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A SMALL COMPLETE CATEGORY

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0. Introduction

This paper is concerned with a remarkable fact. The effective topos contains a small complete subcategory, essentially the familiar category of partial equivalence realtions. This is in contrast to the category of sets (indeed to all Grothendieck toposes) where any small complete category is equivalent to a (complete) poset. Note at once that the phrase 'a small complete subcategory of a topos' is misleading. It is not the subcategory but the internal (small) category which matters. Indeed for any ordinary subcategory of a topos there may be a number of internal categories with global sections equivalent to the given subcategory. The appropriate notion of subcategory is an indexed (or better fibred) one, see 0.1. Another point that needs attention is the definition of completeness (see 0.2). In my talk at the Church's Thesis meeting, and in the first draft of this paper, I claimed too strong a form of completeness for the internal category. (The elementary oversight is described in 2.7.) Fortunately during the writing of [13] my collaborators Edmund Robinson and Giuseppe Rosolini noticed the mistake. Again one needs to pay careful attention to the ideas of indexed (or fibred) categories.

The idea that small (sufficiently) complete categories in toposes might exist, and would provide the right setting in which to discuss models for strong polymorphism (quantification over types), was suggested to me by Eugenio Moggi. And he first realized that the effective topos did indeed contain a small complete category. When, led by Moggi's suggestion, I first came to consider the matter, I realized that the 'result' was staring me in the face. It is just a matter of putting together some well-known facts.

The effective topos is the world of realizability (Kleene [15]) extended from arithmetic to general constructive mathematics. Details are in [11], and the general context in [12] and [21]. The relevant subcategory, called the category of effective objects in [11], is already in Kreisel [16]. Briefly the problem is to show

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that an internal version $\hat{\mathbf{P}}$ of this category has all indexed products in **Eff**. (This is an indexed/internal version of the fact that the effective objects form an exponential ideal [11].) However Girard [10] already shows that the externalization of $\hat{\mathbf{P}}$ has all indexed products over **Sets**. The extension to **Eff** is analogous to tripos theory: the completeness of an indexed poset in **Sets** gives completeness (as the subobject classifier) in the generated topos. (Indeed, the poset reflection of $\hat{\mathbf{P}}$ is well known to be complete, as it corresponds to 'extensional realizability' and so (Pitts [21]) gives a locale in **Eff**.) The conceptually natural way to see the products in $\hat{\mathbf{P}}$ itself is to recognize that these should be exactly the products for a full subcategory of **Eff** indexed over itself. Then the passage from **Sets** to **Eff** is a triviality.

A more abstract approach to the material of this paper has recently emerged, based on an idea of Peter Freyd. Accounts have been prepared in [13] and [3]. Yet other aspects have been considered by Rosolini and Scott. They found an approach to proving completeness based on the identification of the category $\hat{\mathbf{P}}$ with the *internally defined* category of (double negation) closed subquotients of the natural number object.

In the remainder of Section 0, I treat some background on small categories in toposes. Section 1 contains a concrete introduction to the category of effective objects (now called modest sets). Section 2 gives the main proof and consequences are briefly discussed in Section 3.

0.1. Small full subcategories of a topos

The reader can consult [14] for background topos theory and for the notion of a category object in a topos. I prefer to drop the 'object', and talk of a category $\mathbf{C}$ in a topos $\mathscr{E}$ with objects $\mathbf{C}_0 \in \mathscr{E}$ and maps $\mathbf{C}_1 \in \mathscr{E}$.

Some understanding of fibred and indexed categories is essential. The reader can consult [20] or [2]. The telling discussion in Bénabou [1] is strongly recommended! For ease of exposition, I will mainly use the language of indexed categories rather than fibrations. (But note that fibrations describe a more general situation.)

A topos $\mathscr{E}$ is fibred over itself by the codomain map $\mathscr{E}^2 \rightarrow \mathscr{E}$. That is, it is indexed over itself by slice categories

$$E \rightarrow \mathscr{E}/E,$$

reindexing being by pullback. A subcategory $\mathscr{C}$ of $\mathscr{E}$ in the general fibred/indexed sense consists of a full faithful cartesian functor

$$\begin{tikzcd} \mathscr{C} \arrow[rr] \arrow[dr] & & \mathscr{E}^2 \arrow[dl] \\ & \mathscr{E} & \end{tikzcd}$$

(A cartesian functor is one preserving the cartesian morphisms (pure relabellings) of the fibration. Often this functor will be an embedding, but it seems best to allow the more general situation.)

In the corresponding situation for indexed categories, there is a full and faithful functor of indexed categories

$$\mathscr{C}(E) \to \mathscr{E}/E.$$

(Were $\mathscr{C} \to \mathscr{E}^2$ an embedding, $\mathscr{C}(E)$ would be a subcategory of ($\mathscr{E}/E$.)

The Cayley representation (see [9]) will represent any small category as a category of sets and functions. This has an analogue over any base topos. A category $\mathbf{C}$ in $\mathscr{E}$ gives rise *to its externalization* a category $\mathbf{C}$ indexed over $\mathscr{E}$ with fibres

$$\mathbf{C}(E) = \mathscr{E}(E, \mathbf{C}).$$

This can be identified as some category of 'sets' and 'functions' in that there is a faithful Cayley representation $\mathbf{C}(E) \to \mathscr{E}/E$ natural in E. However I do not need this general situation, but rather the case when the internal category $\mathbf{C}$ has a terminal object 1 and the (internal) global sections functor $\mathrm{Hom}_C(1, -)$ represents $\mathbf{C}$ faithfully; that is *the case when* 1 *is generator or* $\mathbf{C}$ *has enough points.* (Unfortunately computer scientists sometimes use *concrete category* here!)

Suppose that the internal category $\mathbf{C}$ in $\mathscr{E}$ has a terminal object 1. Then the (internal) *global sections functor* can be described as a functor of indexed categories

$$\mathbf{C}(E) \to \mathscr{E}/E,$$

as follows.

On objects. The generic family of objects of $\mathscr{E}$ represented in $\mathbf{C}$ is given by the map $p : \mathbf{C}_1(1, -) \to \mathbf{C}_0$ which appears in the pullback

$$\begin{tikzcd} \mathbf{C}_1(1,-) \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{C}_1 \arrow[d, "(d_0, d_1)"] \\ \mathbf{C}_0 \arrow[r, "(\ulcorner 1 \urcorner, \mathrm{id})"'] & \mathbf{C}_0 \times \mathbf{C}_0 \end{tikzcd}$$

where $\ulcorner 1 \urcorner : \mathbf{C}_0 \to \mathbf{C}_0$ is the constant map to the terminal object. Now given an object $R : E \to \mathbf{C}_0$ in $\mathbf{C}(E)$ we take the pullback of p along it to get an object $R \to E$ in $\mathscr{E}/E$

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{C}_1(1,-) \arrow[d, "p"] \\ E \arrow[r, "R"'] & \mathbf{C}_0 \end{tikzcd}$$

On maps. To obtain the generic family of maps of $\mathscr{E}$ consider first the pullbacks

$$\begin{tikzcd} P_0 \arrow[r] \arrow[d, "q"'] & \mathbf{C}_1(1,-) \arrow[d, "p"] \\ \mathbf{C}_1 \arrow[r, "d_0"'] & \mathbf{C}_0 \end{tikzcd} \qquad \begin{tikzcd} P_1 \arrow[r] \arrow[d, "r"'] & \mathbf{C}_1(1,-) \arrow[d, "p"] \\ \mathbf{C}_1 \arrow[r, "d_1"'] & \mathbf{C}_0 \end{tikzcd}$$

P_0 is a subobject of $\mathbf{C}_2$ and composition $m:\mathbf{C}_2 \to \mathbf{C}_1$ restricts to $m:P_0 \to \mathbf{C}_1(1,-)$ so that

$$\begin{tikzcd} P_0 \arrow[r, "m"] \arrow[d, "q"'] & \mathbf{C}_1(1,-) \arrow[d, "p"] \\ \mathbf{C}_1 \arrow[r, "d_1"'] & \mathbf{C}_0 \end{tikzcd}$$

commutes. Thus we get a generic map $s:P_0 \to P_1$. The generic family of maps is

$$\begin{tikzcd} P_0 \arrow[rr, "s"] \arrow[dr, "q"'] & & P_1 \arrow[dl, "r"] \\ & \mathbf{C}_1 & \end{tikzcd}$$

Now given a map $g:E \to \mathbf{C}_1$ in $\mathbf{C}(E)$ with domain $R = d_0 \cdot g$ and codomain $S = d_1 \cdot g$, we pull the above triangle back along g to obtain

$$\begin{tikzcd} g^*P_0 \arrow[rr, "g^*(s)"] \arrow[dr] & & g^*P_1 \arrow[dl] \\ & E & \end{tikzcd}$$

But $g^*P_0 \to E$ and $g^*P_1 \to E$ are (isomorphic to) $R \to E$ and $S \to E$ respectively, so we have a map in $\mathscr{E}/E$ with the right domain and codomain.

Definition. Suppose that $\mathbf{C}$ is a category in a topos $\mathscr{E}$ with (explicit) terminal object. Then $\mathbf{C}$ *is* (or *represents*) *a small full subcategory of* $\mathscr{E}$ if and only if the global sections functor

$$\mathbf{C}(E) \to \mathscr{E}/E$$

is full and faithful.

0.2 Completeness of internal categories

Suppose that $\mathbf{C}$ is an internal category in a topos $\mathscr{E}$, and that one wishes (implausibly enough) to say that $\mathbf{C}$ is complete. There are a number of notions of varying strengths. One could have a scheme giving a limit for each internal diagram in $\mathbf{C}$. (Clearly this should mean 'in every slice'.) Or better, one could ask for a right adjoint to the diagonal functor for each internal category. (Express this in an appropriate internal logic and again it means 'in every slice'.) However since I am not using the internal logic in this paper it seems best to obtain a formulation from the notion of completeness for indexed categories. There is no point in assuming more than pullbacks for the base category.

Definition. Suppose that $\mathscr{C}$ is an indexed category over a category $\mathscr{E}$ with pullbacks. Then $\mathscr{C}$ is *complete* if and only if

(i) *$\mathscr{C}$ has finite limits*, that is each fibre $\mathscr{C}(E)$ has finite limits and they are preserved by the reindexing functors,

(ii) *$\mathscr{C}$ has indexed products*, that is each reindexing functor $\mathscr{C}(\alpha)$ has a right adjoint $\prod \alpha$, such that the Beck–Chevalley condition holds: if

$$\begin{tikzcd} F' \arrow[r, "\gamma"] \arrow[d, "\delta"'] & F \arrow[d, "\alpha"] \\ E' \arrow[r, "\beta"'] & E \end{tikzcd}$$

is a pullback in $\mathscr{E}$, then $\mathscr{C}(\beta)\circ\prod\alpha \cong \prod\delta\circ\mathscr{C}(\gamma)$.

Definition. Suppose that $\mathbf{C}$ is a (small) category in a locally cartesian closed category $\mathscr{E}$. Then $\mathbf{C}$ is *complete* if and only if

(i) $\mathbf{C}$ is equipped with the structure of a category with finite limits,

(ii) the externalization of $\mathbf{C}$ has indexed products.

Remarks. If $\mathbf{C}$ is complete in $\mathscr{E}$, then the externalization of $\mathbf{C}$ is complete, and what is more reindexing preserves the finite limits on the nose. For modelling languages one often wants the Beck–Chevalley condition to hold on the nose also.

Though I do not prove it here, it is worth recording the fact (which is 'almost obvious in the internal logic') that full subcategories (with terminal object) have their limit structure determined by that of the ambient category.

Theorem. *Suppose that $\mathscr{C}$ is a full subcategory of a locally cartesian closed category $\mathscr{E}$ with terminal object so that*

$$\mathscr{C}(E)\to\mathscr{E}/E$$

preserves the terminal object.

(i) *If $\mathscr{C}$ has finite limits, then these agree with those of $\mathscr{E}$. Conversely, if $\mathscr{C}$ is closed under the finite limits of $\mathscr{E}$, then $\mathscr{C}$ has finite limits. Thus*

$$\mathscr{C}(-)\to\mathscr{E}/-$$

preserves and reflects finite limits.

(ii) *If $\mathscr{C}$ is cartesian closed, then that structure agrees with the structure in $\mathscr{E}$. Conversely, if $\mathscr{C}$ is closed under (binary) products and function spaces in $\mathscr{E}$, then*

$\mathcal{C}$ is cartesian closed. Thus

$$\mathcal{C}(-) \to \mathcal{E}-$$

preserves and reflects cartesian closed structure.

(iii) *If $\mathcal{C}$ has indexed products, then these agree with those of $\mathcal{E}$. Conversely, if $\mathcal{C}$ is closed under indexed products in $\mathcal{E}$, then $\mathcal{C}$ has indexed products. Thus*

$$\mathcal{C}(-) \to \mathcal{E}/-$$

preserves and reflects indexed products.

In particular this theorem applies to small full subcategories of a topos $\mathcal{E}$ as defined in 0.1. Unfortunately, as I write, it is not clear whether there are such small categories in a topos, so a weaker notion of completeness is needed. The sense and significance of the following definition will be explained in [13]. (It amounts to a scheme giving the existence of limits.)

Definition. Suppose that there is a full subcategory $s(\mathcal{E})$ of a topos $\mathcal{E}$ such that any object in $\mathcal{E}$ is covered by one in $s(\mathcal{E})$.

(i) If $\mathcal{C}$ is indexed over $\mathcal{E}$ and is complete as a category indexed over $s(\mathcal{E})$, then $\mathcal{C}$ is *weakly complete* over $\mathcal{E}$.

(ii) If $\mathbf{C}$ is a small category in $\mathcal{E}$ whose externalization is weakly complete over $\mathcal{E}$, then $\mathbf{C}$ is *weakly complete* in $\mathcal{E}$

Say that the category $s(\mathcal{E})$ is *sound* for $\mathcal{C}$ or $\mathbf{C}$.

0.3. Notation

The notation is mainly that of [11] with one main exception. I now use an absolute value sign '$|\cdots|$' instead of open face brackets '$[\![\cdots]\!]$' to denote the realizability interpretation of a formula. The full subcategory of **Eff** consisting of all separated objects plays an important part in this paper and is written **Sep**.

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1. The category of modest sets

This section contains a description of a category equivalent to that of the effective objects discussed in [11]. There, the category was introduced as a subcategory of the effective topos, and its own categorical properties were not systematically presented. Here I try to remedy that deficiency by giving a self-contained treatment.

The idea behind the category of effective objects is familiar and stems from Kreisel [16]. An explicitly categorical formulation of the same idea (based on a model for the λ-calculus in place of a more general applicative structure) is in [24]. Originally I used the name *effective objects* as they generalize the *effective operations* in [16]. However this terminology is only good for one realizability topos. So I have adopted Scott's recent suggestion to give the objects of the category the distinctive name of *modest sets*.

1.1. Modest set and effective objects

Definition. A *modest set* consists of a set X together with a partial enumeration $\pi = \pi_X : |X| \to X$ of X. That is to say, $|X|$ is a subset of the natural numbers $\mathbb{N}$, and $\pi : |X| \to X$ is surjective. The notation does not distinguish between a modest set and its underlying set. The subset $|X|$ of $\mathbb{N}$ is the set of *codes* or *indices* of X. (Usually I do not trouble to give π a subscript.)

If X is a modest set and $x \in X$, then set $|x| = \pi^{-1}(x)$. $|x|$ is the set of *codes* or *indices* for x.

A *map* from a modest set X to another Y is a map $f : X \to Y$ of underlying sets such that there is a partial recursive function ϕ with dom $\phi \supseteq |X|$ and $f \circ \pi = \pi \circ \phi : |X| \to Y$: that is so that the following diagram commutes:

$$\begin{tikzcd} {|X|} \arrow[r, "\phi"] \arrow[d, "\pi"'] & {|Y|} \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It seems reasonable to adopt the language of recursive algebra and say that ϕ *tracks* f.

If $f : X \to Y$ is a map of modest sets, then write $|f|$ for the set of indices of partial recursive functions which track f.

Finally, it is clear that the collection of maps contains identities and is closed under (associative) composition, so that the modest sets and maps form a category, the *category* **M** *of modest sets.*

Remark. Our formulation makes it obvious that Eršov's category of enumerations [7] is a full subcategory of the category of modest sets. It consists of those X in **M** with $|X| = \mathbb{N}$.

In [11] the full subcategory of (strictly) effective objects in the effective topos **Eff** is described. There is an obvious connection with **M**.

Proposition. *There is a full and faithful functor* $T:\mathbf{M}\to\mathbf{Eff}$ *whose image is the subcategory of strictly effective objects.*

Proof. A modest set X gives rise to an effective object $(X, =)$ determined by

$$[x \in X] = |x| = \pi^{-1}(\{x\}).$$

The correspondence of the maps is given in Section 7 of [11]: maps between strictly effective objects in **Eff** are exactly given by maps on underlying sets tracked by partial recursive functions. □

1.2. The modest sets are cartesian closed

M has a terminal object 1 defined by

$$1 = \{*\},$$

some one-element set, and

$$|1| = \{0\}.$$

(Equivalently one could set $|1|$ equal to any non-empty set of natural numbers.)

The product in **M** of two modest sets X and Y can be defined by taking the product $X \times Y$ of the underlying sets, setting

$$|X \times Y| = \{\langle n, m\rangle \mid n \in |X|, m \in |Y|\}$$

and defining $\pi : |X \times Y| \to X \times Y$ by

$$\pi\,|\langle n, m\rangle| = (x, y) \quad \text{iff} \quad \pi_X(n) = x \quad \text{and} \quad \pi_Y(m) = y.$$

The projections are the usual ones on the underlying sets which are tracked by functions taking $\langle n, m\rangle$ to n and to m. It is easy to check that this is the categorical product.

If X and Y are modest sets, then a function space can be defined as follows. Set

$$Y^X = \{f \mid f : X \to Y \text{ in } \mathbf{M}\}, \qquad |Y^X| = \bigcup\{|f| : |f| \in Y^X\}$$

(so that $|Y^X|$ is the set of indices for functions which track maps $f : X \to Y$), and define $\pi : |Y^X| \to Y^X$ by

$$\pi(k) = f \quad \text{iff} \quad k \in |f|.$$

(Note that the notation $|f|$ for the indices of functions tracking f which was introduced in 1.1 agrees with the notation $|f| = \pi^{-1}(f)$ also introduced there.)

There is an isomorphism

$$\mathbf{M}(Z, Y^X) \cong \mathbf{M}(Z \times X, Y)$$

(natural in Z and Y, and hence in X) obtained from the natural isomorphism at the level of sets.

Proposition. **M** *is cartesian closed.*

In view of the identification of **M** with the strictly effective objects of **Eff**, this proposition can also be deduced from Section 7 of [11]. Strictly effective objects are closed under product and function space in **Eff**. In particular we have the following result.

Proposition. *The functor* $T:\mathbf{M}\to\mathbf{Eff}$ *preserves the cartesian closed structure.*

1.3. Families of modest sets indexed by a modest set

Suppose that A is a modest set. There are two ways to think of an A-indexed family of modest sets.

(i) There is the traditional category-theoretic picture of a map whose fibres are the members of the family. So an A-indexed family is just a map $f:X\to A$ in **M**.

(ii) There is the traditional set-theoretic picture of a map from A to **M**. To do this forget about the structure so that an A-indexed family is just a collection $\langle X_a \mid a\in A\rangle$ of modest sets indexed by the underlying set of A.

Given a collection as in (ii) with $\pi_a:|X_a|\to X_a$ for each $a\in A$, we construct a modest set X as follows. We set

$$X=\bigsqcup\{X_a \mid a\in A\},$$

the disjoint sum of the X_a,

$$|X|=\{\langle n,m\rangle \mid n\in|a| \text{ some } a\in A \text{ and } m\in|x| \text{ some } x\in X_a\},$$

and define $\pi:|X|\to X$ by $\pi(\langle n,m\rangle))=x$ if and only if $n\in|a|$, $m\in|x|$ and $x\in X_a$. It seems reasonable to adopt Martin–Löf's notation and write $X=\sum_{a\in A}X_a$. There is an obvious map of sets $X\to A$ which sends all of X_a to a and this map is tracked by the function sending $\langle n,m\rangle$ to n. This gives a map $f:X\to A$ of modest sets.

Conversely a map $f:X\to A$ in **M** gives a collection $\langle X_a \mid a\in A\rangle$ of modest sets as follows. Set

$$X_a=\{x \mid f(x)=a\},\qquad |X_a|=\bigcup\{|x| \mid x\in X_a\},$$

and define $\pi_a:|X_a|\to X_a$ to be the restriction of π to $|X_a|$.

Of course functions $f:X\to A$ in **M** are objects of the slice category $\mathbf{M}/A$. The equivalence of the two notions of indexed family can be expressed by imposing an equivalent category structure on the indexed collections.

Definition. Let A be a modest set. Given collections $\{X_a \mid a\in A\rangle$ and $\langle Y_a \mid a\in A\rangle$ of modest sets, a *map* from one to the other (*over* A) is a family of maps $h_a:X_a\to Y_a$ such that there is a partial recursive $\psi:|A|\to\mathbb{N}$ satisfying $n\in|a|$ implies $\psi(n)$ an index for h_a, all $a\in A$. It is easy to give identities and check closure under (associative) composition. Thus collections indexed over A form a category, the *category* $\mathbf{M}^A$ *of collections of modest sets over* A.

Proposition. *The two constructions described above are* (*the object parts of*) *equivalences between the slice category* $\mathbf{M}/A$ *and the category* $\mathbf{M}^A$ *of collections of modest sets.*

(The only apparent difference between objects in $\mathbf{M}^A$ and in $\mathbf{M}/A$ is that those in $\mathbf{M}^A$ contain no information about codes of A. But this comes into the definition of maps. There is an analogous phenomenon already for ordinary sets. To make a fibred set from an indexed family one makes the fibres disjoint.)

1.4. Pullback of modest sets

One natural way to think of pullbacks is in terms of indexed collections. Suppose given a map $h: B \to A$ in $\mathbf{M}$ and an A-indexed collection $\langle X_a \mid a \in A \rangle$ of modest sets. Reindexing gives a B-indexed collection $\langle X_{h(b)} \mid b \in B \rangle$. Clearly reindexing extends to maps and gives a functor

$$\mathbf{M}^h : \mathbf{M}^A \to \mathbf{M}^B.$$

Just as in the category of sets this gives rise to the pullback functor. Compose with the equivalences of 1.3 to get a functor

$$h^* : \mathbf{M}/A \to \mathbf{M}/B,$$

such that

$$\begin{tikzcd} \mathbf{M}/A \arrow[r, "h^*"] \arrow[d, shift right] & \mathbf{M}/B \arrow[d, shift right] \\ \mathbf{M}^A \arrow[r, "\mathbf{M}^h"'] \arrow[u, shift right] & \mathbf{M}^B \arrow[u, shift right] \end{tikzcd}$$

commuts up to natural isomorphism.

A pullback functor $h^* : \mathbf{M}/A \to \mathbf{M}/B$ should be right adjoint to the functor $\sum_h : \mathbf{M}/B \to \mathbf{M}/A$, the *indexed sum along* h, obtained by composition with h. Now given $Y \to B$ in $\mathbf{M}/B$ and $\langle X_a \mid a \in A \rangle$ in $\mathbf{M}^A$, it is easy to see that maps

$$Y \to \Big(\sum_{b \in B} X_{n(b)} \Big) \quad \text{in } \mathbf{M}/B$$

correspond exactly to maps

$$\langle (\textstyle\sum_h Y)_a \mid a \in A \rangle \to \langle X_a \mid a \in A \rangle \quad \text{in } \mathbf{M}^A.$$

The required identification follows from the equivalence in 1.3.

Proposition. $h^* : \mathbf{M}/A \to \mathbf{M}/B$ *is a pullback functor in* $\mathbf{M}$.

An equivalent explicit description of the pullback can be given as follows. Suppose given $f: X \to A$. We set

$$h^*X = \{(b, x) \mid h(b) = f(x)\}$$

(the pullback at the level of sets),

$$|h^*X| = \{\langle n, m\rangle \mid n \in |b|, m \in |x|, h(b) = f(x)\}$$

(effectively, the pullback of $|B|$ and $|X|$ over A), and define $\pi: |h^*X| \to h^*X$ by

$$\pi(\langle n, m\rangle) = (b, x) \quad \text{iff} \quad n \in |b|, m \in |x| \quad \text{and} \quad h(b) = f(x).$$

Not only does **M** have pullbacks, but using a characterization of pullbacks in **Eff** (2.8 of [12]) one sees that they correspond under T to pullbacks in **Eff**. Putting this together with the corresponding result for products (1.2), one has the following result.

Proposition. **M** *has finite limits and* $T: \mathbf{M} \to \mathbf{Eff}$ *preserves finite limits.*

For completeness one should note the following triviality.

Proposition. $T: \mathbf{M} \to \mathbf{Eff}$ *preserves the* Σ*-functors in* **M**.

The following is an important consequence of the preservation properties of T.

Proposition. *The Beck–Chevalley condition holds in* **M**: *if*

$$\begin{tikzcd} B' \arrow[r, "h'"] \arrow[d, "k'"'] & A' \arrow[d, "k"] \\ B \arrow[r, "h"'] & A \end{tikzcd}$$

is a pullback, then $h^* \circ \Sigma_k = \Sigma_{k'} \circ h'^*$.

Proof. The condition holds in the topos **Eff** and $T: \mathbf{M} \to \mathbf{Eff}$ is full and faithful and preserves the structure. □

1.5. Indexed products: the right adjoint to pullback

Given a map $h: B \to A$ in **M**, the *indexed product along* h is a functor $\Pi_h: \mathbf{M}/B \to \mathbf{M}/A$ (if such exists) right adjoint to $h^*: \mathbf{M}/A \to \mathbf{M}/B$. For $g: Y \to B$ in $\mathbf{M}/B$ we expect $\Pi_h \circ Y \in \mathbf{M}/A$ to be such that a fibre $(\Pi_h \circ Y)_a$ is the modest set of *sections of* $g: Y \to B$ over $h^{-1}(a)$. So it is natural to work with indexed collections.

First however note that $h^{-1}(a)$ can be regarded as a modest set where

$$|h^{-1}(a)| = \{n \in |B| \mid h(\pi(n)) = a\},$$

and where $\pi : |h^{-1}(a)| \to h^{-1}(a)$ is just the restriction of π_B. It is in a natural sense a *modest subset* of B, which corresponds to a (canonical) *closed* monic in **Eff** as in [11].)

Suppose that $\langle Y_b \mid b \in B \rangle$ is in $\mathbf{M}^B$ and let $g : Y \to B$ be the corresponding object of $\mathbf{M}/B$ defined in 1.3. For each $a \in A$, define $(\Pi_h \cdot Y)_a$ as a modest subset of the function space $Y^{h^{-1}(a)}$ as follows. Take

$$(\Pi_h \cdot Y)_a = \{k : h^{-1}(a) \to Y \mid g \circ k = \mathrm{id}_{h^{-1}(a)}\},$$
$$|(\Pi_h \cdot Y)_a| = \bigcup \{|k| \mid k \in (\Pi_h \cdot Y)_a\},$$

with π induced from the function space. Note that if there is no b with $h(b) = a$, then as one expects $(\Pi_h \cdot Y)_a$ is (isomorphic to) 1.

A map $\langle X_a \mid a \in A \rangle$ to $\langle (\Pi_h \cdot Y)_a \mid a \in A \rangle$ in $\mathbf{M}^a$ consists of a family $p_a : X_a \to (\Pi_h \cdot Y)_a$ of maps whose indices can be found effectively from codes of a. Given $\langle p_a \mid a \in A \rangle$ define $q_b : X_{h(b)} \to Y_b$ by

$$q_b(x) = \pi_2((p_{h(b)}(x))(b))$$

(where π_2 denotes the second projection from $Y = \{(b, y) \mid y \in Y_b\}$). Clearly $q_b : X_{h(b)} \to Y_b$ is a map of modest sets for each $b \in B$, and indices for q_b can be found effectively from codes for b. Thus $\langle q_b \mid b \in B \rangle$ is a map $\langle X_{h(b)} \mid b \in B \rangle$ to $\langle Y_b \mid b \in B \rangle$ in $\mathbf{M}^B$. Conversely given $\langle q_b \mid b \in B \rangle$ define $p_a : X_a \to (\Pi_h \cdot Y)_a$ by setting

$$(p_a(x))(b) = (b, q_b(x))$$

for $b \in h^{-1}(a)$. Again one can check that $\langle p_a \mid a \in A \rangle$ is a map $\langle X_a \mid a \in A \rangle$ to $\langle (\Pi_h \cdot Y)_a \mid a \in A \rangle$ in $\mathbf{M}^A$. These two constructions are inverse to each other and natural. So the functor $\mathbf{M}^h : \mathbf{M}^A \to \mathbf{M}^B$ has a right adjoint. Now use the equivalence of 1.3.

Proposition. *The pullback functor* $h^* : \mathbf{M}/A \to \mathbf{M}/B$ *has a right adjoint* $\Pi_h : \mathbf{M}/B \to \mathbf{M}/A$.

Note that the Beck–Chevalley condition holds for Σ and hence by taking right adjoints for Π. Again Π-functors in **M** coincide with those in the large category **Eff**. A direct proof would be laborious, but an indirect one will be provided by the analysis in [13].

Proposition. $T : \mathbf{M} \to \mathbf{Eff}$ *preserves* Π*-functors.*

The following theorem encapsulates the results of the last few sections. Recall that a category is *locally cartesian closed* if and only if it has finite limits and each slice category is cartesian closed.

Theorem. **M** *is a locally cartesian closed category.*

Proof. Exponentials in slices came from $\prod$-functors and pullbacks. (See, e.g. [8].) $\square$

It follows from [25] that **M** is a model for the version of Martin-Löf type theory with the unintended extensional equality described in [19].

*1.6. Finite colimits in **M***

Finite colimits can be computed in **M** in a quite straightforward way. **M** has an initial object, 0, the empty set with unique partial enumeration. Given objects X and Y of **M**, take $X + Y$, the coproduct of the underlying sets, put

$$|X+Y| = \{\langle 0, n\rangle \mid n \in |X|\} \cup \{\langle 1, n\rangle \mid n \in |Y|\},$$

and define $\pi : |X+Y| \to X+Y$ by

$$\pi(\langle 0, n\rangle) = x \in X \quad \text{if } n \in |x|, x \in X,$$
$$\pi(\langle 1, n\rangle) = y \in Y \quad \text{if } n \in |y|, y \in Y.$$

There are obvious maps $X \to X+Y$, $Y \to X+Y$ making $X+Y$ the coproduct of X and Y. Finally given maps $f, g : X \rightrightarrows Y$ in **M**, take $\bar{Y}$, the coequalizer of the underlying sets $q : Y \to \bar{Y}$, set $|\bar{Y}| = |Y|$ and define $\pi : |\bar{Y}| \to \bar{Y}$ to be $q \circ \pi_Y$. Then

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} \bar{Y}$$

is a coequalizer diagram in **M**.

Proposition. **M** *has finite colimits.*

For completeness at this point note the following.

Proposition. **M** *has a natural number object and* $T : \mathbf{M} \to \mathbf{Eff}$ *preserves it.*

Proof. The identity on $\mathbb{N}$ gives a natural number object in **M**, and T carries it to the standard $(\mathbb{N}, =)$ in **Eff**. $\square$

The coproducts in **M** are well behaved. 0 is a strict initial object and coproducts are disjoint and stable under pullback. In particular, by an easy extension of [25], **M** provides a model for (the unintended version of) Martin–Löf type theory with sum types. Also coproducts in **M** correspond to those in **Eff** by the description in [12].

Proposition. **M** *has stable (finite) coproducts and* $T : \mathbf{M} \to \mathbf{Eff}$ *preserves them.*

The situation for coequalizers is different. Recall first from [11] the following two facts.

(1) The image of **M** in **Eff** is closed under subobjects in **Eff**.

(2) The image of **M** in **Eff** is *not* closed under all quotients in **Eff** but only under those by a closed equivalence relation.

Take a quotient in **Eff** of an effective object by a non-closed equivalence relation. By (1) its kernel pair lives in (the image of) **M**; so by (2) the coequalizer in **M** of the kernel pair does not map to a coequalizer in **Eff**.

A concrete example based on the failure in (2) may help. Consider the quotient $(\mathbb{N}, \sim)$ of the natural number object $(\mathbb{N}, =)$ in **Eff** where

$$[n \sim m] = \{\langle n, m, k\rangle \mid \forall i \geq k: n \cdot i{\downarrow} \leftrightarrow m \cdot i{\downarrow}\}.$$

(Here "$n \cdot i$" denotes Kleene application of the nth partial recursive function to i and "$\downarrow$" denotes termination.) The relation $\sim$ is not closed for the double negation topology and hence $(\mathbb{N}, \sim)$ is "not a modest set". The kernel pair of the quotient $(\mathbb{N}, =) \to (\mathbb{N}, \sim)$ does live in the category of modest sets and can be represented as follows. Take

$$E = \{(n, m) \mid \exists k\ \langle n, m, k\rangle \in [n \sim m]\},$$
$$|E| = \{\langle n, m, k\rangle \mid \langle n, m, k\rangle \in [n \sim m]\}$$

and $\pi : |E| \to E$ the obvious map $\langle n, m, k\rangle \to (n, m)$. The two obvious projections $E \rightrightarrows \mathbb{N}$ in the category of modest sets **M** are the kernel pair. Their coequalizer in **M** is the set of equivalence classes $\bar{\mathbb{N}}$ under the relation

$$\exists k\ \langle n, m, k\rangle \in [n \sim m],$$

with $\pi : |\bar{\mathbb{N}}| \to \bar{\mathbb{N}}$ determined by setting

$$\|[n]\| = [n]$$

for $[n]$ the equivalence class of n. So $\bar{\mathbb{N}}$ (that is $T(\bar{\mathbb{N}})$) is the separated reflection of $(\mathbb{N}, \sim)$ in **Eff**, obtained by applying double negation to the relation $\sim$. (Why this inevitably happens will be explained in [13].)

*1.7. Monics and epis in **M***

Proposition. *Let $f : X \to Y$ be a map in* **M**.

(i) *f is monic if and only if f is injective on underlying sets.*

(ii) *f is epi if and only if f is surjective on underlying sets.*

Proof. (i) is obvious as 1 is a generator. So the global sections or underlying set functor is faithful and preserves pullbacks (cf. 1.4).

(ii) is not quite so obvious: but the underlying set functor is faithful and preserves pushouts by 1.6. $\square$

The functor $T : \mathbf{M} \to \mathbf{Eff}$ preserves monics as it is faithful and preserves pullbacks. However, 1.6 shows that the dual argument does not work for epis. For a counterexample let 2 be the modest set $\pi : \{0, 1\} \to \{0, 1\}$ with π the identity and Σ the modest set $\pi : \mathbb{N} \to \{\top, \bot\}$ where π is determined by setting

$$\pi^{-1}(T) = \{n \mid n \cdot n\downarrow\}.$$

Then 2 is a coproduct of 1 with itself while Σ (or rather $T(\Sigma)$) is the r.e. subobject classifier in **Eff** considered by Rosolini [23]. A bijection from $\{0, 1\}$ to $\{\top, \bot\}$ is a map of modest sets which is clearly (monic and) epi. However there is no surjection from 2 to Σ in **Eff** (otherwise 2 would be isomorphic to Σ in **Eff**).

Proposition. *$T : \mathbf{M} \to \mathbf{Eff}$ preserves monics. But $\mathbf{M}$ is not balanced (monic and epi does not imply iso) and T does not preserve epis.*

1.8. Surjective monic factorization in $\boldsymbol{M}$

T does preserve some epis, the *surjections* (or *stable extremal epis*) in **M**. Consider $f : X \to Y$ in **M** satisfying:

(*) there is a partial recursive $\psi : |Y| \to |X|$ such that the following diagram commutes:

$$\begin{tikzcd} {|X|} \arrow[d, "\pi"'] & {|Y|} \arrow[l, "\psi"'] \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Note the following facts about (*)

Lemma 1. *Maps satisfying (*) are stable under pullback.*

Proof. Straightforward on the basis of 1.4 $\square$

Lemma 2. *Any map $X \xrightarrow{f} Y$ in $\mathbf{M}$ factors as $X \xrightarrow{\bar{f}} \bar{X} \xrightarrow{m} Y$ with $\bar{f}$ satisfying (*) and m monic. (And by Lemma 1 this factorization is stable.)*

Proof. Given $f : X \to Y$ in **M**, let $X \xrightarrow{\bar{f}} \bar{X} \xrightarrow{m} Y$ be the surjection injection factorization at the level of sets. Set $|\bar{X}| = |X|$ and define $\pi : |\bar{X}| \to \bar{X}$ to be $\bar{f} \circ \pi_X$. This gives the factorization in **M**. $\square$

Lemma 3. *If f satisfies (*) then f is extremal epi.*

Proof. Suppose that $X \xrightarrow{g} Z \xrightarrow{m} Y$ is a factorization of $X \xrightarrow{f} Y$ with m monic. As f is a surjection on underlying sets, m is an isomorphism on underlying sets. There are partial recursive $\phi : |X| \to |Z|$ (as g is a map) and $\psi : |Y| \to |X|$ (as f satisfies (*)) so that

$$\begin{tikzcd} {|X|} \arrow[r, "\phi"] \arrow[d, "\pi"'] & {|Z|} \arrow[d, "\pi"] \\ X \arrow[r, "g"'] & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} {|X|} \arrow[d, "\pi"'] & {|Y|} \arrow[l, "\psi"'] \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commute. But then

$$\begin{tikzcd} |Y| \arrow[r, "\phi \cdot \psi"] \arrow[d, "\pi"'] & |Z| \arrow[d, "\pi"] \\ Y \arrow[r, "m^{-1}"'] & Z \end{tikzcd}$$

commutes so that m has an inverse in **M**. □

Lemma 4. *Any extremal epi satisfies* (*).

Proof. Take its factorization. □

The following proposition follows from the lemmas.

Proposition. **M** *has* (*stable*) *surjective-monic factorization.*

One can readily read the following lemma off from the characterization of epis (equivalently surjections) in **Eff**.

Lemma 5. *f satisfies* (*) *if and only if Tf is epi in* **Eff**.

(This gives an alternative proof of Lemma 3.)

Proposition. $T:\mathbf{M}\to\mathbf{Eff}$ *preserves surjective-monic factorizations.*

Recall that a category is *coherent* (or logical in the terminology of Makkai and Reyes [18]) just when it has finite limits, surjective-monic factorization and stable finite sups of subobjects. However this last follows from stable finite coproducts and surjective monic factorization (stable images in [18] where see the proof of 3.3.10). Of course T preserves all relevant structure.

Theorem. **M** *is a coherent category and* $T:\mathbf{M}\to\mathbf{Eff}$ *is a coherent functor.*

Finally note that as there are epis which are not surjections, **M** is *not* a pretopos. A pretopos closely related to **M** will be described in [13].

1.9. The category of partial equivalence relations

It is easy to see that **M** is an essentially small category.

A *partial equivalence relation* R is a symmetric and transitive relation; so R is only reflexive on its field. R can be identified with its set of equivalence classes which partition some subset of the domain of the relation. Context should make clear which aspect of R is meant. Given a partial equivalence relation R on $\mathbb{N}$, let $|R|$ be the field of the relation and $\pi:|R|\to R$ the quotient map taking a natural number to its equivalence class. Let **P** denote the full subcategory of **M** whose objects are $\pi:|R|\to R$ for some partial equivalence relation R. Clearly **P** is (isomorphic to) the familiar category of partial equivalence relations (see for example [24] or [11], and also [16] and [10]).

Proposition. *The embedding* $\mathbf{P}\to\mathbf{M}$ *is a (weak) equivalence of categories.*

Proof. Take $\pi:|X|\to X$ in **M**. Define R by $R(n, m)$ if and only if $\pi(n)=\pi(m)$. Then the corresponding $\pi:|R|\to R$ is isomorphic to $\pi:|X|\to X$ in **M**. □

Clearly **P** is a small category equivalent to **M**, so that **M** is essentially small.

2. An internal category of modest sets

The small category **P** described in 1.9 is an internal category in **Sets**, and so is mapped by the inclusion $\Delta:\mathbf{Sets}\to\mathbf{Eff}$ to an internal category in **Eff**. However this category $\Delta\mathbf{P}$ will not usefully represent the category of modest sets: if A is a modest set in **Eff**, then $\mathbf{Eff}(A, \Delta P)$ is not equivalent to $\mathbf{M}/A$. Compare it with the category of $\mathbf{M}^A$ of collections of modest sets over A introduced in 1.3. $\mathbf{Eff}(A, \Delta\mathbf{P})$ has a reasonable collection of objects but quite the wrong maps: there is no uniformity as required in the definition of 1.3.

However there is a subcategory $\hat{\mathbf{P}}$ of $\Delta\mathbf{P}$ which does represent the modest sets well. It has (in **Eff**) all the good categorical properties of **M**. Moreover, and this is conceptually vital, it is a full subcategory of **Eff** in the sense of the introduction. The essential image of the externalization of $\hat{\mathbf{P}}$ in $\mathbf{Eff}/(X, =)$ form the modest sets indexed over $(X, =)$. As indexed over **Sets**, $\hat{\mathbf{P}}$ is complete, a fact essentially in [10]. It is a short step from that to the weak completeness of $\hat{\mathbf{P}}$ in **Eff**. Thus this section closes by establishing the existence of a small complete category in the effective topos.

2.1. *The internal category* $\hat{\mathbf{P}}$

Definition. $\hat{\mathbf{P}}$ is the subcategory of $\Delta\mathbf{P}$ defined as follows. The objects of $\hat{\mathbf{P}}$ are those of $\Delta\mathbf{P}$ so that

$$(\hat{\mathbf{P}})_0=\Delta\mathbf{P}_0.$$

The maps of $\hat{\mathbf{P}}$ are given as a canonical monic

$$(\hat{\mathbf{P}})_1\rightarrowtail\Delta\mathbf{P}_1$$

with existence predicate E defined by

$$E(f)=|f|=\{a \mid a \text{ an index for } f\}$$

where $f:R\to S$ in **P**. (That is, $E(f)=\{a \mid nRm \Rightarrow a\cdot n\, S\, a\cdot m, \text{ and } a \text{ induces } f\}$.)

It is easy to check in **Eff** that $(\hat{\mathbf{P}})_1$ contains identities and is closed under composition so that $\hat{\mathbf{P}}$ is indeed a subcategory of $\Delta\mathbf{P}$. In fact, $\hat{\mathbf{P}}$ is defined so that there is an immediate connection with the categories $\mathbf{M}^A$ defined in 1.3 (see 2.3).

2.2. *Properties of* $\hat{\mathbf{P}}$

Express the categorical properties of **M** and hence **P**, discussed in Section 1, in terms of additional structure, and these properties can all be described in finite limit logic (see [6] for example). Hence they hold for the internal category $\Delta\mathbf{P}$ in **Eff**. The fact that $\hat{\mathbf{P}}$ inherits these properties is a consequence of the effective way in which the structure is defined in **M** and **P**. Note that $\hat{\mathbf{P}}$ itself has separated equality and so the analysis of [11] applies to simplify the realizability interpretation.

I will discuss briefly the case of finite limits as that is crucial to this paper. The other properties are left to the conscientious reader. The terminal object is easy, so consider pullbacks. By [11] it suffices

(i) to take codes a, b for maps f, g with common codomain in **P** and compute codes for the pullback diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & S \arrow[d, "g"] \\ R \arrow[r, "f"'] & T; \end{tikzcd}$$

(ii) to take codes for a commuting diagram

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[d, "h"'] & S \arrow[d, "g"] \\ R \arrow[r, "f"'] & T, \end{tikzcd}$$

that is, codes c, d for the maps h, k, and compute a code for the universal factorization $V \to U$.

Assume the usual definition of U as a subobject of the product $R \times S$. Now (i) is easy. The codes needed are those for the maps $U \to R$ and $U \to S$: for these, take indices for the first and second projections. (It is a little misleading that these are independent of a and b.) As for (ii), what is required is a code for the function

$$\lambda x\,.\,\langle c \cdot x, d \cdot x \rangle$$

which is effectively computable from c and d. Note that there is enough uniformity to ensure that $\hat{\mathbf{P}}$ has finite limits in the strong sense that it comes equipped with the relevant structure (adjoints to the 'diagonal functors').

Theorem. *The internal category* $\hat{\mathbf{P}}$ *in* **Eff**

(i) *has finite limits,*

(ii) *is locally cartesian closed,*

(iii) *has finite colimits and a natural number object,*

(iv) *is a coherent category with stable sums.*

2.3. *The externalization of* $\hat{\mathbf{P}}$

By definition, the objects of $\hat{\mathbf{P}}$ form a set or sheaf (for the double negation topology), while the maps of $\hat{\mathbf{P}}$ are at least separated (again for the double negation topology). Thus the sheafification functor a and the separated reflection functor s can be used to describe the externalizaton of $\hat{\mathbf{P}}$.

Let $(X, =)$ be an arbitrary object of **Eff**. Note that there are two ways to describe a sheafification $a(X, =)$. We can factor X out by the partial equivalence relation defined by

$$x \sim x' \quad \text{iff} \quad |x = x'| \neq \emptyset;$$

then

$$a(X, =) \cong \Delta(X/\sim).$$

Or define on X a new equality $=_a$, where

$$|x =_a x'| = \begin{cases} \mathbb{N} & \text{if } |x = x'| \neq \emptyset, \\ \emptyset & \text{otherwise;} \end{cases}$$

then

$$a(X, =) \cong (X, =_a).$$

Analogously, there are two ways to describe a separated reflection $s(X, =)$, so that there is an epic-monic factorization

$$(X, =) \twoheadrightarrow s(X, =) \rightarrowtail a(X, =).$$

$s(X, =)$ can be described by a canonical monic on $\Delta(X/\sim)$ namely by the relation

$$[x] \to \bigcup \{|x'| \mid x \sim x'\}$$

for $[x] \in X/\sim$. Or define on X a new equality $=_s$ where

$$|x =_s x'| = \begin{cases} |x| \times |x'| & \text{if } |x = x'| \neq \emptyset, \\ \emptyset & \text{otherwise;} \end{cases}$$

then

$$s(X, =) \cong (X, =_s).$$

As $\hat{\mathbf{P}}_0$ is a sheaf, maps $R : (X, =) \to \hat{\mathbf{P}}_0$ are in (natural) bijective correspondence with maps $R : a(X, =) \to \hat{\mathbf{P}}_0$. Assume for simplicity that $|x| \neq \emptyset$ for all $x \in X$. Then such a map is given by an X-indexed family $(R_x)_{x \in X}$ of partial equivalence relations such that

$$|x_1 = x_2| \neq \emptyset \quad \text{implies} \quad R_{x_1} = R_{x_2}.$$

As described in the introduction, such a map gives rise to an object $R \to (X, =)$ of the slice category, by applying the internal global sections functor. Direct calculation gives a representation as follows: R has as underling set

$$\{(x, r) \mid x \in X, r \in R_x\}$$

with equality given by

$$|(x_1, r_1) = (x_2, r_2)| = |x_1 = x_2| \times |r_1 = r_2|.$$

(This makes sense as if $|x_1 = x_2| \neq \emptyset$ then r_1, r_2 are in $R_{x_1} = R_{x_2}$ and hence $|r_1 = r_2|$ makes sense. Otherwise, $|(x_1, r_1) = (x_2, r_2)|$ is empty.) The map $R \to X$ has the obvious graph

$$((x, r), y) \to |x = y| \times |r|.$$

As $\hat{\mathbf{P}}_1$ is separated, maps $g:(X, =) \to \hat{\mathbf{P}}_1$ are in natural bijective correspondence with maps $g: s(X, =) \to \hat{\mathbf{P}}_1$. Assume still that $|x| \neq \emptyset$ for all $x \in X$. Then by Section 6.3 of [11] such a g is an X-indexed family $(g_x)_{x \in X}$ in $\mathbf{P}_1$ such that

$$|x_1 = x_2| \neq \emptyset \quad \text{implies} \quad g_{x_1} = g_{x_2},$$

and such that there is a code c with

$$\alpha \in |x| \quad \text{implies} \quad c \cdot a \in |g_x|.$$

As described in the introduction there is an induced map $g: R \to S$ over $(X, =)$, where $R \to (X, =)$ and $S \to (X, =)$ arise from the domain and codomain of g. Direct calculation gives the graph of g:

$$((x, r), (y, s)) \to |x = y| \wedge |g_x(r) = s|,$$

where of course

$$|g_x(r) = s| = \begin{cases} |r| \times |s| & \text{if } g_x(r) = s, \\ \emptyset & \text{otherwise.} \end{cases}$$

Set

$$\mathbf{P}(X, =) = \mathbf{Eff}((X, =), \hat{\mathbf{P}}),$$

so that $\mathbf{P}$ is the externalization of $\hat{\mathbf{P}}$, usually written $[\hat{\mathbf{P}}]$ in the indexed category literature. Then the above is a description of the global sections functor at $(X, =)$,

$$\mathbf{P}(X, =) \to \mathbf{Eff}/(X, =).$$

These are the components of a functor of indexed categories $\mathbf{P}(-) \to \mathbf{Eff}/-$ from $\mathbf{P}(-)$ to the standard fibration of $\mathbf{Eff}$ over itself. (As explained in the introduction, reindexing in $\mathbf{P}(-)$ inevitably corresponds to pulling back in $\mathbf{Eff}/-$.)

Recall the functor $T : \mathbf{M} \to \mathbf{Eff}$. For A a modest set, define $\mathbf{P}(A)$ by

$$\mathbf{P}(A) = \mathbf{Eff}(TA, \hat{\mathbf{P}}).$$

(Regarding $\mathbf{M}$ as a subcategory of $\mathbf{Eff}$, $\mathbf{P}(A)$ is $\hat{\mathbf{P}}$-indexed over A.) Clearly $\mathbf{P}(-)$ is a category indexed over $\mathbf{M}$. Inspection of 1.3 gives the following proposition.

Proposition. *For A a modest set, there is an equivalence functor*

$$\mathbf{P}(A) \to \mathbf{M}^A,$$

and hence an equivalence functor

$$\mathbf{P}(A) \to \mathbf{M}/A.$$

This gives rise to a (weak) equivalence of indexed categories

$$\mathbf{P}(-) \to \mathbf{M}/-$$

(from $\mathbf{P}(-)$ to the standard fibration of $\mathbf{M}$ over itself).

Remarks. (1) For a general discussion of equivalence of indexed categories, the reader may consult [2].

(2) There are other reasons for regarding $\hat{\mathbf{P}}$ as a good internal represenation of the modest sets, but for the moment the above proposition must suffice.

2.4. $\hat{\mathbf{P}}$ *is a full subcategory of* $\mathbf{Eff}$

Recall from the introduction that an internal category is a full subcategory if and only if the global sections functor is full and faithful. In the case of $\hat{\mathbf{P}}$, faithfulness is easy and the problem is to show that the functor is full.

Take $(X, =)$ in $\mathbf{Eff}$ with $|x| \neq \emptyset$ for all $x \in X$. Suppose that R, S are in $\mathbf{P}(X, =)$ with corresponding indexed families $(R_x)_{x \in X}$ and $(S_x)_{x \in X}$ and maps $R \to (X, =)$, $S \to (X, =)$ as described in 2.3. Take a map $[G]$ (with graph G) such that the following diagram commutes:

$$\begin{tikzcd} R \arrow[rr, "{[G]}"] \arrow[dr] & & S \arrow[dl] \\ & (X, =) & \end{tikzcd}$$

Claim. *There is a (unique) family of maps $g_x : R_x \to S_x$ in $\mathbf{P}_1$ such that*

$$|x_1 = x_2| \neq \emptyset \quad \textit{implies} \quad g_{x_1} = g_{x_2},$$

and such that there is a code c with

$$a \in |x| \quad \textit{implies} \quad c \cdot a \in |g_x|,$$

which induces the map $[G]$ as described in 2.3.

To prove the claim, just hack it out! Write down the conditions that $G((x, r), (y, s))$ is strict, extensional, single-valued, total, and makes the above diagram commute. The first two conditions mainly provide hygiene. The third and fourth provide for each $r \in R_x$ a unique $s \in S_y$, and the fifth ensures that $S_y = S_x$. This gives the maps $g_x : R_x \to S_x$ and it is routine to check that G is equivalent to

$$|x = y| \wedge |g_x(r) = s|$$

as described in 2.3.

Theorem. *The global sections functor*

$$\mathbf{P}(-) \to \mathbf{Eff}/\sim$$

is full and faithful so that $\hat{\mathbf{P}}$ *is a full subcategory of* **Eff**.

2.5. The indexed category of all modest sets in Eff

It is natural now to consider the essential image of $\mathbf{P}(-)$ in $\mathbf{Eff}/-$.

Definition. For $(X, =)$ an object of **Eff**, let $\mathbf{M}(X, =)$ be the full subcategory of $\mathbf{Eff}/(X, =)$ whose objects are isomorphic to those in the image of

$$\mathbf{P}(X, =) \to \mathbf{Eff}/(X, =).$$

$\mathbf{M}(X, =)$ is the *category of modest sets* over $(X, =)$. $\mathbf{M}(-)$ is the *indexed category of modest sets* (the category of modest sets indexed over **Eff**).

It is clear that $\mathbf{M}(-)$ is indeed an indexed category, and that its (Grothendieck) fibration *embeds* by a full faithful cartesian functor into $\mathbf{Eff}^2 \to \mathbf{Eff}$. Note that $\mathbf{M}(-)$ is *essentially small.* Any $R \to (X, =)$ in $\mathbf{M}(X, =)$ is a pullback of the map $G \to \Delta\mathbf{P}_0$ corresponding to the identity $\Delta\mathbf{P}_0 \to \Delta\mathbf{P}_0$. Note also that there is an obvious connection with the category $\mathbf{M}$ of Section 1. For an object A of $\mathbf{M}$ there is an equivalence

$$\mathbf{M}/A \to \mathbf{M}(TA)$$

(where $T : \mathbf{M} \to \mathbf{Eff}$ is in 1.1). The following triviality is worth stating as a separate proposition.

Proposition. $\mathbf{M}(-)$ *has finite limits and* $\mathbf{M}(-) \to \mathbf{Eff}/-$ *preserves them.*

Proof. $\hat{\mathbf{P}}$ has finite limits and hence so do $\mathbf{P}(-)$ and $\mathbf{M}(-)$. The preservation is a consequence of generalities in 0.2, but could be shown directly. □

Recall that the objects of $\hat{\mathbf{P}}$ form a sheaf and the maps a separated object. These facts have consequences for the indexed category $\mathbf{M}(-)$. Any $R : (X, =) \to \hat{\mathbf{P}}_0$ factors uniquely through $s(X, =)$ and $a(X, =)$ (see Section 2.3) and thus determines a diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & R'' \arrow[d] \\ (X, =) \arrow[r] & s(X, =) \arrow[r] & a(X, =), \end{tikzcd}$$

where the squares are pullbacks and R, R', R'' are modest over $(X, =)$, $s(X, =)$, $a(X, =)$ respectively. Also any $g:(X, =)\to \hat{\mathbf{P}}_1$ factors uniquely through $s(X, =)$. The information can be summed up as follows.

Proposition. (i) *Composition along* $(X, =)\to s(X, =)$ *induces an isomorphism of categories*

$$\mathbf{P}s(X, =)\xrightarrow{\cong}\mathbf{P}(X, =).$$

Consequently pullback along $(X, =)\to s(X, =)$ *induces an equivalence*

$$\mathbf{M}s(X, =)\xrightarrow{\sim}\mathbf{M}(X, =).$$

(ii) *Composition along* $s(X, =)\rightarrowtail a(X, =)$ *induces an isomorphism of objects*

$$(\mathbf{P}a(X, =))_0\to(\mathbf{P}s(X, =))_0.$$

Consequently pullback along $s(X, =)\rightarrowtail a(X, =)$ *induces a functor*

$$\mathbf{M}a(X, =)\to\mathbf{M}s(X, =)$$

which is essentially surjective on objects.

2.6. *The category* $\hat{\mathbf{P}}$ *indexed over* **Sets**

Recall from [11] the embedding $\Delta:\mathbf{Sets}\to\mathbf{Eff}$. For I a set, define

$$\mathbf{P}\Delta(I)=\mathbf{P}(\Delta I)=\mathbf{Eff}(\Delta I, \hat{\mathbf{P}}).$$

This category $\mathbf{P}\Delta(-)$ indexed over **Sets** is familiar to logicians.

The following analysis is a special case of that of 2.3. As $\hat{\mathbf{P}}$ is a subcategory of $\mathbf{P}\Delta$ in **Eff** (and as Δ is full and faithful) one can regard $\mathbf{P}\Delta(I)$ as a subcategory of $\mathbf{P}^I$ (the I-fold product of $\mathbf{P}$ in **Sets**.) The objects of $\mathbf{P}\Delta(I)$ are then I-indexed families $(R_i)_{i\in I}$ of partial equivalence relations. An I-indexed family $(f_i)_{i\in I}$ of maps $f_i:R_i\to S_i$, regarded as a map $\Delta I\to\Delta\mathbf{P}_1$, factors through $(\hat{\mathbf{P}})_1$ if and only if $\bigcap\{|f_i|:i\in I\}\neq\emptyset$. These are the maps in $\mathbf{P}\Delta(I)$. Clearly $\mathbf{P}\Delta(-)$ is thus a category indexed over **Sets**. Given $\alpha:J\to I$, $\mathbf{P}\Delta(\alpha)$ is just reindexing:

$$\mathbf{P}\Delta(\alpha)(R_i)_{i\in I}=(R_{\alpha(j)})_{j\in J},$$

$$\mathbf{P}\Delta(\alpha)(f_i)_{i\in I}=(f_{\alpha(j)})_{j\in J}.$$

This indexed category is the (by now standard?) model for higher order polymorphism introduced in Girard's thesis. (Those with access to [10] will see that, in order to deal with the Dialectica Interpretation, Girard uses a variant where types come equipped with canonical elements. The difference is not significant.)

In essence the two propositions of this section are proved in [10].

Proposition. *Each* $\mathbf{P}\Delta(I)$ *is cartesian closed and reindexing preserves this structure. So* $\mathbf{P}\Delta(-)$ *is an indexed cartesian closed category over* **Sets**. *Furthermore the functor*

$$\mathbf{P}\Delta(-) \to \mathbf{Eff}/\Delta(-)$$

preserves the cartesian closed structure.

Proof. The cartesian closed structure is given pointwise in each $\mathbf{P}\Delta(I)$. The preservation is easy to check directly. □

Also for each $\alpha : J \to I$ in **Sets**, one can define right and left adjoints $\Pi_{\Delta\alpha}$ and $\Sigma_{\Delta\alpha}$ to $\mathbf{P}\Delta(\alpha)$ just as in tripos theory (see [12]). Set

$$|\alpha(j) = i| = \begin{cases} \mathbb{N} & \text{if } \alpha(j) = i, \\ \emptyset & \text{otherwise,} \end{cases}$$

and then

$$\prod_{\Delta\alpha} (S_j)_{j\in J}) = (\textstyle\bigcap_j [|\alpha(j) = i| \to S_j])_{i\in I},$$

$$\sum_{\Delta\alpha} ((S_j)_{j\in J}) = (\textstyle\bigcup_j |a(j) = i| \times S_j)_{i\in I}.$$

(Here $[|\alpha(j) = i| \to S_j]$ denotes the function space in $\mathbf{P}$. The "$|\alpha(j) = i|$" is redundant in the definition of Σ, but not in that of Π.)

Proposition. *As an indexed category over* **Sets**, $\mathbf{P}\Delta(-)$ *is complete. Also the functor*

$$\mathbf{P}\Delta(-) \to \mathbf{Eff}/\Delta(-)$$

preserves finite limits and indexed products.

Proof. By 2.1 finite limits exist and are preserved under reindexng. Preservation is easy (though here we could use 0.2 on $\mathbf{P}(-) \to \mathbf{Eff}/(-)$. The right adjoints $\Pi_{\Delta\alpha}$ have just been described, and the Beck–Chevalley condition is routine. That $\Pi_{\Delta\alpha}$ as defined gives the right product in **Eff** can again be checked by direct calculation. (It is really in the internal logic.) □

It is worth noting here that the Beck–Chevalley condition holds 'on the nose' and not just 'up to isomorphism'.

2.7. The completeness of $\mathbf{M}(-)$ over ***Eff***

The reader should have in mind the discussion of completeness in 0.2. (I do not need to use the results stated there.)

Proposition. *As a subcategory of* **Eff**/−, *the indexed category* $\mathbf{M}(-)$ *is closed under* $\prod_h$ *for all maps h of sheaves.*

Proof. Recall from [11] that the image of Δ:**Sets**→**Eff** is equivalent to the category of sheaves. $\mathbf{P}(-)\to\mathbf{M}(-)$ is an equivalence so the result follows from 2.6. □

The first step is to extend this result to all maps between separated objects. Recall from 2.5 the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d, hook] & R' \arrow[r] \arrow[d, hook] & R'' \arrow[d] \\ (X,=) \arrow[r] & s(X,=) \arrow[r] & a(X,=) \end{tikzcd}$$

induced by $R:(X,=)\to\Delta\mathbf{P}_0$. Write τ for $s(X,=)\to a(X,=)$ as a (monic) map in **Eff**, and E (or $E\to a(X,=)$) for the corresponding modest set over $a(X,=)$. (It is obviously a modest set, in fact a subobject of 1 over $a(X,=)$.)

Lemma. *Over* $a(X,=)$, $R'\cong R''\times E$, *so since* $\mathbf{M}(a(X,=))$ *is closed under products in* **Eff**, $\sum_\tau R'$ *is modest.*

Lemma. *There is an isomorphism of objects over* $a(X,=)$

$$\prod_\tau R'\cong\Big(\sum_\tau R'\Big)^E(\cong R''^E),$$

and hence as $\mathbf{M}(a(X,=))$ *is closed under exponentials in each fibre of* **Eff**/(−), $\prod_\tau R'$ *is modest.*

Proof. The isomorphism is trivial category theory. □

Proposition. *If* $f:(X,=)\to(Y,=)$ *is a map of separated objects in* **Eff**, *then* $\prod_f R$ *is modest (over* $(Y,=)$*).*

Proof. As we are dealing with separated objects it is sufficient to show that $\prod_{sf} R'$ is modest (over $s(Y,=)$). Write ρ for the map $s(Y,=)\to a(Y,=)$. One has isomorphisms

$$\prod_{sf} R' \cong \rho^* \prod_{\rho} \prod_{sf} R' \qquad \text{(as } \rho \text{ monic)}$$

$$\cong \rho^* \prod_{af} \prod_{\tau} R' \qquad \text{(as } af \cdot \tau = \rho \cdot sf\text{)}.$$

But $\prod_\tau R'$ is modest by the second lemma, and $\mathbf{M}(-)$ is closed under $\prod_{af}$ (af a map of sheaves) and of course under pullback. □

Recall that **Sep** denotes the full subcategory of **Eff** consisting of all separated objects. The following results are immediate.

Theorem. *As categories indexed over* **Sep**, $\mathbf{M}(-)$ *is a complete subcategory of* **Eff**/–.

Corollary. $\mathbf{M}(-)$ *is a weakly complete subcategory of* **Eff**/– *indexed over* **Eff**.

Proof. Any object in **Eff** is covered by a separated object, so **Sep** is sound for $\mathbf{M}(-)$ in the sense of Section 0.2.

Of course, as $\mathbf{M}(-)$ is the essential image of $\mathbf{P}(-)$ in **Eff**/–, $\mathbf{P}(-)\to\mathbf{M}(-)$ is an equivalence, so the final result is obvious. □

Theorem. *The small category* $\hat{\mathbf{P}}$ *is weakly complete in* **Eff**.

The mistake which led me to believe that $\hat{\mathbf{P}}$ is complete in the strong sense is sufficiently instructive to be worth spelling out. For any $(X, =)$ in **Eff**,

$$\mathbf{P}(s(X, =)) \to \mathbf{P}(X, =)$$

is an isomorphism. Hence for any map $f:(X, =)\to(Y, =)$ there is a right adjoint $\prod_f$ to reindexing obtained from $\prod_{sf}$. Thus $\mathbf{P}(-)$ (or indeed $\mathbf{M}(-)$) has all $\prod$-functors. So why is $\mathbf{P}(-)$ not complete? The problem is the Beck–Chevalley condition. It does not transfer from separated objects to all objects as the separated reflection does not preserve pullbacks. Since many people never bother to check (or even to mention!) the Beck–Chevalley condition, I hope my oversight can serve as *an awful warning*.

3. Conclusion

This section contains some general remarks about small complete categories, and some brief comments on (problems connected with) the way in which such categories model strongly polymorphic systems. Because there are still unsolved problems associated with it, I am leaving the notion of completeness vague. So this section is just an impressionistic sketch.

3.1. Small category theory

It is customary to remark that basic category theory is constructive in character. (This is quite misleading in fact as is shown for example by the number of constructively inequivalent notions of completeness.) The discovery of interesting *small* weakly complete categories, albeit in a constructive setting, does provide new applications for the results of standard category theory. Tiresome size restrictions (for example, the solution set condition) are no longer needed, so the theorems seem more natural. The following is a good example.

Proposition. *Any small complete category is cocomplete.*

Proof. I sketch the argument for a weak notion of completeness and leave it to the reader to check the result for stronger notions. Suppose $\mathbf{C}$ is small complete and $D:J\to\mathbf{C}$ is a small diagram. Let $\mathrm{Coc}(D)$ be the category of cocones under D and let $P:\mathrm{Coc}(D)\to\mathbf{C}$ be the obvious projection. Then define $\varinjlim_J D$ to be $\varprojlim_{\mathrm{Coc}(D)} P$: construction of the colimit cocone is obvious. □

Remarks. (1) Classically one does not stress this aspect of the adjoint functor theorem. The proposition applies only to complete posets (lattices), that is, posets where all infima exist; and it says that they are cocomplete, that is, all suprema exist.

(2) For quite general reasons (as observed in 0.2) the limit structure of the indexed category $\mathbf{M}(-)$ is the same as that in $\mathbf{Eff}$. But the same is not true of the colimit structure. The ΔI coproduct of 1 in $\mathbf{Eff}$ is ΔI, which is not in $\mathbf{M}$, if I has two or more elements.

A clearer application of the adjoint functor theorem is the following.

Proposition. *Suppose that* $\mathbf{C}$ *is a small complete subcategory of a locally cartesian closed category (including the terminal object). Then there is a reflection 'from* $\mathscr{E}$ *to* $\mathbf{C}$': *that is, the functor*

$$\mathbf{C}(-)\to\mathscr{E}/-$$

of indexed categories has a left adjoint

$$\mathscr{E}/-\to\mathbf{C}(-).$$

Proof. As observed in 0.2, $\mathbf{C}(-)\to\mathscr{E}/-$ preserves all limits; and $\mathbf{C}$ is small so the result follows by an indexed adjoint functor theorem (cf. [20]). □

Recall now a general notion of T-algebra. If $T:\mathbf{C}\to\mathbf{C}$ is any endo-functor on a category, the category of T-algebras has as object maps $\theta:TX\to X$ in $\mathbf{C}$. Maps from $\theta:TX\to X$ to $\phi:TY\to Y$ are maps $\alpha:X\to Y$ in $\mathbf{C}$ such that the following diagram commutes:

$$\begin{tikzcd} TX \arrow[r, "T\alpha"] \arrow[d, "\theta"'] & TY \arrow[d, "\phi"] \\ X \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

The following are easy exercises in constructive category theory.

Proposition. *If $\mathbf{C}$ is a (small or large) complete category and $T:\mathbf{C}\to\mathbf{C}$ is any functor, then the category of T-algebras is complete.*

Corollary. *If $\mathbf{C}$ is a small complete category and $T:\mathbf{C}\to\mathbf{C}$ is any functor, then there is an initial T-algebra.*

Proof. This is the 'theorem on the existence of initial objects'. □

In the language of computer science one would say that the functor T has a least fixed point: if $\theta:TX\to X$ is initial then θ is an isomorphism.

The above corollary is related to the celebrated paper of Reynolds [22]. In essence, Reynolds first gives an indexed category version of a proof of the existence of weak initial objects in a (small) category with small products. (He does this in the language of the second-order λ-calculus.) Then he derives the existence of an initial object in a category which also has equializers. So in effect Reynolds proves the above corollary in the more general setting for polymorphism described by Seely [26].

One final point derived from [22] is worth making. The functor $[[-\to 2]\to 2]$ is covariant on $\hat{\mathbf{P}}$ and hence has a (least) fixed point. It follows that there is a modest set A and an isomorphism from A to $[[A\to 2]\to 2]$. In the case of Kleene realizability such a modest set seems hard to understand. (For function realizability it would be the natural number object.) Indeed the existence of such an object (and others like it) remained unnoticed until the perspective described in this paper was developed.

3.2. Models for the polymorphic lambda calculus

In this section I will do no more than sketch how small complete categories in toposes model strong type theories.

The best place to start is with the higher order polymorphic lambda calculus, that is, the system F_ω of Girard [10]. There is a clear description of the type system under the name 'PL-theory' in [26]. Seely has also made a choice of categorical structure, a 'PL-category', to correspond to the type theory. Essentially, a PL-category is a hyperdoctrine (Lawvere [17]) over a cartesian closed category with a generic object in some fibre. In Seely's notation, it is a category $\underset{\sim}{G}$ indexed over a category $\underset{\sim}{S}$ satisfying certain conditions.

Suppose $\mathbf{C}$ is a small weakly complete cartesian closed category in a topos $\mathscr{E}$. Suppose there is a category $s(\mathscr{E})$ sound for $\mathbf{C}$ in the sense of 0.2 which is cartesian closed, and such that $\mathbf{C}_0$ is in $s(\mathscr{E})$. Then say that '$\mathbf{C}$ is sound in $\mathscr{E}$'. Take (economically) $\underset{\sim}{S}$ to be the cartesian closed subcategory of $\mathscr{E}$ generated by $\mathbf{C}_0$ (or extravagantly take $\underset{\sim}{S}$ to be $s(\mathscr{E})$ itself). Then for $A \in \underset{\sim}{S}$ define

$$\underset{\sim}{G}(A) = \mathscr{E}(A, \mathbf{C}).$$

Showing that $(\underset{\sim}{S}, \underset{\sim}{G})$ is a PL-category is straightforward except for one point. To reflect manifest properties of syntax, a PL-category is defined so that relevant Beck–Chevally conditions hold 'on the nose'. Of course, if $\underset{\sim}{S}$ is the subcategory Δ(**Sets**) of **Eff** and $\mathbf{C}$ is $\hat{\mathbf{P}}$, then as noted in 2.6, the Beck–Chevalley condition does hold 'on the nose'. But this is the exception not the rule: for example 'up to isomorphism' is the best one can do over **Eff** (see 2.8).

Fortunately the indexed category version of weak completeness is strong for small categories: one can obtain internal right adjoints for diagonal functors. I will treat here the simplest aspect. Let A be an object of $s(\mathscr{E})$, let e denote the evaluation map

$$e : \mathbf{C}_0^A \times A \to \mathbf{C}_0$$

and p the projection

$$p : \mathbf{C}_0^A \times A \to \mathbf{C}_0^A.$$

Then $e \in \mathscr{E}(\mathbf{C}_0^A \times A, \mathbf{C}_0)$ and hence $\prod_p e \in \mathscr{E}(\mathbf{C}_0^A, \mathbf{C}_0)$. One can easily show that $\prod_p e$ is the object part of an (internal) functor

$$\prod : \mathbf{C}^A \to \mathbf{C},$$

which is right adjoint to the diagonal. Now one can define product along all projections of the form $A \times B \to B$ by

$$\mathscr{E}(A \times B, \mathbf{C}) \cong \mathscr{E}(B, \mathbf{C}^A) \xrightarrow{\mathscr{E}(B, \prod)} \mathscr{E}(B, \mathbf{C}).$$

The relevant cases of the Beck–Chevalley condition have become trivial. If $f : B_2 \to B_1$ in $\mathscr{E}$, then

$$\begin{array}{ccccc}
\mathscr{E}(A \times B_1, \mathbf{C}) & \cong & \mathscr{E}(B_1, \mathbf{C}^A) & \longrightarrow & \mathscr{E}(B_1, \mathbf{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{E}(A \times B_2, \mathbf{C}) & \cong & \mathscr{E}(B_2, \mathbf{C}^A) & \longrightarrow & \mathscr{E}(B_2, \mathbf{C})
\end{array}$$

commutes ('on the nose'), as

$$\mathscr{E}(f, \mathbf{C}) \cdot \mathscr{E}(B_1, \Pi) = \mathscr{E}(f, \Pi) = \mathscr{E}(B_2, \Pi) \cdot \mathscr{E}(f, \mathbf{C}^A).$$

Whence the expected result follows.

Proposition. *If* $\mathbf{C}$ *is a small sound complete, cartesian closed category in a topos* $\mathscr{E}$*, then* $(\underline{S}, \underline{G})$ *(as defined earlier) forms a PL-category.*

Corollary. *A small sound complete cartesian closed category in a topos gives rise to a model of the higher order polymorphic lambda calculus.*

Corollary (Girard [10]). $\mathbf{P}\Delta(-)$ *over* **Sets** *provides a model of the higher order polymorphic lambda calculus.*

The strongest known type theory which can be modelled along the lines indicated above is the theory of constructions of Coquand and Huet [4] (see also Coquand [5]). One way to see that theory as an extension of higher order polymorphic lambda calculus is as follows:

(i) Types are not only indexed over orders but also over types and are closed under corresponding products.

(ii) Orders are not only closed under function space, but can be indexed over both types and orders and are closed under the corresponding products.

Condition (i) can be satisfied by modelling types as an indexed locally cartesian closed category (cf. Seely [25]). Condition (ii) can be satisfied if types are special orders and orders are locally cartesian closed. (In fact sensible assumptions about sums will force this.) In the references the distinction types/orders is the distinction proposition/types.

Suppose that $\mathbf{C}$ is a small weakly complete full subcategory of a topos $\mathscr{E}$, which is locally cartesian closed. Suppose also that there is a category $s(\mathscr{E})$ sound for $\mathbf{C}$ in the sense of 0.2 which is locally cartesian closed, and which contains (in each fibre over an object of $s(\mathscr{E})$) the essential image of $\mathbf{C}(-)$. Given such a situation, the categories $\mathbf{C}(-)$ and $s(\mathscr{E})/-$ indexed over $s(\mathscr{E})$ can be used to represent types and orders. Problems involving substitution and the Beck–Chevalley condition have still to be dealt with. There are a number of ways, and I do not know which is best. However one way or another one has the following result.

Theorem. *The situation just described provides a model for the theory of constructions.*

Corollary. *The category* $\hat{\mathbf{P}}$ *in* **Eff** *provides a model for the theory of constructions.*

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