow$. Then the classifying map $\bot(E) \to \Sigma$ is left adjoint to the map $\bot(\uparrow): \Sigma \to \bot(E)$.*

Proof As $\uparrow$ is the top element of E the map $\uparrow: 1 \to E$ has $E \to 1$ as left adjoint. As $\bot$ is order-preserving in the sense of 4.5.2, it preserves adjunctions. Whence the result. □

Corollary 5.3.2 *The evaluation map $e_\infty: \Sigma^{\bar{\omega}} \to \Sigma$ is left adjoint to the constant map $\Sigma \to \Sigma^{\bar{\omega}}$.*

This shows, at least for objects of the form Σ^A and their retracts, that $\infty \in \bar{\omega}$ represents the suprema of ω-chains.

Proposition 5.3.3 *The supremum map $\bigvee: \Sigma^\omega \to \Sigma$ is equal to the composite $\Sigma^\omega \cong \Sigma^{\bar{\omega}} \xrightarrow{e_\infty} \Sigma$.*

Since sups of ω-chains are represented, they are automatically preserved (even in the internal sense) by maps $\Sigma^A \to \Sigma^B$. In fact the following are equivalent:

(i) $\omega \to \bar{\omega}$ induces an isomorphism $\Sigma^{\bar{\omega}} \to \Sigma^\omega$.

(ii) $\mathsf{S} \models (\forall f: \Sigma^A \to \Sigma^B)(\forall (R_n) \in (\Sigma^A)^\omega)\, f(\bigvee(R_n)) = \bigvee(f(R_n))$.

We have just seen that (i) implies (ii), while (ii) implies (i) because $\bar{\omega}$ is the closure of ω under sups of ω-chains. We could also give an equivalent formulation of Axiom 9 in terms of a notion of *intrinsic limit* of ω-chains; but this makes more sense in connection with Axiom 10.

5.4 Directed unions

As we saw in connection with Axiom 7, there may be more to closure under unions than the existence of a suitable left adjoint. Hence there is a further infinitary axiom.

Axiom 10 *The collection of Σ-subsets of an object is closed under unions of increasing (and* N*-indexed) sequences.*

(As with Axiom 6 we could easily express this extra information in diagrammatic terms.) Obviously we can combine Axioms 7 and 10.

Proposition 5.4.1 *The class of Σ-subsets is closed under* N*-indexed unions.*

As a result we have a union map $\bigcup: \Sigma^{\mathsf{N}} \to \Sigma$. Using it we can exhibit N as a regular subobject of Σ^{N}. The essential map $\Sigma^{\mathsf{N}} \to \Sigma$ is that which takes $(p_n)_{n\in\mathsf{N}}$ to $\bigcup\{p_n \cap p_m | n \neq m\}$. If we pair this map with $\bigcup$ and $k_\perp$ with $k_\top$, we get two maps $\Sigma^{\mathsf{N}} \rightrightarrows \Sigma^2$.

Proposition 5.4.2 *There is an equalizer diagram* $\mathsf{N} \to \Sigma^{\mathsf{N}} \rightrightarrows \Sigma^2$*, where the maps* $\Sigma^{\mathsf{N}} \rightrightarrows \Sigma^2$ *are as just described.*

Corollary 5.4.3 N*-indexed partitions of an object X into Σ-subsets correspond to maps* $X \to \mathsf{N}$.

A further consequence of Axiom 10 is a form of the Rice-Shapiro Theorem.

Theorem 5.4.4 *(Rice-Shapiro) Σ-subsets of an object are Scott open with respect to ω-chains:*

$$\forall U \in \Sigma^A . \forall (a_n) \in A^\omega . \bigvee (A_n) \in U \Rightarrow \exists n . a_n \in U.$$

A full development of material in this area is contained in Rosolini's Thesis [10].

6 A category of predomains

6.1 The category of Σ-replete objects

A number of people have considered candidates for a good category of predomains within a topos (usually the effective topos). Phoa [8] considered the category of complete Σ-spaces, which consists of those objects which are complete with respect (N-indexed) sequences increasing with respect to the intrinsic order. Freyd et al [1] considered (in effect) the subcategory consisting of those complete Σ-spaces which are regular subobjects of a power of Σ. Here we consider a notion which is in some sense canonical. There are two perspectives which one can take. Since the object Σ^A of all Σ-subsets of A is to play the crucial conceptual role in the theory, we should consider those objects which are (in some sense) determined by their Σ-subsets. Alternatively we could argue that we need the object Σ and good completeness properties, but should not take more objects than we are forced to take. These two perspectives are equivalent; they lead to the same good category, the category of replete objects.

We can give an account of the category of replete objects by means of some internal category theory applied to the internal category S. The definitions which follow should be understood in that sense. (I hope that my giving translations into the internal logic of sets will help rather than confuse.)

Definition 5 A map $g\colon Q \to \Sigma$ is Σ-*equable* if the induced map $\Sigma^Q \to \Sigma^P$ is an isomorphism. (That is, in the internal logic of sets, any map $P \to \Sigma$ extends uniquely to a map $P \to \Sigma$.) We write **Equ** for the class of Σ-equable maps. A map $f\colon A \to B$ is Σ-*replete* if for any Σ-equable map $g\colon P \to Q$ the diagram

$$\begin{tikzcd} A^Q \arrow[r] \arrow[d] & A^P \arrow[d] \\ B^Q \arrow[r] & B^P \end{tikzcd}$$

is a pullback. (That is, in the internal logic of sets, given any commutative square as indicated below, there is a unique fill-in $Q \longrightarrow A$

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \arrow[dl] \\ A \arrow[r] & B \end{tikzcd}$$

making the two triangles commute.) We write **Rep** for the class of all Σ-replete maps.

For the most part it is sufficient to restrict attention to the (internal) full subcategory of all objects A such that $A \to 1$ is a replete map. We write **R** for this category, the category of *replete objects*. (Of course maps between replete objects are automatically replete maps.)

Clearly (**Equ**, **Rep**) forms a prefactorization on S, and with enough (internal) completeness on that category we could show that it is a factorization system. However using rather little of the internal completeness of the ambient topos, we get what we chiefly need.

Theorem 6.1.1 *$\mathbf{R}$ is a reflective subcategory of* S.

Proof The unit of the reflection $A \to r(A)$ appears in $A \to r(A) \to \Sigma^{(\Sigma^A)}$ as the largest subobject of $\Sigma^{(\Sigma^A)}$ such that $A \to r(A)$ is Σ-equable. The image (and indeed the regular image) of A is necessarily such a subobject. □

Unfortunately it does not seem easy to give a simple concrete description of the replete objects in (for example) the effective topos. Indeed though Freyd was (of course) aware of the possibility of the above definition, he prefered for this reason to work with the more concrete category in [1]. Paul Taylor independently came to regard the replete objects as important and has carefully analyzed different equivalent descriptions of $r(A)$.

The significance of **R** is indicated by the following standard characterization.

Theorem 6.1.2 *$\mathbf{R}$ is the least internally full reflective subcategory of* S *which contains* Σ.

Note that the categories considered by Phoa [8] and Freyd et al [1] within the effective topos are complete and contain Σ. Hence they contain the category **R** of replete objects. However one can show that **R** is strictly contained in them. Rosolini and Scott considered a more topologically motivated category of *σ-spaces* (see [10] and [11]). In a Scott topos, Johnstone's example (see [5]) shows that the σ-spaces are strictly contained in the category of replete objects. Whether or not this holds in the effective topos remains an open question.

6.2 Simple closure properties of R

As R is reflective in S it inherits some properties.

Theorem 6.2.1 *The category* R *is as complete and cocomplete as is our initial category* S. *In particular under the assumptions of 1.3* R *is cartesian closed and closed under products indexed over separated objects.*

Warning: Local cartesian closedness is not a completeness property. It does not follow that R is locally cartesian closed. The problem was first made explicit by Thomas Streicher, see [12] where there is a counterexample.

Corollary 6.2.2 (i) *The objects* 0 *and* 2 *are in* R, *and* R *is closed under finite coproducts.*

(ii) *The object* N *is in* R *and* R *is closed under coproducts (internally) indexed by* N.

Proof R is closed under finite limits. Hence 0 is in yR by axiom 3, while 2 is in R by Proposition 4.6.1. More generally $A + B$ lies in an equalizer diagram of form

$$(A + B) \longrightarrow (A_\perp \times B_\perp) \rightrightarrows \Sigma \times \Sigma,$$

and so is in R. This deals with the first part. A similar argument involving Proposition 5.4.2 establishes the second part. □

The construction of R ensures that epimorphisms in R are easy to detect.

Proposition 6.2.3 *A map* $e: A \to B$ *in* R *is an epimorphism (in* R*) if and only if the induced map* $\Sigma^e: \Sigma^B \to \Sigma^A$ *is a monomorphism.*

Proof One direction is familiar enough. Take the pushout

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "e"'] & B \arrow[d, "f_0"] \\ B \arrow[r, "f_1"'] & C \end{tikzcd}$$

in R so that $C = (B +_A B)$. If e is epi then $f = f_0 = f_1$ are (equal) isos. Hence in the corresponding pullback

$$\begin{tikzcd} \Sigma^C \arrow[r] \arrow[d] & \Sigma^B \arrow[d] \\ \Sigma^B \arrow[r] & \Sigma^A \end{tikzcd}$$

the map $\Sigma^f: \Sigma^C \to \Sigma^B$ is an iso. Thus $\Sigma^e: \Sigma^B \to \Sigma^A$ is a mono. Conversely, if Σ^e is mono then Σ^f is an iso. But $f: B \to C$ is a map between replete objects. So if Σ^f is an iso, then f must be an iso. □

6.3 Lifts of replete objects

Next we mention a theorem which is needed for applications, and whose proof is pleasingly algebraic in character.

Theorem 6.3.1 *If A is replete, then so are $\perp(A)$ and $\top(A)$.*

Proof We give a proof for the case of $\perp(A)$ in the style of the internal category theory which we have been using. First we note the following trivial lemma. Suppose that $f: C \to D$ is Σ-equable and that $D' \subseteq_\Sigma D$ is a Σ-subset. Consider the pullback

$$\begin{tikzcd} C \arrow[r, "f"] & D \\ C' \arrow[u] \arrow[r, "f'"'] & D' \arrow[u] \end{tikzcd}$$

of $D' \to D$ along f. Then the map $f': C' \to D'$ is Σ-equable. Suppose now that $C \to D$ is Σ-equable and that we are given a map $C \to \perp(A)$. Clearly there is a unique map $D \to \Sigma$ making the diagram

$$\begin{tikzcd} D \arrow[drr] & & \\ C \arrow[u] \arrow[r] & \perp(A) \arrow[r] & \Sigma \end{tikzcd}$$

commute. If we pull this diagram back along $t: 1 \to \Sigma$, we obtain

$$\begin{tikzcd} D' \arrow[drr] & & \\ C' \arrow[u] \arrow[r] & A \arrow[r] & 1 \end{tikzcd}$$

By the lemma $C' \to D'$ is Σ-equable and so there is a unique map $D' \to A$ such that

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & D' \arrow[dl] \\ A & \end{tikzcd}$$

commutes. But this map corresponds to a Σ-partial map $D \rightharpoonup A$ and so induces a unique map $D \to \perp A$ fitting into the diagram of pullbacks:

$$\begin{tikzcd} C \arrow[r] & D \arrow[r] & \perp(A) \\ C' \arrow[u] \arrow[r] & D' \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

By the uniqueness property of $\perp(A)$, the map $D \to \perp(A)$ is the unique extension of the map $C \to \perp(A)$. □

6.4 Fixed points

Repleteness of an object is of course a kind of completeness condition. Indeed Axiom 9, which says in effect that the map $\omega \to \bar{\omega}$ is equable, has the immediate effect that replete objects have limits of ω-chains. This allows us to develop a traditional theory of fixed points. (Another somewhat fuller treatment has been given by Paul Taylor.)

Definition 6 An object A in a cartesian closed category C is in *the fixed point category* just when for any $I \in$ C, the power A^I is a *fixed point object*: any map $A^I \to A^I$ has a fixed point.

It does not matter for the purposes of this definition whether we take 'fixed point object' in the internal or external sense.

Proposition 6.4.1 *Suppose that A is replete and has a bottom element. Then every map $A \to A$ has a* least *fixed point. Hence as powers of replete objects with bottom are again replete objects with bottom, such an A is automatically in the fixed point category.*

Proof The usual proof works! □

Thus if we take as category of domains the objects of R with bottom element, we can interpret within it all definitions of functions by recursion. We have taken the first steps in a synthetic domain theory.

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