ENTROPY PRODUCTION IN GAUSSIAN THERMOSTATS

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ABSTRACT. We show that an arbitrary Anosov Gaussian thermostat on a surface is dissipative unless the external field has a global potential. This result is obtained by studying the cohomological equation of more general thermostats using the methods in [3].

1. INTRODUCTION

Gaussian thermostats provide interesting models in nonequilibrium statistical mechanics [6, 9, 21]. Given a closed Riemannian manifold (M, g) and a vector field E (the *external field*) on M, the *Gaussian thermostat* (or isokinetic dynamics, cf. [13]) is given by the differential equation

$$(1) \qquad \frac{D\dot{\gamma}}{dt} = E(\gamma) - \frac{\langle E(\gamma), \dot{\gamma} \rangle}{|\dot{\gamma}|^2} \dot{\gamma}.$$

This equation defines a flow ϕ on the unit sphere bundle SM of M which reduces to the geodesic flow when $E = 0$.

In general, Gaussian thermostats are not volume preserving and the purpose of the present paper is to characterize precisely those Anosov Gaussian thermostats in 2 degrees of freedom which do not preserve any smooth measure.

When ϕ is Anosov and $\dim M = 2$ a result of E. Ghys [11] ensures that ϕ is topologically conjugate to the geodesic flow of a metric of constant negative curvature and thus ϕ is transitive and topologically mixing. For such a flow it is well known (cf. [14, Chapter 20]) that there exists a unique Gibbs state ρ associated with the Hölder continuous potential $-\frac{d}{dt}\big|_{t=0} \log J_t^u$, where J_t^u is the *unstable Jacobian* of ϕ. The measure ρ is characterized by being the maximum of

$$\nu \mapsto h_\nu(\phi) - \int \frac{d}{dt}\bigg|_{t=0} \log J_t^u \, d\nu$$

where ν runs over all ϕ-invariant Borel probability measures and $h_\nu(\phi)$ is the measure theoretic entropy of ϕ with respect to ν. The unique measure ρ is called the *SRB measure* of ϕ. If τ is a probability measure which is absolutely continuous with respect to the Liouville measure of SM, then ρ is also the weak limit of $\frac{1}{T}\int_0^T \phi^*\tau \, dt$ as $T \to \infty$.

The entropy production of the state ρ is given by (cf. [20])

$$e_\phi(\rho) := -\int \mathrm{div} F \, d\rho = -\sum \text{Lyapunov exponents}$$

where F is the infinitesimal generator of ϕ and $\mathrm{div} F$ is the divergence of F with respect to any volume form in SM.

Fix a volume form Θ on SM. Any other volume form can be written as $f\Theta$ for some smooth positive function f. If we let $L_F\Theta$ be the Lie derivative of Θ along F, then

$$L_F(f\Theta) = d(i_F f\Theta) = F(f)\Theta + fL_F\Theta = F(f)\Theta + f\mathrm{div}F\Theta.$$

Hence if $\widetilde{\mathrm{div}}F$ denotes the divergence of F with respect to $f\Theta$ we have

$$\tag{2} \widetilde{\mathrm{div}}F = F(\log f) + \mathrm{div}F.$$

In other words the two divergences are flow cohomologous (and thus e_ϕ is well defined for any ϕ-invariant measure).

Ruelle [20] has shown that $e_\phi(\rho) \geq 0$ with equality if and only if ρ is also the SRB measure of the flow ϕ_{-t}. If ρ is an SRB measure for both ϕ_t and ϕ_{-t} then the theory of Gibbs states for Anosov flows (cf. [14, Proposition 20.3.10]) implies that $-\frac{d}{dt}\big|_{t=0} \log J_t^u$ and $\frac{d}{dt}\big|_{t=0} \log J_t^s$ are cohomologous (and the coboundary is the derivative along the flow of a Hölder continuous function). It follows that ϕ preserves an absolutely continuous invariant measure with positive continuous density (and this measure would have to be ρ). An application of the smooth Livšic theorem [15, Corollary 2.1] shows that ϕ preserves an absolutely continuous invariant measure with positive continuous density if and only if ϕ preserves a smooth volume form. Using (2) we see that $e_\phi(\rho) = 0$ if and only if $\mathrm{div}F$ is a flow coboundary and we can take $\mathrm{div}F$ with respect to any volume form.

Let θ be the 1-form dual to E, i.e., $\theta_x(v) = \langle E(x), v\rangle$. An easy calculation (see Lemma 3.2) shows that if we consider in SM the volume form determined by the canonical contact 1-form, then $\mathrm{div}F(x,v) = -\theta_x(v)$. Thus $e_\phi(\rho) = 0$ if and only if there is a smooth solution u to the cohomological equation

$$\tag{3} F(u) = \theta.$$

We will show as a consequence of a more general result to be stated below that if $\dim M = 2$ then (3) holds if and only if θ is an exact form, i.e. if and only if E has a global potential. Thus we obtain:

Theorem A. *An Anosov Gaussian thermostat on a closed surface has zero entropy production if and only if the external field E has a global potential.*

A system with $e_\phi(\rho) > 0$ is referred to as *dissipative*. Dissipative Gaussian thermostats provide a large class of examples to which one can apply the Fluctuation Theorem of G. Gallavotti and E.G.D. Cohen [7, 8, 5] (extended to Anosov flows by G. Gentile [10]) and this theorem is perhaps one of the main motivations for determining precisely which thermostats are dissipative. Observe that Gaussian thermostats are reversible in the sense that the flip $(x,v) \mapsto (x,-v)$ conjugates ϕ_t with ϕ_{-t} (just as in the case of geodesic flows). We recall that the *chaotic hypothesis* of Gallavotti and Cohen asserts that for systems out of equilibrium, physically correct macroscopic results will be obtained by assuming that the microscopic dynamics is uniformly hyperbolic.

In [22], M. Wojtkowski proved Theorem A assuming that E has a local potential (i.e. θ is closed) and in [1], F. Bonetto, G. Gentile and V. Mastropietro proved the

theorem for the case of a metric of constant negative curvature and θ a small harmonic 1-form. We emphasize that we do not make any assumptions on g or E except that the underlying isokinetic dynamics is Anosov. Conditions under which the Anosov property holds have been given in [22, 23].

We now explain for which Anosov systems we can understand the cohomological equation (3) completely.

Let M be a closed manifold endowed with a Riemannian metric g. We consider a *generalized isokinetic thermostat*. This consists of a semibasic vector field $E(x, v)$, that is, a smooth map $TM \ni (x, v) \mapsto E(x, v) \in TM$ such that $E(x, v) \in T_x M$ for all $(x, v) \in TM$. As before the equation

$$\frac{D\dot{\gamma}}{dt} = E(\gamma, \dot{\gamma}) - \frac{\langle E(\gamma, \dot{\gamma}), \dot{\gamma} \rangle}{|\dot{\gamma}|^2} \dot{\gamma}.$$

defines a flow ϕ on the unit sphere bundle SM. These generalized thermostats are no longer reversible unless $E(x, v) = E(x, -v)$.

Suppose now that M is a closed oriented surface. We can write

$$E(x, v) = \kappa(x, v)v + \lambda(x, v)iv$$

where i indicates rotation by $\pi/2$ according to the orientation of the surface and κ and λ are smooth functions. The evolution of the thermostat on SM can now be written as

$$(4) \qquad \frac{D\dot{\gamma}}{dt} = \lambda(\gamma, \dot{\gamma})\, i\dot{\gamma}.$$

If λ does not depend on v, then ϕ is the *magnetic flow* associated with the magnetic field $\lambda\Omega_a$, where Ω_a is the area form of M. Of course, magnetic flows are Hamiltonian. If λ depends linearly on v, we obtain the Gaussian thermostat (1).

Let $\pi : SM \to M$ be the canonical projection.

Theorem B. *Let M be a closed oriented surface and consider a generalized isokinetic thermostat (4). Suppose the flow ϕ is Anosov and let F be the vector field generating ϕ. Let $h \in C^\infty(M)$ and let θ be a smooth 1-form on M. Then the cohomological equation*

$$F(u) = h \circ \pi + \theta$$

has a solution $u \in C^\infty(SM)$ if and only if $h = 0$ and θ is exact.

Note that by the smooth Livšic theorem [15] saying that $h \circ \pi + \theta = F(u)$ is equivalent to saying that $h \circ \pi + \theta$ has zero integral over every closed orbit of ϕ.

Theorem B was proved in [3] for the case of magnetic flows (i.e. λ depends only on x). It was surprising for us that the theorem also holds for systems that do not preserve a smooth measure. The proof is also based on establishing a Pestov identity as in [2, 4] for geodesic flows, but some unexpected cancellations take place producing in the end formulas which are just what one needs to prove the theorem. Earlier proofs of Theorem B for some geodesic and magnetic flows using Fourier analysis can be found in [12, 18].

Finally we note that Theorem A also holds if we allow magnetic forces. Indeed Theorem B holds for a generalized thermostat and $\mathrm{div}F = -\theta$ even when we have a magnetic field present. The extension of Theorem A to *isoenergetic* thermostats (i.e. in the presence of potential forces) is discussed in Remark 5.1.

2. Preliminaries

Let M be a closed oriented surface, SM the unit sphere bundle and $\pi : SM \to M$ the canonical projection. The latter is in fact a principal S^1-fibration and we let V be the infinitesimal generator of the action of S^1.

Given a unit vector $v \in T_x M$, we will denote by iv the unique unit vector orthogonal to v such that $\{v, iv\}$ is an oriented basis of $T_x M$. There are two basic 1-forms α and β on SM which are defined by the formulas:

$$\alpha_{(x,v)}(\xi) := \langle d_{(x,v)}\pi(\xi), v \rangle;$$

$$\beta_{(x,v)}(\xi) := \langle d_{(x,v)}\pi(\xi), iv \rangle.$$

The form α is the canonical contact form of SM whose Reeb vector field is the geodesic vector field X. The volume form $\alpha \wedge d\alpha$ gives rise to the Liouville measure $d\mu$ of SM.

A basic theorem in 2-dimensional Riemannian geometry asserts that there exists a unique 1-form ψ on SM (the connection form) such that $\psi(V) = 1$ and

$$(5) \qquad\qquad\qquad d\alpha = \psi \wedge \beta$$

$$(6) \qquad\qquad\qquad d\beta = -\psi \wedge \alpha$$

$$(7) \qquad\qquad\qquad d\psi = -(K \circ \pi)\,\alpha \wedge \beta$$

where K is the Gaussian curvature of M. In fact, the form ψ is given by

$$\psi_{(x,v)}(\xi) = \left\langle \frac{DZ}{dt}(0), iv \right\rangle,$$

where $Z : (-\varepsilon, \varepsilon) \to SM$ is any curve with $Z(0) = (x, v)$ and $\dot{Z}(0) = \xi$ and $\frac{DZ}{dt}$ is the covariant derivative of Z along the curve $\pi \circ Z$.

For later use it is convenient to introduce the vector field H uniquely defined by the conditions $\beta(H) = 1$ and $\alpha(H) = \psi(H) = 0$. The vector fields X, H and V are dual to α, β and ψ and as a consequence of (5–7) they satisfy the commutation relations

$$(8) \qquad\qquad [V, X] = H, \quad [V, H] = -X, \quad [X, H] = KV.$$

Equations (5–7) also imply that the vector fields X, H and V preserve the volume form $\alpha \wedge d\alpha$ and hence the Liouville measure. Note that the flow of H is given by $R^{-1} \circ g_t \circ R$, where $R(x, v) = (x, iv)$ and g_t is the geodesic flow.

3. An integral identity

Henceforth (M, g) is a closed oriented surface and X, H, and V are the same vector fields on SM as in the previous section.

Let λ be the smooth function on SM given by (4), and let

$$F = X + \lambda V$$

be the generating vector field of the generalized thermostat.

From (8) we obtain:

$$[V, F] = H + V(\lambda)V, \quad [V, H] = -F + \lambda V, \quad [F, H] = -\lambda F + (K - H(\lambda) + \lambda^2)V.$$

Lemma 3.1 (The Pestov identity). *For every smooth function $u : SM \to \mathbb{R}$ we have*

$$2Hu \cdot VFu = (Fu)^2 + (Hu)^2 - (K - H(\lambda) + \lambda^2)(Vu)^2$$
$$+ F(Hu \cdot Vu) + V(\lambda)Hu \cdot Vu - H(Fu \cdot Vu) + V(Fu \cdot Hu).$$

Proof. Using the commutation formulas, we deduce:

$$2Hu \cdot VFu - V(Hu \cdot Fu)$$
$$= Hu \cdot VFu - VHu \cdot Fu$$
$$= Hu \cdot (FVu + [V, F]u) - Fu \cdot (HVu + [V, H]u)$$
$$= Hu \cdot (FVu + Hu + V(\lambda)Vu) - Fu \cdot (HVu - Fu + \lambda Vu)$$
$$= (Fu)^2 + (Hu)^2 + (FVu)(Hu) - (HVu)(Fu) - \lambda Fu \cdot Vu + Hu \cdot V(\lambda)Vu$$
$$= (Fu)^2 + (Hu)^2 + F(Vu \cdot Hu) - H(Vu \cdot Fu) - [F, H]u \cdot Vu$$
$$- \lambda Fu \cdot Vu + Hu \cdot V(\lambda)Vu$$
$$= (Fu)^2 + (Hu)^2 + F(Vu \cdot Hu) + V(\lambda)Hu \cdot Vu - H(Vu \cdot Fu)$$
$$- (K - H(\lambda) + \lambda^2)(Vu)^2$$

which is equivalent to the Pestov identity. $\qquad\square$

Now let $\Theta := \alpha \wedge d\alpha$. This volume form generates the Liouville measure $d\mu$.

Lemma 3.2. *We have:*

(9) $$\qquad\qquad\qquad\qquad L_F\Theta = V(\lambda)\Theta;$$

(10) $$\qquad\qquad\qquad\qquad L_H\Theta = 0;$$

(11) $$\qquad\qquad\qquad\qquad L_V\Theta = 0.$$

Proof. Note that for any vector field Y, $L_Y\Theta = d(i_Y\Theta)$. Since $i_V\Theta = -\alpha \wedge \beta = -\pi^*\Omega_a$, where Ω_a is the area form of M, we see that $L_V\Theta = 0$. Similarly, $L_X\Theta = L_H\Theta = 0$. Finally $L_F\Theta = L_X\Theta + L_{\lambda V}\Theta = d(i_{\lambda V}\Theta) = V(\lambda)\Theta$.

$\qquad\square$

Below we will use the following consequence of Stokes theorem. Let N be a closed oriented manifold and Θ a volume form. Let X be a vector field on N and $f : N \to \mathbb{R}$ a smooth function. Then

$$(12) \qquad \int_N X(f)\,\Theta = -\int_N f\,L_X\Theta.$$

Integrating the Pestov identity over SM against the Liouville measure $d\mu$, and using (10) and (11) we obtain:

$$2\int_{SM} Hu \cdot VFu\,d\mu = \int_{SM}(Fu)^2\,d\mu + \int_{SM}(Hu)^2\,d\mu$$
$$+ \int_{SM}(F(Hu \cdot Vu) + V(\lambda)Hu \cdot Vu)\,d\mu$$
$$- \int_{SM}(K - H(\lambda) + \lambda^2)(Vu)^2\,d\mu.$$

Using (12) and (9) we get:

$$\int_{SM}(F(Hu \cdot Vu) + V(\lambda)Hu \cdot Vu)\,d\mu = 0$$

and thus

$$(13) \qquad 2\int_{SM} Hu \cdot VFu\,d\mu = \int_{SM}(Fu)^2\,d\mu + \int_{SM}(Hu)^2\,d\mu$$
$$- \int_{SM}(K - H(\lambda) + \lambda^2)(Vu)^2\,d\mu.$$

We will derive one more integral identity. By the commutation relations, we have

$$FVu = VFu - Hu - V(\lambda)Vu.$$

Therefore,

$$(FVu)^2 = (VFu)^2 + (Hu)^2 + (V(\lambda))^2(Vu)^2$$
$$-2VFu \cdot Hu - 2VFu \cdot V(\lambda)Vu + 2V(\lambda)Vu \cdot Hu.$$

Thus using again the commutation relations:

$$(FVu)^2 = (VFu)^2 + (Hu)^2 + (V(\lambda))^2(Vu)^2$$
$$-2VFu \cdot Hu - 2FVu \cdot V(\lambda)Vu - 2(V(\lambda))^2(Vu)^2.$$

Since

$$F(V(\lambda)(Vu)^2) = 2V(\lambda)Vu \cdot FVu + (Vu)^2F(V(\lambda))$$

we obtain:

$$(FVu)^2 = (VFu)^2 + (Hu)^2 - (V(\lambda))^2(Vu)^2$$
$$-2VFu \cdot Hu - F(V(\lambda)(Vu)^2) + (Vu)^2F(V(\lambda)).$$

Integrating this equation we obtain:

$$(14) \qquad 2\int_{SM} Hu \cdot VFu \, d\mu = \int_{SM} (VFu)^2 \, d\mu + \int_{SM} (Hu)^2 \, d\mu$$
$$- \int_{SM} (FVu)^2 \, d\mu + \int_{SM} F(V(\lambda))(Vu)^2 \, d\mu$$

since by (12) and (9) we get:

$$\int_{SM} \{ F(V(\lambda)(Vu)^2) + (V(\lambda))^2 (Vu)^2 \} \, d\mu = 0.$$

Combining (13) and (14) we arrive at the final integral identity of this section:

Theorem 3.3.

$$(15) \qquad \int_{SM} (FVu)^2 \, d\mu - \int_{SM} \mathbb{K}(Vu)^2 \, d\mu = \int_{SM} (VFu)^2 \, d\mu - \int_{SM} (Fu)^2 \, d\mu,$$

where $\mathbb{K} := K - H(\lambda) + \lambda^2 + F(V(\lambda))$.

Of course this identity holds without any assumption on the underlying dynamics. In the next section we will show how to use the Anosov hypothesis to rewrite the left hand side of (15) in terms of the stable or unstable bundles. At this point the proof differs from the one presented in [3]. We can no longer estimate the left hand side of (15) using closed orbits and the non-negative Livšic theorem [16, 19] since in our context the Liouville measure is not necessarily invariant.

4. Using the Anosov property

Recall that the Anosov property means that $T(SM)$ splits as $T(SM) = \mathbb{R}F \oplus E^u \oplus E^s$ in such a way that there are constants $C > 0$ and $0 < \rho < 1 < \eta$ such that for all $t > 0$ we have

$$\|d\phi_{-t}|_{E^u}\| \leq C\,\eta^{-t} \quad \text{and} \quad \|d\phi_t|_{E^s}\| \leq C\,\rho^t.$$

The subbundles are then invariant and Hölder continuous and have smooth integral manifolds, the stable and unstable manifolds, which define a continuous foliation with smooth leaves.

Let us introduce the weak stable and unstable bundles:

$$E^+ = \mathbb{R}F \oplus E^s,$$
$$E^- = \mathbb{R}F \oplus E^u.$$

Lemma 4.1. *For any* $(x,v) \in SM$, $V(x,v) \notin E^\pm(x,v)$.

Proof. Let $\Lambda(SM)$ be the bundle over SM such that at each point $(x,v) \in SM$ consists of all 2-dimensional subspaces W of $T_{(x,v)}SM$ with $F(x,v) \in W$.

The map $(x,v) \mapsto \mathbf{V} := \mathbb{R}F(x,v) \oplus \mathbb{R}V(x,v)$ is a section of $\Lambda(SM)$ and its image is a codimension one submanifold that we denote by Λ_V. Similarly the map $(x,v) \mapsto \mathbb{R}F(x,v) \oplus \mathbb{R}H(x,v)$ is a section of $\Lambda(SM)$ and its image is a codimension one submanifold that we denote by Λ_H.

The flow ϕ naturally lifts to a flow ϕ^* acting on $\Lambda(SM)$ via its differential. Let F^* be the infinitesimal generator of ϕ^*.

Claim. F^* is transversal to Λ_V.

To prove the claim we define a function $m : \Lambda(SM) \setminus \Lambda_H \to \mathbb{R}$ as follows. If $W \in \Lambda(SM) \setminus \Lambda_H$, then $H \notin W$. Thus there exists a unique $m = m(W)$ such that $mH + V \in W$. Clearly m is smooth and $\Lambda_V = m^{-1}(0) \subset \Lambda(SM) \setminus \Lambda_H$. Fix $(x,v) \in SM$ and set $m(t) := m(\phi_t^*(\mathbf{V}(x,v)))$. By the definition of m, there exist functions $x(t)$ and $y(t)$ such that

$$m(t)H(t) + V(t) = x(t)F(t) + y(t)d\phi_t(V).$$

Equivalently

$$m(t)d\phi_{-t}(H(t)) + d\phi_{-t}(V(t)) = x(t)F + y(t)V.$$

Differentiating with respect to t and setting $t = 0$ (recall that $m(0) = 0$) we obtain:

$$\dot{m}(0)H + [F, V] = \dot{x}(0)F + \dot{y}(0)V.$$

But $[V, F] = H + V(\lambda)V$. Thus $\dot{m}(0) = 1$ which proves the Claim.

From the Claim it follows that Λ_V determines an oriented codimension one cycle in $\Lambda(SM)$ and by duality it defines a cohomology class $\mathfrak{m} \in H^1(\Lambda(SM), \mathbb{Z})$. Set $E = E^\pm$. Given a continuous closed curve $\alpha : S^1 \to SM$, the *index* of α is $\nu(\alpha) := \langle \mathfrak{m}, [E \circ \alpha] \rangle$ (i.e. $\nu = E^*\mathfrak{m} \in H^1(SM, \mathbb{Z})$). The index of α only depends on the homology class of α. Since E is ϕ-invariant, the Claim also ensures that if γ is any closed orbit of ϕ, then $\nu(\gamma) \geq 0$.

Recall that according to Ghys [11] we know that ϕ is topologically conjugate to the geodesic flow of a metric of constant negative curvature. In particular, every homology class in $H_1(SM, Z)$ contains a closed orbit of ϕ. Thus ν must vanish.

If there exists $(x,v) \in SM$ for which $V(x,v) \in E(x,v)$, then using that every point of ϕ is non-wandering, we can produce exactly as in [17, Lemma 2.49] a closed curve $\alpha : S^1 \to SM$ with $\nu(\alpha) > 0$. This contradiction shows the lemma.
$\square$

Remark 4.2. The reader will recognize that the index that appears in the proof of the lemma reduces to the Maslov index when ϕ is Hamiltonian. The proof of the lemma also follows the presentation in [17, Chapter 2] of analogous results for geodesic flows.

The lemma implies that there exist unique continuous functions $r^\pm$ on SM such that

$$H + r^+V \in E^+,$$

$$H + r^-V \in E^-.$$

Note that the Anosov property implies that $r^+ \neq r^-$ everywhere. Below we will need to use that the functions $r^\pm$ satisfy a Riccati type equation along the flow. Note that $r^\pm$ are smooth along ϕ because $E^\pm$ are ϕ-invariant.

Lemma 4.3. *Let $r = r^{\pm}$. Then*

$$F(r - V(\lambda)) + r(r - V(\lambda)) + \mathbb{K} = 0.$$

Proof. Let $E = E^{\pm}$. Fix $(x, v) \in SM$, flow along ϕ and set

$$\xi(t) := d\phi_{-t}(H(t) + r(t)V(t)).$$

By the definition of r, $\xi(t) \in E(x, v)$ for all t. Differentiating with respect to t and setting $t = 0$ we obtain:

$$\dot{\xi}(0) = [F, H] + F(r)V + r[F, V].$$

Using that

$$[V, F] = H + V(\lambda)V, \quad [F, H] = -\lambda F + (K - H(\lambda) + \lambda^2)V$$

we have

$$\dot{\xi}(0) = -\lambda F - rH + \left\{F(r) + K - H(\lambda) + \lambda^2 - V(\lambda)r\right\}V.$$

Replacing H by $\xi(0) - rV$ yields:

$$\dot{\xi}(0) + r\xi(0) + \lambda F = \left\{r^2 + F(r) + K - H(\lambda) + \lambda^2 - V(\lambda)r\right\}V.$$

Since $\dot{\xi}(0) + r\xi(0) + \lambda F \in E$ we must have

$$r^2 + F(r) + K - H(\lambda) + \lambda^2 - V(\lambda)r = 0$$

which is the desired equation since $\mathbb{K} = K - H(\lambda) + \lambda^2 + F(V(\lambda))$. $\qquad\square$

Here is the main result of this section:

Theorem 4.4. *Let $\psi : SM \to \mathbb{R}$ be a smooth function and suppose ϕ is Anosov. Then for $r = r^{\pm}$*

$$\int_{SM} (F\psi)^2 \, d\mu - \int_{SM} \mathbb{K}\psi^2 \, d\mu = \int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 \, d\mu \geq 0.$$

Moreover,

$$\int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 \, d\mu = 0$$

if and only if $\psi = 0$.

Proof. Let us expand $[F(\psi) - r\psi + \psi V(\lambda)]^2$:

$$[F(\psi) - r\psi + \psi V(\lambda)]^2 = [F(\psi)]^2 + \psi^2 r^2 + \psi^2[V(\lambda)]^2$$
$$- 2F(\psi)\psi r + 2F(\psi)\psi V(\lambda) - 2\psi^2 r V(\lambda).$$

Using that (see Lemma 4.3)

$$F(r - V(\lambda)) + r(r - V(\lambda)) + \mathbb{K} = 0$$

we obtain:

$$[F(\psi) - r\psi + \psi V(\lambda)]^2 = [F(\psi)]^2 - \mathbb{K}\psi^2$$
$$- F((r - V(\lambda))\psi^2) + \psi^2[V(\lambda)]^2 - \psi^2 rV(\lambda).$$

If we integrate the last equality with respect to the Liouville measure μ we obtain as desired:

$$\int_{SM} (F\psi)^2 \, d\mu - \int_{SM} \mathbb{K}\psi^2 \, d\mu = \int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 \, d\mu$$

since by (12) and (9) we have the following cancellation:

$$\int_{SM} \{-F((r - V(\lambda))\psi^2) + \psi^2[V(\lambda)]^2 - \psi^2 rV(\lambda)\} \, d\mu = 0.$$

Suppose now

$$\int_{SM} [F(\psi) - r\psi + \psi V(\lambda)]^2 \, d\mu = 0$$

which implies

$$F(\psi) - r\psi + \psi V(\lambda) = 0$$

everywhere. Since this holds for $r = r^\pm$ we deduce:

$$(r^+ - r^-)\psi = 0.$$

But for an Anosov flow $r^+ - r^- \neq 0$ everywhere, thus $\psi = 0$.

$\square$

5. PROOF OF THEOREM B

Let us now prove Theorem B. If $Fu = h \circ \pi + \theta$, then it is easy to see that the right-hand side of (15) is nonpositive. Indeed, since μ is invariant under $v \mapsto -v$ and $v \to iv$ we have

$$\int_{SM} \theta_x(v) \, d\mu = 0 \quad \text{and} \quad \int_{SM} (\theta_x(v))^2 \, d\mu = \int_{SM} (\theta_x(iv))^2 \, d\mu.$$

But $VFu = \theta_x(iv)$ and thus

$$\int_{SM} (VFu)^2 \, d\mu - \int_{SM} (Fu)^2 \, d\mu = -\int_{SM} (h \circ \pi)^2 \, d\mu \leq 0.$$

Setting $\psi = Vu$, we get

(16) $$\int_{SM} \{(F\psi)^2 - \mathbb{K}\psi^2\} \, d\mu \leq 0.$$

By Theorem 4.4 this happens if and only if $\psi = 0$. This would give $Vu = 0$, which says that $u = f \circ \pi$ where f is a smooth function on M. But in this case, since $d\pi_{(x,v)}(F) = v$ we have $Fu = df_x(v)$. This clearly implies the claim of the theorem.

Remark 5.1. Suppose that we include potential forces in our dynamics, that is, we consider the *isoenergetic thermostat*:

$$(17) \qquad \frac{D\dot{\gamma}}{dt} = -\nabla W + E(\gamma) - \frac{\langle E(\gamma), \dot{\gamma} \rangle}{|\dot{\gamma}|^2} \dot{\gamma}$$

on the energy level $\frac{1}{2}|v|^2 + W(x) = k$ (we assume that $|v|$ does not vanish on the energy level). Wojtkowski has pointed out [24, Theorem 2.4] that the dynamics of (17) reparametrized by arc-length defines a flow on SM which coincides with the isokinetic thermostat with external field

$$\widetilde{E} := \frac{-\nabla W + E}{2(k - W)} = \frac{1}{2}\nabla(\log(k - W)) + \frac{E}{2(k - W)}.$$

Since the vanishing of entropy production and the Anosov property are unaltered by smooth time changes we conclude applying Theorem A to $\widetilde{E}$ that an Anosov isoenergetic thermostat has zero entropy production if and only if $E/2(k - W)$ has a global potential.

The question of whether Theorem B extends to higher dimension is more delicate. We hope to discuss this topic elsewhere.

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