Linear λ-Calculus and Categorical Models Revisited

Nick Benton[1], Gavin Bierman[1], Valeria de Paiva[1] and Martin Hyland[2]

[1] Computer Laboratory, University of Cambridge, UK
[2] Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, UK

1 Intuitionistic Linear Logic

Girard's Intuitionistic Linear Logic [7] is a refinement of Intuitionistic Logic, where formulae must be used exactly once. In other words, the familiar Weakening and Contraction rules of Gentzen's sequent calculus [17] are removed. To regain the expressive power of Intuitionistic Logic, these rules are returned, but in a controlled manner. A logical operator, '!', is introduced which allows a formula to be used as many times as required (including zero).

In this paper we shall consider *multiplicative exponential* linear logic (MELL), i.e. the fragment which has multiplicative conjunction or tensor, $\otimes$, linear implication, $\multimap$, and the logical operator "exponential", !. We recall the rules for MELL in a sequent calculus system in Fig. 1. We use capital Greek letters Γ, Δ for sequences of formulae and A, B for single formulae. The *Exchange* rule simply allows the permutation of assumptions.

The '! rules' have been given names by other authors. $!_{\mathcal{L}-1}$ is called *Weakening*, $!_{\mathcal{L}-2}$ *Contraction*, $!_{\mathcal{L}-3}$ *Dereliction* and $(!_{\mathcal{R}})$ *Promotion*[1]. (We shall use these terms throughout this paper.) In the *Promotion* rule, $!\Gamma$ means that every formula in the set Γ is modal, in other words, if Γ is the set $\{A_1, A_2, \ldots A_n\}$, then $!\Gamma$ denotes the set $\{!A_1, !A_2, \ldots !A_n\}$.

2 Categorical considerations and term assignment

The sequent calculus is best thought of as providing not proofs themselves, but a meta-theory concerning proofs. Hence a formulation in these terms does not always provide clear clues as to how it should be enriched to a term assignment system. Fortunately we can use the general form of a categorical model (of the proof theory) of the logic to derive an appropriate term assignment system for the sequent calculus formulation of this logic.

The fundamental idea of the categorical treatment of proof theory is that propositions should be interpreted as the objects of a category (or multicategory, or polycategory) and proofs should be interpreted as maps; operations transforming proofs into proofs then correspond (if possible) to natural transformations (between appropriate hom-functors) in the categorical sense. The maps modelling proofs are built up using these categorical operations and so the problem of a term assignment is

[1] Girard, Scedrov and Scott [8] prefer to call this rule *Storage*.

$$\frac{}{A \vdash A}\ \mathit{Identity}$$

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}\ \mathit{Exchange}$$

$$\frac{\Gamma \vdash B \qquad B, \Delta \vdash C}{\Gamma, \Delta \vdash C}\ \mathit{Cut}$$

$$\frac{\Gamma \vdash A}{\Gamma, I \vdash A}\ (I_{\mathcal{L}}) \qquad \frac{}{\vdash I}\ (I_{\mathcal{R}})$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ (\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ (!_{\mathcal{L}-1}) \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (!_{\mathcal{L}-2})$$

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ (!_{\mathcal{L}-3}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!_{\mathcal{R}})$$

Fig. 1. Multiplicative Exponential Linear Logic

essentially the problem of providing a syntax expressing these operations. Here we carry out this programme for MELL.

Deriving the term formation rules

Since we are dealing with sequents $\Gamma \vdash A$, in principle we should deal with multicategories. However it simplifies things to assume at once that the multicategorical structure is represented by a tensor product $\bullet$, so that we are dealing with a monoidal category [13]. We shall write $\langle\rangle$ for the unit of this tensor product. To simplify the presentation we use the same symbols both for propositions of linear logic and for their denotations in our monoidal category. The idea then is that a sequent of form

$$C_1, C_2, \ldots, C_n \vdash A$$

will be interpreted as a map $C_1 \bullet C_2 \bullet \ldots \bullet C_n \to A$ from the tensor product of the C_i to A. (Thus a coherence result is assumed [11].) When Γ is the sequence $C_1, C_2, \ldots, C_n$, we write $\Gamma \to A$ for this map. We seek to enrich the sequent judgement to a term assignment judgement of the form

$$x_1 : C_1, x_2 : C_2, \ldots, x_n : C_n \vdash e : A$$

where the x_i are (distinct) variables and e is a term; usually we suppress (irrelevant) variables and write $\Gamma \vdash e : A$ for this term assignment.

The whole process is based upon some simple assumptions about the interpretation of the basic structural rules, and a simple procedure for dealing with the logical rules, which we describe in turn.

2.1 Structural Rules

The sequent representing the *Identity* rule is interpreted as the (canonical) identity arrow $A \xrightarrow{1} A$ from A to A. The corresponding rule of term formation is $x : A \vdash x : A$. The rule of *Exchange* we interpret by assuming that we have a symmetry for the tensor product $\bullet$ (making our model a *symmetric* monoidal category). We henceforth suppress *Exchange* and the corresponding symmetry; thus we really consider multisets of formulae, and as a result no term forming operations result from this rule. The *Cut* rule

$$\frac{\Gamma \vdash A \qquad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \; \mathit{Cut}$$

is then interpreted as a generalized form of composition: if the maps $\Gamma \xrightarrow{f} A$ and $A \bullet \Delta \xrightarrow{g} B$ are the interpretations of hypotheses of the rule, then the composite

$$\Gamma \bullet \Delta \xrightarrow{\;f \bullet 1_\Delta\;} A \bullet \Delta \xrightarrow{\;g\;} B$$

is the interpretation of the conclusion. We take as the corresponding rule of term formation a textual substitution:

$$\frac{\Gamma \vdash f : A \qquad x : A, \Delta \vdash g : B}{\Gamma, \Delta \vdash g[f/x] : B} \; \mathit{Cut}$$

One should note that the contexts Γ and Δ are disjoint; namely the variables which occur in Γ do not occur in Δ. This restriction holds for all the binary multiplicative rules.

2.2 Logical rules for Multiplicatives

We shall make the assumption that any logical rule corresponds to an operation on maps of the category which is *natural* in (the interpretations of) the components of the sequents which remain unchanged during the application of a rule. Composition corresponds to *Cut* so clearly the logical significance is that we are assuming that our operations commute (where appropriate) with *Cut*.

We start by considering the connective $\otimes$. The $(\otimes_{\mathcal{L}})$ rule

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A\otimes B \vdash C}\,(\otimes_{\mathcal{L}})$$

gives an operation taking maps $\Gamma \bullet A \bullet B \to C$ to maps $\Gamma \bullet (A\otimes B) \to C$. An appropriate syntax is

$$\frac{\Gamma, x : A, y : B \vdash f : C}{\Gamma, z : A\otimes B \vdash \mathsf{let}\ z\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ f : C}\,(\otimes_{\mathcal{L}})$$

where we understand that the variables x and y are bound in the term $\mathsf{let}\ z\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ f$. Naturality in Γ is clear since we may substitute for the corresponding variables, whilst naturality in C gives rise to an equation

$$g[\mathsf{let}\ z\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ f/w] = \mathsf{let}\ z\ \mathsf{be}\ x\otimes y\ \mathsf{in}\ g[f/w] \qquad (1)$$

The $(\otimes_{\mathcal{R}})$ rule

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A\otimes B}\,(\otimes_{\mathcal{R}})$$

gives an operation taking arrows $\Gamma \to A$ and $\Delta \to B$ to an arrow $\Gamma\bullet\Delta \to A\otimes B$. This might suggest a quite complex syntax, but fortunately our naturality assumptions imply that this operation is completely determined by a map $A \bullet B \to A\otimes B$. It follows that an appropriate syntax is

$$\frac{\Gamma \vdash e : A \qquad \Delta \vdash f : B}{\Gamma, \Delta \vdash e\otimes f : A\otimes B}\,(\otimes_{\mathcal{R}})$$

The $(I_{\mathcal{L}})$ rule

$$\frac{\Gamma \vdash A}{\Gamma, I \vdash A}\,(I_{\mathcal{L}})$$

gives an operation taking maps $\Gamma \to A$ to maps $\Gamma \bullet I \to A$. An appropriate syntax is

$$\frac{\Gamma \vdash e : A}{\Gamma, x : I \vdash \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ e : A}\,(I_{\mathcal{L}})$$

so that in effect we simply introduce a dummy free variable for the assumption I. Naturality in Γ is clear since we may substitute for the corresponding (free) variables. However naturality in A gives rise to an equation

$$f[\mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ e/y] = \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ f[e/y] \qquad (2)$$

The $(I_{\mathcal{R}})$ rule

$$\frac{}{\vdash I}\,(I_{\mathcal{R}})$$

gives simply a map $\langle\rangle \to I$. An appropriate syntax is

$$\frac{}{\vdash * : I}\,(I_{\mathcal{R}})$$

Our treatment of the $(\multimap_{\mathcal{L}})$ rule

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, A \multimap B, \Delta \vdash C}\,(\multimap_{\mathcal{L}})$$

follows traditional treatments of the left implication rule in sequent systems (which all involve a Yoneda Lemma argument). It follows from our naturality assumptions by a straightforward application of a Yoneda Lemma that an operation as above is determined by its action on a pair of identity arrows. Thus it is enough to give an operation of application $\mathsf{app}\colon A \bullet (A \multimap B) \longrightarrow B$. Then given arrows $e\colon \Gamma \to A$, $f\colon B \bullet \Delta \to C$ the required arrow $\Gamma \bullet (A \multimap B) \bullet \Delta \to C$ is the composite

$$\Gamma \bullet (A \multimap B) \bullet \Delta \xrightarrow{\;e \bullet 1 \bullet 1\;} A \bullet (A \multimap B) \bullet \Delta \xrightarrow{\;\mathsf{app} \bullet 1\;} B \bullet \Delta \xrightarrow{\;f\;} C$$

and an appropriate syntax is

$$\frac{\Gamma \vdash e : A \qquad \Delta, x : B \vdash f : C}{\Gamma, g : A \multimap B, \Delta \vdash f[(ge)/x] : C}\,(\multimap_{\mathcal{L}})$$

All the naturality assumptions are now dealt with by substitution. The $(\multimap_{\mathcal{R}})$ rule

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,(\multimap_{\mathcal{R}})$$

gives an operation taking an arrow $\Gamma \bullet A \to B$ to an arrow $\Gamma \to A \multimap B$. This is a form of abstraction and an appropriate syntax is

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x\colon A.e : A \multimap B}\,(\multimap_{\mathcal{R}})$$

2.3 Logical rules for the connective '!'

Next we consider the '!' connective. The left rules are reasonably straightforward, the right rule is a bit more involved. We consider the *Dereliction* and *Promotion* rules first.

Dereliction **and** *Promotion.* Consider the *Dereliction* rule

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\,\textit{Dereliction}$$

Since it gives an operation taking an arrow $\Gamma \bullet A \to B$ to an arrow $\Gamma \bullet !A \to B$, an appropriate syntax is

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma, z :!A \vdash \mathsf{let}\ z\ \mathsf{be}\ !x\ \mathsf{in}\ e : B}\,\textit{Dereliction}$$

and indeed this is the syntax given by Abramsky [1]. With this formulation naturality in B gives rise to an equation

$$f[\mathsf{let}\ z\ \mathsf{be}\ !x\ \mathsf{in}\ e/y] = \mathsf{let}\ z\ \mathsf{be}\ !x\ \mathsf{in}\ f[e/y]$$

However it is a consequence of naturality that our operation is determined by its effect on identity arrows, thus it is enough to give a map $!A \xrightarrow{\varepsilon} A$. Then given an arrow $e\colon \Gamma \bullet A \to B$, the required arrow $\Gamma \bullet !A \to B$ is the composite

$$\Gamma \bullet !A \xrightarrow{\ 1 \bullet \varepsilon\ } \Gamma \bullet A \xrightarrow{\ e\ } B$$

so another appropriate syntax (and the one we shall use in what follows as it surpresses further naturality equations) is

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma, z : !A \vdash e[\mathsf{derelict}(z)/x] : B}\ \textit{Dereliction}$$

Next consider the problematic *Promotion* rule

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ \textit{Promotion}$$

This gives an operation (of *Promotion*) taking an arrow $!\Gamma \to A$ to an arrow $!\Gamma \to !A$. Now it is not a priori clear what form of naturality should be assumed for this rule. If we assume that the operation should be natural in $!\Gamma$, then Abramsky's rule [1, Section 3],

$$\frac{\overline{x} : !\Gamma \vdash e : A}{\overline{x} : !\Gamma \vdash !e : !A}$$

would give an appropriate syntax[2]. However nothing in the idea of a categorical model suggests this assumption. (Note in passing that the categorically appealing assumption would be that ! is a functor and that we have naturality in Γ.) The important point to realize is that if the operation is not natural in $!\Gamma$, then the operation should not preserve substitution for the free variables implicitly declared in $!\Gamma$. Hence we are restricted to giving an operation on 'higher-order' terms, where the variables which appear initially must be bound and fresh variables introduced. These considerations lead to the term assignment rule

$$\frac{\overline{x} : !\Gamma \vdash e : A}{\overline{y} : !\Gamma \vdash \mathsf{promote}\ \overline{y}\ \mathsf{for}\ \overline{x}\ \mathsf{in}\ e : !A}\ \textit{Promotion}$$

We do not claim that there is a clear reason in terms of the category theory given so far to prefer one rule to the other, but we choose our rule simply so as to avoid any premature assumptions.

[2] This assumption has the effect that in the categorical model, which we shall consider later, the comonad is *idempotent*: a point noted by Wadler [18].

Weakening **and** *Contraction***.** Finally we consider the *Weakening* and *Contraction* rules. The rule

$$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ \textit{Weakening}$$

gives an operation taking an arrow $\Gamma \to B$ to an arrow $\Gamma \bullet !A \to B$. An appropriate syntax is

$$\frac{\Gamma \vdash e : B}{\Gamma, z :!A \vdash \mathsf{discard}\ z\ \mathsf{in}\ e : B}\ \textit{Weakening}$$

where we have simply introduced a fresh dummy variable of type $!A$. Naturality in Γ is as before clear since we may substitute for the corresponding variables. Naturality in B gives rise to an equation

$$f[\mathsf{discard}\ z\ \mathsf{in}\ e/y] = \mathsf{discard}\ z\ \mathsf{in}\ f[e/y] \tag{3}$$

The *Contraction* rule

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ \textit{Contraction}$$

gives an operation taking an arrow $\Gamma \bullet !A \bullet !A \to B$ to an arrow $\Gamma \bullet !A \to B$. An appropriate syntax is

$$\frac{\Gamma, x :!A, y :!A \vdash e : B}{\Gamma, z :!A \vdash \mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ e : B}\ \textit{Contraction}$$

where we understand that the variables x and y are bound in the term $\mathsf{copy}\ z \mathsf{as}\ x, y \mathsf{in}\ e$. Naturality in Γ is clear since we may substitute for the corresponding variables, while naturality in B gives rise to an equation

$$f[\mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ e/w] = \mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ f[e/w] \tag{4}$$

This concludes our derivation of a term assignment system for MELL from general considerations of the form of a categorical model. We display this system of term assignment in Fig. 2. We stress that rather elementary assumptions and unsophisticated categorical observations have been used in this analysis. However, our analysis has not only led us to a term assignment system, but has also uncovered a series of *naturality equations*, which are listed in Fig. 3.

3 Linear Natural Deduction

In the previous section we have provided a term assignment for a sequent calculus presentation of linear logic. Here we briefly consider a corresponding natural deduction formulation. In such a system a deduction is a derivation of a proposition from a finite set of assumption packets by means of inference rules. In intuitionistic logic these packets consist of (possibly empty) multisets of propositions. The restriction needed to make the derivations linear is that packets contain exactly one proposition, i.e. a packet is now equivalent to a proposition. Whereas before we typically had rules discharging many packets of an assumption we now only discharge the one. Thus we can label every proposition with a unique natural number.

$$x : A \vdash x : A$$

$$\frac{\Gamma \vdash e : A \qquad \Delta, x : A \vdash f : B}{\Gamma, \Delta \vdash f[e/x] : B}\ \mathit{Cut}$$

$$\frac{\Gamma \vdash e : A \qquad \Delta, x : B \vdash f : C}{\Gamma, g : A \multimap B, \Delta \vdash f[(ge)/x] : C}\ (\multimap_{\mathcal{L}}) \qquad \frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x{:}\, A.e : A \multimap B}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma \vdash e : A}{\Gamma, x : I \vdash \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ e : A}\ (I_{\mathcal{L}}) \qquad \frac{}{\vdash * : I}\ (I_{\mathcal{R}})$$

$$\frac{\Delta, x : A, y : B \vdash f : C}{\Delta, z : A \otimes B \vdash \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f : C}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \vdash e : A \qquad \Delta \vdash f : B}{\Gamma, \Delta \vdash e \otimes f : A \otimes B}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \vdash e : B}{\Gamma, z :!A \vdash \mathsf{discard}\ z\ \mathsf{in}\ e : B}\ \mathit{Weakening}$$

$$\frac{\Gamma, x :!A, y :!A \vdash e : B}{\Gamma, z :!A \vdash \mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ e : B}\ \mathit{Contraction}$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma, z :!A \vdash e[\mathsf{derelict}(z)/x] : B}\ \mathit{Dereliction}$$

$$\frac{\overline{x} :!\Gamma \vdash e : A}{\overline{y} :!\Gamma \vdash \mathsf{promote}\ \overline{y}\ \mathsf{for}\ \overline{x}\ \mathsf{in}\ e :!A}\ \mathit{Promotion}$$

Fig. 2. Term Assignment System for sequent calculus

Others have considered systems of natural deduction for linear logic [15, 18, 14]. Our main contribution is in our treatment of the *Promotion* rule. Previous authors formulated it as the following:

$$\begin{array}{c} !A_1 \cdots !A_n \\ \vdots \\ \dfrac{B}{!B}\ \mathit{Promotion} \end{array}$$

Clearly this rule is not closed under substitution. To ensure that the rule enjoys closure under substitution we use the following formulation:

$$f[\mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ e/y] = \mathsf{let}\ x\ \mathsf{be}\ *\ \mathsf{in}\ f[e/y]$$

$$f[\mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ g/w] = \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f[g/w]$$

$$f[\mathsf{discard}\ z\ \mathsf{in}\ e/y] = \mathsf{discard}\ z\ \mathsf{in}\ f[e/y]$$

$$f[\mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ e/w] = \mathsf{copy}\ z\ \mathsf{as}\ x, y\ \mathsf{in}\ f[e/w]$$

Fig. 3. Naturality Equations

$$\dfrac{\begin{array}{c}\vdots \\ !A_1\end{array} \quad \ldots \quad \begin{array}{c}\vdots \\ !A_n\end{array} \qquad \begin{array}{c}[!A_1 \cdots !A_n] \\ \vdots \\ B\end{array}}{!B}\ \mathit{Promotion}$$

One should be aware that this rule carries an implicit side condition that not only must *all* assumptions be exponential, but that *all* are discharged (and re-introduced). Our subsequent term assignment is given in Fig. 4. We note at once a significant property of the term assignment system for linear natural deduction. Essentially the terms code the derivation trees so that any valid term assignment has a *unique* derivation.

Theorem 1 (Unique Derivation). *For any term t and proposition A, if there is a valid derivation of the form $\Gamma \vdash t : A$, then there is a unique derivation of $\Gamma \vdash t : A$.*

Proof. By induction on the structure of t. □

As mentioned above, our system enjoys closure under substitution.

Theorem 2 Substitution. *If $\Gamma \vdash a : A$ and $\Delta, x : A \vdash b : B$ then $\Gamma, \Delta \vdash b[a/x] : B$*

Proof. By induction on the derivation $\Delta, x : A \vdash b : B$. □

As one would expect there is an exact equivalence between the natural deduction and sequent calculus formulations (indeed the substitution property is essential for this). The details of this equivalence are given in [2].

4 Cut Elimination

In this section we consider cut elimination for the sequent calculus formulation of MELL, extended or decorated with terms. Suppose that a derivation in the term assignment system of Fig. 2 contains a cut:

$$x : A \vdash x : A$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x{:}\,A.e : A \multimap B}\,(\multimap_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : A \multimap B \qquad \Delta \vdash f : A}{\Gamma, \Delta \vdash ef : B}\,(\multimap_{\mathcal{E}})$$

$$\vdash * : I \qquad \frac{\Gamma \vdash e : A \qquad \Delta \vdash f : I}{\Gamma, \Delta \vdash \mathsf{let}\ f\ \mathsf{be}\ *\ \mathsf{in}\ e : A}\,(I_{\mathcal{E}})$$

$$\frac{\Gamma \vdash e : A \qquad \Delta \vdash f : B}{\Gamma, \Delta \vdash e \otimes f : A \otimes B}\,(\otimes_{\mathcal{I}}) \qquad \frac{\Gamma \vdash e : A \otimes B \qquad \Delta, x : A, y : B \vdash f : C}{\Gamma, \Delta \vdash \mathsf{let}\ e\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f : C}\,(\otimes_{\mathcal{E}})$$

$$\frac{\Delta_1 \vdash e_1 :!A_1 \quad \cdots \quad \Delta_n \vdash e_n :!A_n \qquad x_1 :!A_1, \ldots, x_n :!A_n \vdash f : B}{\Delta_1, \ldots, \Delta_n \vdash \mathsf{promote}\ e_1, \ldots, e_n\ \mathsf{for}\ x_1, \ldots, x_n\ \mathsf{in}\ f :!B}\,\mathit{Promotion}$$

$$\frac{\Gamma \vdash e :!A \qquad \Delta \vdash f : B}{\Gamma, \Delta \vdash \mathsf{discard}\ e\ \mathsf{in}\ f : B}\,\mathit{Weakening}$$

$$\frac{\Gamma \vdash e :!A \qquad \Delta, x :!A, y :!A \vdash f : B}{\Gamma, \Delta \vdash \mathsf{copy}\ e\ \mathsf{as}\ x, y\ \mathsf{in}\ f : B}\,\mathit{Contraction}$$

$$\frac{\Gamma \vdash e :!A}{\Gamma \vdash \mathsf{derelict}(e) : A}\,\mathit{Dereliction}$$

Fig. 4. Term Assignment System for Linear Natural Deduction

$$\frac{\dfrac{}{\Gamma \vdash e : A}\,D_1 \qquad \dfrac{}{\Delta, x : A \vdash f : B}\,D_2}{\Gamma, \Delta \vdash f[e/x] : B}\,\mathit{Cut}$$

If $\Gamma \vdash e : A$ is the direct result of a rule D_1 and $\Delta, x : A \vdash f : B$ the result of a rule D_2, we say that the cut is a (D_1, D_2)-cut. A step in the process of eliminating cuts in the derivation tree will replace the subtree with root $\Gamma, \Delta \vdash f[e/x]{:}\,B$ with a tree with root of the form $\Gamma, \Delta \vdash t : B$. The terms in the remainder of the tree may be affected as a result.

Thus to ensure that the cut elimination process extends to derivations in the term assignment system, we must insist on an equality $f[e/x] = t$, which we can read from left to right as a term reduction. In fact we must insist on arbitrary substitution instances of the equality, as the formulae in Γ and Δ may be subject to cuts in the derivation tree below the cut in question. In this section we are mainly concerned to describe the equalities/reductions which result from the considerations just described. Note, however, that we cannot be entirely blithe about the process of eliminating cuts at the level of the propositional logic. As we shall see, not every apparent possibility for eliminating cuts should be realized in practice.

As things stand there are 11 rules of the sequent calculus aside from *Cut* (and *Exchange*) and hence 121 a priori possibilities for (D_1, D_2)-cuts. Fortunately most of these possibilities are not computationally meaningful in the sense that they have no effect on the terms. We say that a cut is *insignificant* if the equality $f[e/x] = t$ we derive from it as above is actually an identity (up to α-equivalence) on terms (so in executing the cut the term at the root of the tree does not change).

Note that any cut involving an axiom rule

$$\frac{}{x : A \vdash x : A}\ \textit{Identity}$$

is insignificant; and the cut just disappears (hence instead of 121 we must now account for 100 cases). These 100 cases of cuts we will consider as follows: 40 cases of cuts the form (R, D) as we have 4 right rules and 10 others; 24 cases of cuts of the form (L, R) as we have 6 left-rules and 4 right ones and finally 36 cases of cuts of the form (L, L). Let us consider these three groups in turn.

Firstly we observe that there is a large class of insignificant cuts of the form (R, D) where R is a right rule: $(\otimes_\mathcal{R})$, $(I_\mathcal{R})$, $(\multimap_\mathcal{R})$, *Promotion*. Indeed all such cuts are insignificant with the following exceptions:

- *Principal cuts.* These are the cuts of the form $((\otimes_\mathcal{R}), (\otimes_\mathcal{L}))$, $((I_\mathcal{R}), (I_\mathcal{L}))$, $((\multimap_\mathcal{R}), (\multimap_\mathcal{L}))$, (*Promotion*, *Dereliction*), (*Promotion*, *Weakening*), (*Promotion*, *Contraction*) where the cut formula is introduced on the right and left of the two rules.
- Cases of the form $(R, \textit{Promotion})$ where R is a right rule. Here we note that cuts of the form $((\otimes_\mathcal{R})), \textit{Promotion})$, $((I_\mathcal{R})), \textit{Promotion})$ and $((\multimap_\mathcal{R}), \textit{Promotion})$ cannot occur; so the only possibility is (*Promotion*, *Promotion*).

Next any cut of the form (L, R) where L is one of the left rules $(\otimes_\mathcal{L})$, $(I_\mathcal{L})$, $(\multimap_\mathcal{L})$, *Weakening*, *Contraction*, *Dereliction* and R is one of the simple right rules $(\otimes_\mathcal{R})$, $(I_\mathcal{R})$, $(\multimap_\mathcal{R})$ is insignificant (18 cases). Also cuts of the form $((\multimap_\mathcal{L}), \textit{Promotion})$ and (*Dereliction*, *Promotion*) are insignificant (2 cases). There remain four further cases of cuts of the form $(L, \textit{Promotion})$ where L is a left rule.

Lastly we have to consider the 36 cuts of the form (L_1, L_2), where the L_i are both left rules. Again we derive some benefit from our rules for $(\multimap_\mathcal{L})$ and *Dereliction*: cuts of the form $((\multimap_\mathcal{L}), L)$ and $(\textit{Dereliction}, L)$ are insignificant. There are 24 remaining cuts of interest.

We now summarize the cuts of which we need to take some note. They are:

- Principal cuts. There are six of these.
- Secondary cuts. The single (strange) form of cut (*Promotion*, *Promotion*) and the four remaining cuts of form $(L, \textit{Promotion})$ where L is a left rule other than $(\multimap_\mathcal{L})$ or (*Dereliction*).

– Commutative cuts. The twenty-four remaining cuts of the form (L_1, L_2) just described. These correspond almost[3] case by case to the commutative conversions for natural deduction (considered in [3]) and are not considered further here.

4.1 Principal Cuts

We do not dwell on the cases of principal cuts involving tensor, the constant I and linear implication as they are standard. We shall consider in detail the principal cuts involving the *Promotion* rule.

• (*Promotion*, *Dereliction*)-**cut**. The derivation

$$\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\,\mathit{Promotion} \qquad \dfrac{B, \Delta \vdash C}{!B, \Delta \vdash C}\,\mathit{Dereliction}}{!\Gamma, \Delta \vdash C}\,\mathit{Cut}$$

is reduced to

$$\dfrac{!\Gamma \vdash B \qquad B, \Delta \vdash C}{!\Gamma, \Delta \vdash C}\,\mathit{Cut}$$

This reduction yields the following term reduction

$$(f[\mathsf{derelict}(q)/p])[\mathsf{promote}\ y_i\ \mathsf{for}\ x_i\ \mathsf{in}\ e/q] = f[e/p]$$

• (*Promotion*, *Weakening*)-**cut**. The derivation

$$\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\,\mathit{Promotion} \qquad \dfrac{\Delta \vdash C}{!B, \Delta \vdash C}\,\mathit{Weakening}}{!\Gamma, \Delta \vdash C}\,\mathit{Cut}$$

is reduced to

$$\dfrac{\Delta \vdash C}{!\Gamma, \Delta \vdash C}\,\mathit{Weakening*}$$

where *Weakening** corresponds to many applications of the *Weakening* rule.

This gives the term reduction

$$\mathsf{discard}\ (\mathsf{promote}\ e_i\ \mathsf{for}\ x_i\ \mathsf{in}\ f)\ \mathsf{in}\ g = \mathsf{discard}\ e_i\ \mathsf{in}\ g$$

• (*Promotion*, *Contraction*)-**cut**. The derivation

$$\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\,\mathit{Promotion} \qquad \dfrac{!B, !B, \Delta \vdash C}{!B, \Delta \vdash C}\,\mathit{Contraction}}{!\Gamma, \Delta \vdash C}\,\mathit{Cut}$$

is reduced to

[3] The exceptions are the cases where $(\multimap_{\mathcal{L}})$ is the (second) rule above the cut. In these cases we obtain slightly stronger rules.

$$\cfrac{\cfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\, Promotion \qquad \cfrac{\cfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\, Promotion \quad !B, !B, \Delta \vdash C}{!\Gamma, !B, \Delta \vdash C}\, Cut}{\cfrac{!\Gamma, !\Gamma, \Delta \vdash C}{!\Gamma, \Delta \vdash C}\, Contraction^*}\, Cut$$

or to the symmetric one where we cut against the other $!B$ first. This gives the term reduction

$$\begin{aligned}&\mathsf{copy}\ (\mathsf{promote}\ e_i\ \mathsf{for}\ x_i\ \mathsf{in}\ f)\ \mathsf{as}\ y, y'\ \mathsf{in}\ g = \\ &\mathsf{copy}\ e_i\ \mathsf{as}\ z_i, z_i'\ \mathsf{in}\ g[\mathsf{promote}\ z_i\ \mathsf{for}\ x_i\ \mathsf{in}\ f/y, \mathsf{promote}\ z_i'\ \mathsf{for}\ x_i\ \mathsf{in}\ f/y']\end{aligned}$$

As would be expected these principal cuts correspond to the β-reductions which can be derived from the natural deduction system outlined in Section 3 (and detailed in [3]).

let $f \otimes g$ be $x \otimes y$ in h	$= h[f/x, g/y]$
let $*$ be $*$ in h	$= h$
$h[(\lambda x\colon A.f)g/y]$	$= h[f[g/x]/y]$
$(f[\mathsf{derelict}(q)/p])[\mathsf{promote}\ y_i\ \mathsf{for}\ x_i\ \mathsf{in}\ e/q]$	$= f[e/p]$
discard (promote e_i for x_i in f) in g	$=$ discard e_i in g
copy (promote e_i for x_i in f) as y, y' in g	$=$ copy e_i as z_i, z_i' in g[promote z_i for x_i in f/y, promote z_i' for x_i in f/y']

Fig. 5. Principal reduction rules

4.2 Secondary Cuts

We now consider the cases where the *Promotion* rule is on the right of a cut rule. The first case is the 'strange' case of cutting *Promotion* against *Promotion*, then we have the four cases $(\otimes_{\mathcal{L}})$, $(I_{\mathcal{L}})$, *Weakening* and *Contraction* against the rule *Promotion*. Here we discuss only the 'strange' case of

• (*Promotion*, *Promotion*)**-cut**. The derivation

$$\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\, Promotion \qquad \dfrac{!B, !\Delta \vdash C}{!B, !\Delta \vdash !C}\, Promotion}{!\Gamma, !\Delta \vdash !C}\, Cut$$

reduces to

$$\dfrac{\dfrac{\dfrac{!\Gamma \vdash B}{!\Gamma \vdash !B}\, Promotion \qquad !B, !\Delta \vdash C}{!\Gamma, !\Delta \vdash C}\, Cut}{!\Gamma, !\Delta \vdash !C}\, Promotion$$

Note that it is always possible to permute the cut upwards, as all the formulae in the antecedent are modal. This gives the term reduction

$$\mathsf{promote}\ (\mathsf{promote}\ z\ \mathsf{for}\ x\ \mathsf{in}\ f)\ \mathsf{for}\ y\ \mathsf{in}\ g = \mathsf{promote}\ w\ \mathsf{for}\ z\ \mathsf{in}\ (g[\mathsf{promote}\ z\ \mathsf{for}\ x\ \mathsf{in}\ f/y])$$

We present all the term equalities given by the secondary cuts in Fig. 6. Observe that the last four equations are particular instances of the naturality equations described in Section 2, while the first encapsulates the naturality of the Kleisli operation of *Promotion*. One is tempted to suggest that perhaps the reason why the rule *Promotion* gives us reductions with some sort of computational meaning is because this rule is not clearly either a left or a right rule. It introduces the connective on the right (so it is mainly a right rule), but it imposes conditions on the context on the left. Indeed there does not appear to be any analogous reductions in natural deduction.

$$\begin{aligned}
\mathsf{promote}\ (\mathsf{promote}\ z\ \mathsf{for}\ x\ \mathsf{in}\ f)\ \mathsf{for}\ y\ \mathsf{in}\ g &= \mathsf{promote}\ w\ \mathsf{for}\ z\ \mathsf{in} \\
&\qquad g[\mathsf{promote}\ z\ \mathsf{for}\ x\ \mathsf{in}\ f/y] \\
\mathsf{promote}\ (\mathsf{discard}\ x\ \mathsf{in}\ f)\ \mathsf{for}\ y\ \mathsf{in}\ g &= \mathsf{discard}\ x\ \mathsf{in}\ (\mathsf{promote}\ f\ \mathsf{for}\ y\ \mathsf{in}\ g) \\
\mathsf{promote}\ (\mathsf{copy}\ x\ \mathsf{as}\ y, z\ \mathsf{in}\ f)\ \mathsf{for}\ y\ \mathsf{in}\ g &= \mathsf{copy}\ x\ \mathsf{as}\ y, z\ \mathsf{in}\ (\mathsf{promote}\ f\ \mathsf{for}\ y\ \mathsf{in}\ g) \\
\mathsf{promote}\ (\mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f)\ \mathsf{for}\ w\ \mathsf{in}\ g &= \mathsf{let}\ z\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ (\mathsf{promote}\ f\ \mathsf{for}\ w\ \mathsf{in}\ g) \\
\mathsf{promote}\ (\mathsf{let}\ z\ \mathsf{be}\ *\ \mathsf{in}\ f)\ \mathsf{for}\ w\ \mathsf{in}\ g &= \mathsf{let}\ z\ \mathsf{be}\ *\ \mathsf{in}\ (\mathsf{promote}\ f\ \mathsf{for}\ w\ \mathsf{in}\ g)
\end{aligned}$$

Fig. 6. Secondary reduction rules

5 The Categorical Model

Much work has been done on providing such (categorical) models of Intuitionistic Linear Logic. Here we shall just mention the work of Seely [16] and de Paiva [4]. With a view to understanding what is involved here, let us consider the traditional analysis of the proof theory of some basic intuitionistic logic via the notion of a cartesian closed category. (Lambek and Scott [12] is a good source for this material.) In that case, the basic normalization process gives rise to β-equality on the terms of the typed λ-calculus. The β-equality rule is valid in any cartesian closed category, but the attractive categorical assumption of being cartesian closed amounts to requiring $\beta\eta$-equality, that is, to a further 'extensionality' assumption. Thus one way to understand what we do is that we make a minimal number of attractive simplifying assumptions about the basic categorical set up introduced in Section 2 which at least entail the (desired) equalities between proofs. In this section we simply discuss the categorical assumptions we make and give the resulting equations.

5.1 Categorical interpretation of the multiplicatives

We start by considering the connective $\otimes$. The categorical significance of the β-rule for $\otimes$ is that any map of the form $\Gamma \bullet A \bullet B \to C$ factors canonically through the map $A \bullet B \xrightarrow{\otimes} A \otimes B$ which results from the instance of the $(\otimes_{\mathcal{R}})$ rule

$$\frac{A \vdash A \qquad B \vdash B}{A, B \vdash A \otimes B}\,(\otimes_{\mathcal{R}})$$

The simplifying 'extensionality' assumption is then that this factorization is *unique*. This can be expressed by saying that (generalized) composition with $A \bullet B \to A \otimes B$ induces a natural isomorphism between maps

$$\frac{\Gamma \bullet (A \otimes B) \to C}{\Gamma \bullet A \bullet B \to C}$$

In other words that the operation of composing with $A \bullet B \to A \otimes B$ provides an inverse to the $(\otimes_{\mathcal{L}})$-operation taking maps $\Gamma \bullet A \bullet B \to C$ to maps $\Gamma \bullet (A \otimes B) \to C$. Thus we may as well assume that the logical $\otimes$ coincides with $\bullet$. We get two equations expressing this isomorphism. One of these equations is, of course, the β-rule for tensor:

$$\mathsf{let}\ u \otimes v\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f = f[u/x, v/y] \tag{5}$$

The other can be regarded as an η-equality:

$$\mathsf{let}\ u\ \mathsf{be}\ x \otimes y\ \mathsf{in}\ f[x \otimes y/z] = f[u/z] \tag{6}$$

The case of I is like that for $\otimes$. Thus (generalized) composition with $\langle\rangle \to I$ induces a natural isomorphism between maps

$$\frac{\Gamma \bullet I \to C}{\Gamma \bullet \langle\rangle \to C}$$

We identify $\langle\rangle$ and I, and use I both as a logical operator and to interpret the empty sequence on the left hand side of a sequent. As before we get two equations expressing the natural isomorphism. One is the β-rule and the other can again be regarded as an η-equality:

$$\mathsf{let}\ *\ \mathsf{be}\ *\ \mathsf{in}\ f = f \tag{7}$$

$$\mathsf{let}\ u\ \mathsf{be}\ *\ \mathsf{in}\ f[*/z] = f[u/z] \tag{8}$$

The β-rule for $\multimap$ has a slightly more complicated interpretation, it means that any map $f\colon A\otimes B \to C$ factors as

$$A\otimes B \xrightarrow{1\otimes\mathsf{cur}(f)} A\otimes(A\multimap C) \xrightarrow{\mathsf{app}} C$$

where $\mathsf{app}\colon A\otimes(A\multimap C) \to C$ is the map that results from an instance of the $(\multimap_{\mathcal{L}})$ rule

$$\frac{A \vdash A \qquad C \vdash C}{A, A\multimap C \vdash C}\ (\multimap_{\mathcal{L}})$$

Again the natural simplifying assumption is that the factorization is *unique*, which means that there exists a natural isomorphism between maps

$$\begin{array}{c} A\otimes B \longrightarrow C \\ \hline\hline A \longrightarrow B\multimap C \end{array}$$

Thus $\multimap$ provides us with a closed structure on our category corresponding to the tensor $\otimes$. Again we have two equations to express our natural isomorphism. One is the β-rule and the other is the (linear form of the) traditional η-rule:

$$(\lambda x.f)e = f[e/x] \tag{9}$$

$$\lambda x.fx = f \tag{10}$$

5.2 Categorical interpretation of *Dereliction* and *Promotion*

Now we consider the meaning of the β-rule for ! involving *Dereliction*. The categorical import of this rule is that any map $!\Gamma \to A$ factors in a canonical way as a composite

$$!\Gamma \longrightarrow !A \xrightarrow{\varepsilon_A} A$$

where $!A \xrightarrow{\varepsilon_A} A$ is the canonical map obtained by *Dereliction* from the identity as described in Section 2. Given any proof $\Gamma \vdash B$ there is obviously a canonical two-step process that transforms it into a proof $!\Gamma \vdash !B$ by applying the *Dereliction* rule (several times) followed by *Promotion*.

$$\dfrac{\dfrac{\Gamma \vdash B}{!\Gamma \vdash B}\ \mathit{Dereliction}^*}{!\Gamma \vdash !B}\ \mathit{Promotion}$$

If $\Gamma \xrightarrow{f} B$ interprets the original proof, we write the resulting arrow as $!\Gamma \xrightarrow{!f} !B$. As a preliminary simplification, we assume that this definition gives the extension of ! to a multicategorical functor. This amounts to the assumption that ! is a *monoidal* functor [5]; that is, the functor ! comes equipped with a natural transformation

$$m_{A,B}\colon !A \otimes !B \to !(A \otimes B)$$

(natural in A and B) and a morphism $m_I\colon I \to !I$ making a standard collection of diagrams commute. Note that the β-rule for *Dereliction* certainly implies that for any $f\colon \Gamma \to A$, the equation $!f;\varepsilon_A = \varepsilon_\Gamma; f$ holds. Either composite gives the effect of *Dereliction* on f. This shows that $\varepsilon\colon ! \to 1$ will be a multicategorical natural transformation and so a monoidal natural transformation.

We need one further piece of structure. We apply the *Promotion* rule to the axiom $!A \vdash !A$ to obtain the derivation

$$\frac{!A \vdash !A}{!A \vdash !!A}\ \textit{Promotion}$$

In other words, from an identity arrow $!A \longrightarrow !A$ we can get a canonical arrow $\delta_A\colon !A \to !!A$. With the equations to hand we know rather little about δ. One can easily check that the composite

$$!A \xrightarrow{\ \delta_A\ } !!A \xrightarrow{\ \varepsilon_{!A}\ } !A$$

is the identity on $!A$, and that is one of the triangle identities for a comonad, but that is about it. However it is compelling to add to our preliminary assumption that ! is a monoidal functor, the assumption that δ (as well as ε) is a monoidal natural transformation and that $(!, \varepsilon, \delta)$ forms a comonad on our category. Note that given a monoidal comonad $(!, \varepsilon, \delta)$, the *Promotion* rule can be interpreted as follows: given a map $f\colon !C_1 \otimes \ldots \otimes !C_n \longrightarrow A$ we obtain the 'promoted' map as the composite

$$!C_1 \otimes \ldots \otimes !C_n \xrightarrow{\ \delta\ } !!C_1 \otimes \ldots \otimes !!C_n \xrightarrow{\ m\ } !(!C_1 \otimes \ldots \otimes !C_n) \xrightarrow{\ !f\ } !A$$

We can formulate the conditions that $(!, \varepsilon, \delta)$ be a monoidal comonad directly in terms of the basic operations given by MELL. In addition to the β-equality (equation (11) below) we obtain:

$$\mathsf{derelict}(\mathsf{promote}\ e_i\ \mathsf{for}\ x_i\ \mathsf{in}\ f) = f[e_i/x_i] \tag{11}$$

$$\mathsf{promote}\ z\ \mathsf{for}\ x\ \mathsf{in}\ (\mathsf{derelict}(x)) = z \tag{12}$$

$$\begin{aligned}&\mathsf{promote}\ (\mathsf{promote}\ z_i\ \mathsf{for}\ x_i\ \mathsf{in}\ f), w_j\ \mathsf{for}\ y, y_j\ \mathsf{in}\ g = \\ &\mathsf{promote}\ z_i, w_j\ \mathsf{for}\ z_i', y_j\ \mathsf{in}\ (g[\mathsf{promote}\ z_i'\ \mathsf{for}\ x_i\ \mathsf{in}\ f/y]).\end{aligned} \tag{13}$$

Equation (12) can be thought of as an η-rule, as it provides a kind of uniqueness of the factorization mentioned above; equation (13) expresses an appropriate form of naturality of the operation of *Promotion* and it arises from a secondary cut elimination.

5.3 Categorical interpretation of *Weakening* and *Contraction*

We finally consider the categorical significance of the β-rules involving *Weakening* and *Contraction*. To do so let us first introduce a further canonical pair of maps. Using *Weakening* (and the right rule for I) we have a deduction

$$\frac{\vdash I}{!A \vdash I}\ \textit{Weakening}$$

which gives a canonical map $!A \xrightarrow{e_A} I$ (where e is used to remind the reader that this map corresponds to 'erasing' the assumption). From the rules ($\otimes_{\mathcal{R}}$) and *Contraction* we obtain

$$\frac{\dfrac{!A \vdash !A \qquad !A \vdash !A}{!A, !A \vdash !A \otimes !A}\ (\otimes_{\mathcal{R}})}{!A \vdash !A \otimes !A}\ \textit{Contraction}$$

which gives a canonical map $!A \xrightarrow{d_A} !A \otimes !A$ (again d is used to hint at 'duplication' of assumptions).

It follows from the β- and η-rules for $\otimes$ and I as well as from the naturality assumptions on *Contraction* and *Weakening* described in Section 2 that any map $\Gamma \otimes !A \xrightarrow{f} B$ arising from the use of the rule of *Weakening* is the composite

$$\Gamma \otimes !A \xrightarrow{1 \otimes e_A} \Gamma \otimes I \cong \Gamma \xrightarrow{\overline{f}} B$$

Similarly the effect of the rule of *Contraction* is that any map $!A \otimes \Gamma \xrightarrow{f} B$ arising from the use of *Contraction* is the composite

$$!A \otimes \Gamma \xrightarrow{d_A \otimes 1_\Gamma} !A \otimes !A \otimes \Gamma \xrightarrow{\overline{f}} B$$

The β-equalities for *Contraction* and *Weakening* namely,

$$\textsf{discard}\ (\textsf{promote}\ e_i\ \textsf{for}\ x_i\ \textsf{in}\ t)\ \textsf{in}\ u = \textsf{discard}\ e_i\ \textsf{in}\ u \tag{14}$$

$$\begin{aligned}&\textsf{copy}\ (\textsf{promote}\ e_i\ \textsf{for}\ x_i\ \textsf{in}\ t)\ \textsf{as}\ y, z\ \textsf{in}\ u = \\ &\textsf{copy}\ e_i\ \textsf{as}\ x_i', x_i''\ \textsf{in}\ u[\textsf{promote}\ x_i'\ \textsf{for}\ x_i\ \textsf{in}\ t/y, \textsf{promote}\ x_i''\ \textsf{for}\ x_i\ \textsf{in}\ t/z]\end{aligned} \tag{15}$$

say that maps obtained using the rule *Promotion* preserve the structure (on objects of the form $!A$) given by e and d. It follows at once that the canonical morphisms (e and d) are natural transformations. One might also expect that e and d give structure on the coalgebras, or (what amounts to the same thing) that they are themselves maps of coalgebras. This leads to the equations

$$\textsf{promote}\ e, e_i\ \textsf{for}\ x, x_i\ \textsf{in discard}\ x\ \textsf{in}\ t = \textsf{discard}\ e\ \textsf{in promote}\ e_i\ \textsf{for}\ x_i\ \textsf{in}\ t \tag{16}$$

$$\begin{aligned}&\textsf{promote}\ e, e_i\ \textsf{for}\ z, z_i\ \textsf{in copy}\ z\ \textsf{as}\ x, y\ \textsf{in}\ t = \\ &\textsf{copy}\ e\ \textsf{as}\ x', y'\ \textsf{in promote}\ x', y', e_i\ \textsf{for}\ x, y, z_i\ \textsf{in}\ t\end{aligned} \tag{17}$$

We believe that there is some computational sense to this interplay between *Promotion* on the one hand, and *Weakening* and *Contraction* on the other. Furthermore our intuitions about the processes of discarding and copying suggest strongly that the natural transformations e and d give rise to the structure of a (commutative) comonoid on the free !-coalgebras. (As a consequence all coalgebras have (and all maps of coalgebras preserve) the structure of a (commutative) comonoid.)

5.4 The categorical model of Intuitionistic Linear Logic

Much of the *categorical* analysis that we have just given is quite familiar, though the corresponding equational calculus seems new (if only because our syntax is new). We note however that (following Seely [16]) it has become standard to analyze the categorical meaning of *Weakening* and *Contraction* in terms of the relationship between the additives and the multiplicatives. Our analysis dispenses with additives and hence gives a more general account of the force of the exponentials. Even in the presence of the additives our formulation is not equivalent to Seely's and it certainly covers cases of interest not covered by his. To sum up the analysis in this section we give the following definition.

Definition 3. A categorical model for MELL[4] consists of:

1. a symmetric monoidal closed (multi)category (modelling tensor and linear implication);
2. together with a comonad $(!, \varepsilon, \delta)$ with the following properties:
   (a) the functor part '!' of the comonad is a monoidal functor and ε and δ are monoidal natural transformations,
   (b) every (free) !-coalgebra carries naturally the structure of a commutative comonoid[5] in such a way that coalgebra maps are comonoid maps.

We have indicated in the text above what are the equations in our term assignment system corresponding to this notion of categorical model. We display all these equations in Fig. 7.

The connection between these equations and the notion of a categorical model can be made precise along the following lines. First assume that we have a signature given by a collection of ground types and of typed function symbols. From this data, types and terms in context are defined inductively by the clauses of Fig. 2 giving rise to what we call a *term logic* for MELL.

Next assume that $\mathbf{C}$ is a categorical model for MELL. Then in particular $\mathbf{C}$ has the structure outlined in Sect. 2. Here $\langle\rangle$ and $\bullet$ are identified with the I and $\otimes$ of $\mathbf{C}$. The required operations for I, $\otimes$, and $\multimap$ are given by standard operations in a symmetric monoidal closed category. As explained in Sect. 5.2 the map $\varepsilon\colon !A \to A$ used to interpret the *Dereliction* rule can be identified with the counit $\varepsilon_A\colon !A \to A$, while the operation of *Promotion* involves the comultiplication δ and the map which gives the monoidal structure of the functor !. Finally as explained in Sect. 5.3 the operations for *Weakening* and *Contraction* are given in terms of the comonoid structure on the (free) coalgebras.

4 Note that in our formulation it is not necessary to consider the additives to model the exponential.

5 This means not only that each !-coalgebra $(A, h_A\colon A \to !A)$ comes equipped with morphisms $e\colon A \to I$ and $d\colon A \to A \otimes A$ but also that e and d are coalgebra maps.

let $*$ be $*$ in e	$= e$
let u be $*$ in $f[*/z]$	$= f[u/z]$
let $e \otimes t$ be $x \otimes y$ in u	$= u[e/x, t/y]$
let u be $x \otimes y$ in $f[x \otimes y/z]$	$= f$
$(\lambda x\colon A.t)e$	$= t[e/x]$
$\lambda x\colon A.tx$	$= t$
derelict(promote e_i for x_i in t)	$= t[e_i/x_i]$
promote z for x in derelict(x)	$= z$
promote (promote z_i for x_i in f), w_j for y, y_j in g	$=$ promote z_i, w_j for z'_i, y_j in (g[promote z'_i for x_i in f/y])
discard (promote e_i for x_i in t) in u	$=$ discard e_i in u
promote e, e_i for x, x_i in discard x in t	$=$ discard e in promote e_i for x_i in t
copy (promote e_i for x_i in t) as y, z in u	$=$ copy e_i as x'_i, x''_i in u[promote x'_i for x_i in t/y, promote x''_i for x_i in t/z]
promote e, e_i for z, z_i in copy z as x, y in t	$=$ copy e as x', y' in promote x', y', e_i for x, y, z_i in t
copy e as x, y in discard x in t	$= t[e/y]$
copy e as x, y in discard y in t	$= t[e/x]$
copy e as x, y in t	$=$ copy e as y, x in t
copy e as x, w in copy w as y, z in t	$=$ copy e as w, z in copy w as x, y in t
f[let z be $*$ in e/w]	$=$ let z be $*$ in $f[e/w]$
f[let z be $x \otimes y$ in e/w]	$=$ let z be $x \otimes y$ in $f[e/w]$
f[discard z in e/w]	$=$ discard z in $f[e/w]$
f[copy z as x, y in e/w]	$=$ copy z as x, y in $f[e/w]$

Fig. 7. Categorical equalities

Given the structure outlined in Sect. 2, for any interpretation of a signature Σ in $\mathbf{C}$ there is a standard inductive definition of the interpretation of types and of terms in context of the term logic given by Σ in $\mathbf{C}$. The steps in this inductive definition[6] were considered in Sect. 2 and for the convenience of the reader we present an indication of these steps in Fig. 8.

$$A \to A$$

$$\frac{\Gamma \to A \qquad A \bullet \Delta \to B}{\Gamma \bullet \Delta \to B}\ \mathit{Cut}$$

$$\frac{\Gamma \to A}{\Gamma \bullet I \to A}\ (I_{\mathcal{L}}) \qquad \frac{}{\langle\rangle \to I}\ (I_{\mathcal{L}})$$

$$\frac{\Gamma \bullet A \bullet B \to C}{\Gamma \bullet (A \otimes B) \to C}\ (\otimes_{\mathcal{L}}) \qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma \bullet \Delta \to A \otimes B}\ (\otimes_{\mathcal{R}})$$

$$\frac{\Gamma \to A \qquad \Delta \bullet B \to C}{\Gamma \bullet (A \multimap B) \bullet \Delta \to C}\ (\multimap_{\mathcal{L}}) \qquad \frac{\Gamma \bullet A \to B}{\Gamma \to A \multimap B}\ (\multimap_{\mathcal{R}})$$

$$\frac{\Gamma \to B}{\Gamma \bullet !A \to B}\ \mathit{Weakening} \qquad \frac{\Gamma \bullet !A \bullet !A \to B}{\Gamma \bullet !A \to B}\ \mathit{Contraction}$$

$$\frac{\Gamma \bullet A \to B}{\Gamma \bullet !A \to B}\ \mathit{Dereliction} \qquad \frac{!\Gamma \to A}{!\Gamma \to !A}\ \mathit{Promotion}$$

Fig. 8. (Outline of the) interpretation of Term Logic

The interpretation is sound and complete in the following sense.

Theorem 4.

1. *(Soundness) For any signature and interpretation of the corresponding system in a categorical model for Intuitionistic Linear Logic (all the equational consequences of) the equations in Fig. 7 hold in the sense that the interpretation of either term gives the same map in the category.*
2. *(Completeness) For any signature there is a categorical model for Intuitionistic Linear Logic and an interpretation of the system in it with the following property:*

[6] Note that strictly speaking the induction is on the *derivation* (in the sequent calculus) of $\Gamma \vdash e : A$. Hence one has to show that the interpretation in $\mathbf{C}$ is independent of the derivation. It is laborious to show this directly and the result also follows from a consideration of the equivalent natural deduction formulation sketched in Sect. 3.

– *If $\Gamma \vdash t{:}A$ and $\Gamma \vdash s{:}A$ are derivable in the system then t and s are interpreted as the same map $\Gamma \to A$ just when $t = s{:}A$ is provable from the equations in Fig. 7 (in typed equational logic).*

We can make some comments on the proof of soundness and completeness. We derived the equations of Fig. 7 from a consideration of the categorical model. So the proof of soundenss amounts to filling in the details of that derivation. As all too often the proof of completeness is given by a construction of a categorical term model. One has to check that the equations given are sufficient to establish all the properties of a categorical model as exhibited in Defn. 3.

Now we try to make clear the force of our definition in terms of a discussion of (the background to) Girard's translation of intuitionistic propositional logic into linear logic. We start by recalling some folklore results about the Eilenberg-Moore category of coalgebras.

Theorem 5.

1. *If a symmetric monoidal category $\mathbf{C}$ is equipped with a monoidal comonad $(!, \varepsilon, \delta)$, then the tensor product of $\mathbf{C}$ induces a symmetric monoidal structure on the category of coalgebras $\mathbf{C}_!$.*
2. – *If, furthermore, $\mathbf{C}$ is symmetric monoidal closed, then all free coalgebras are 'exponentiable' in $\mathbf{C}_!$ (in the sense appropriate to the monoidal structure); what is more any power of a free coalgebra is a free coalgebra. So the full subcategory of finite tensor products of free coalgebras forms a symmetric monoidal closed category containing the category of free coalgebras.*
   – *If, in addition, the (Kleisli) category of free coalgebras is closed under the tensor product in $\mathbf{C}_!$, then the category of free coalgebras is symmetric monoidal closed.*
3. *If on the other hand $\mathbf{C}$ is symmetric monoidal closed and $\mathbf{C}_!$ has equalizers of coreflexive pairs of arrows then $\mathbf{C}_!$ is symmetric monoidal closed.*

We make clear what is the force of our stipulation that every (free) !-coalgebra carries naturally the structure of a commutative comonoid in such a way that coalgebra maps are comonoid maps.

Theorem 6.

1. *If a symmetric monoidal category $\mathbf{C}$ is equipped with a comonad $(!, \varepsilon, \delta)$ satisfying part 2(b) of Definition 1, then the tensor product induced on the category $\mathbf{C}_!$ of coalgebras is a categorical product.*
2. *If, furthermore, $\mathbf{C}$ is symmetric monoidal closed, then all free coalgebras are exponentiable in $\mathbf{C}_!$ (in the standard sense); and so the full subcategory of exponentiable objects forms a cartesian closed category (containing the category of free coalgebras).*
3. *If, in addition, the (Kleisli) category of free coalgebras is closed under the product in $\mathbf{C}_!$, then the category of free coalgebras is cartesian closed. In particular this follows when $\mathbf{C}$ has finite products $(1, \&)$ and we have the natural isomorphisms*

$$I \cong\, !I$$
$$!A \otimes !B \cong\, !(A \& B)$$

4. *If, on the other hand,* $\mathbf{C}_!$ *has equalizers of coreflexive pairs of arrows then* $\mathbf{C}_!$ *is cartesian closed.*

This theorem, which in essence goes back to Fox [6], is the basis for the Girard translation of intuitionistic logic into Intuitionistic Linear Logic. In the usual formulation this translation is based on *3*, that is on the natural isomorphisms introduced by Seely [16], and so essentially takes place in the category of free coalgebras. (This option is still available in cases where the relevent natural isomorphisms do not hold.) However, the general theorem demonstrates that at the proof theoretic (computational) level a more subtle analysis (which involves the full category of coalgebras) is possible.

6 Conclusions and Future Work

We have described a new term assignment system for a sequent calculus version of MELL, based on a generic idea of a categorical model for MELL. This term assignment system, unlike its predecessors has an exact correspondence with a Linear Natural Deduction system which satisfies the essential property of closure under substitution (unlike all previous proposals). Using this term assignment system and an analysis of the process of cut-elimination we produced some β-equalities. Further analysis of the β-equalities as well as the judicious addition of some extra 'extensionality' assumptions (similar to the usual ones in Categorical Proof Theory) provided a precise notion of a categorical model, more general than the traditional one for Intuitionistic Linear Logic. For this general notion of categorical model we have soundness and completeness.

But we can identify a number of areas which need to be covered in the future. Clearly we need to consider the *additive* connectives. We should also like to consider quantifiers within this framework. Especially we should like to consider some of the many variants of Intuitionistic Linear Logic that have been proposed [10, 9, 8].

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