

## RIEMANN SURFACES EXAMPLES 1

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at [g.p.paternain@dpmms.cam.ac.uk](mailto:g.p.paternain@dpmms.cam.ac.uk).

1. Let  $U = \mathbb{C} \setminus ([-1, 0] \cup [1, \infty))$  and let  $\gamma$  be a closed curve in  $U$ . Using standard properties of winding numbers, show that (i)  $n(\gamma, 1) = 0$ , and (ii)  $n(\gamma, 0) = n(\gamma, -1)$ .
2. Let  $P(w_0, w_1, \dots, w_s; z)$  be a polynomial in the  $s+1$  complex variables  $w_0, w_1, \dots, w_s$ , where the coefficients of  $P$  are holomorphic on  $\mathbb{C}$ . Thus

$$P(f(z), f^{(1)}(z), \dots, f^{(s)}(z); z) = 0$$

is a differential equation, which we abbreviate to  $P(f) = 0$ . If  $(f, D)$  is a function element with  $P(f) = 0$  in  $D$  and if  $(g, D') \approx (f, D)$  is an analytic continuation, then show that  $P(g) = 0$  in  $D'$ . Give an example of a differential equation and function elements as above, where  $D' = D$  but  $g \neq f$  on  $D$ .

3. Let  $\pi : \tilde{X} \rightarrow X$  be a covering map of topological spaces (recalling here that the spaces are assumed connected and Hausdorff), and  $f : \tilde{X} \rightarrow \tilde{X}$  a continuous map such that  $\pi f = \pi$ . Show that  $f$  has no fixed points unless it is the identity.
4. Show that the power series  $f(z) = \sum_{n>1} \frac{1}{n(n-1)} z^n$  defines an analytic function  $(1-z) \log(1-z) + z$  on the unit disc  $D$ . Deduce that the function element  $(f, D)$  defines a complete analytic function on  $\mathbb{C} \setminus \{1\}$ , but does not extend to an analytic function on  $\mathbb{C} \setminus \{1\}$ .
5. Show that the power series  $f(z) = \sum z^{2^n} / 2^n$  has the unit circle as a natural boundary.
6. Show that atlases being equivalent is an equivalence relation on the set of atlases. Show that any conformal structure on a Riemann surface contains a maximal atlas.
7. Let  $T$  be the complex torus  $\mathbb{C}/\langle 1, \tau \rangle$ , and let  $Q_1 \subset \mathbb{C}$  be the open parallelogram with vertices  $0, 1, \tau, 1+\tau$ , and  $Q_2$  the translation of  $Q_1$  by  $(1+\tau)/2$ . Let  $U_1, U_2$  denote the open subsets of  $T$  given by projection of  $Q_1, Q_2$  respectively, and let  $\phi_1 : U_1 \rightarrow Q_1$ ,  $\phi_2 : U_2 \rightarrow Q_2$  be the charts obtained by taking the inverse maps. Describe explicitly the transition function

$$\phi_2 \phi_1^{-1} : \phi_1(U_1 \cap U_2) \rightarrow \phi_2(U_1 \cap U_2).$$

8. By considering the singularity at  $\infty$  or otherwise, show that any injective analytic map  $f : \mathbb{C} \rightarrow \mathbb{C}$  has the form  $f(z) = az + b$ , for some  $a \in \mathbb{C}^*$  and  $b \in \mathbb{C}$ . Find the injective analytic maps  $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ .
9. Let  $\Lambda = \langle \tau_1, \tau_2 \rangle$  be a lattice in  $\mathbb{C}$  and let  $T = \mathbb{C}/\Lambda$  be the corresponding complex torus. Let  $\Lambda'$  denote the lattice  $\langle 1, \tau_2/\tau_1 \rangle$  and  $T' = \mathbb{C}/\Lambda'$ . Show that the Riemann surfaces  $T$  and  $T'$  are analytically isomorphic (i.e. conformally equivalent).
10. Define an equivalence relation  $\sim$  on  $\mathbb{C}^*$  by  $z \sim w$  iff  $z = 2^s w$  for some  $s \in \mathbb{Z}$ . Show that the quotient space  $R = \mathbb{C}^*/\sim$  has the natural structure of a compact Riemann surface, and that  $R$  is analytically isomorphic to a complex torus.
11. (The identity principle for Riemann surfaces) Let  $R, S$  be Riemann surfaces, and  $f, g : R \rightarrow S$  be analytic maps between them. Set  $E = \{z \in R : f(z) = g(z)\}$ ; show that either  $E = R$  or  $E$  contains only isolated points.

**12.** Let  $D \subset \mathbb{C}$  be an open disc and  $u$  a harmonic function on  $D$ . Define a complex valued function  $g$  on  $D$  by  $g = u_x - iu_y$ ; show that  $g$  is analytic. If  $z_0$  denotes the centre of the disc, define a function  $f$  on  $D$  by

$$f(z) = u(z_0) + \int_{z_0}^z g,$$

the integral being taken over the straight line segment. Show that  $f$  is analytic with  $f' = g$ , and that  $u = \Re f$ .

**13.** Suppose  $u, v$  are harmonic functions on a Riemann surface  $R$  and  $E = \{z \in R : u(z) = v(z)\}$ . Show that either  $E = R$ , or  $E$  has empty interior. Give an example to show that  $E$  does not in general consist of isolated points.

**14.** Let  $\{a_1, a_2, a_3, a_4\}$  and  $\{b_1, b_2, b_3, b_4\}$  both be sets of four distinct points in  $\mathbb{C}_\infty$ . Show that any analytic isomorphism

$$f : \mathbb{C}_\infty \setminus \{a_1, a_2, a_3, a_4\} \rightarrow \mathbb{C}_\infty \setminus \{b_1, b_2, b_3, b_4\}$$

extends to an analytic isomorphism  $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ . Using your answer to Question 8, find a necessary and sufficient condition for  $\mathbb{C} \setminus \{0, 1, a\}$  to be conformally equivalent to  $\mathbb{C} \setminus \{0, 1, b\}$ , where  $a, b$  are complex numbers distinct from 0 and 1.

**15.** Let  $f(z)$  be the complex polynomial  $z^3 - z$ ; consider the subspace  $R$  of  $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$  given by the equation  $w^2 = f(z)$ , where  $(z, w)$  denote the coordinates on  $\mathbb{C}^2$ , and let  $\pi : R \rightarrow \mathbb{C}$  be the restriction of the projection map onto the first factor. Show that  $R$  has the structure of a Riemann surface, on which  $\pi$  is an analytic map. If  $g$  denotes the projection onto the second factor, show that  $g$  is also an analytic map.

By deleting three appropriate points from  $R$ , show that  $\pi$  yields a covering map from the resulting Riemann surface  $R_0 \subset R$  to  $\mathbb{C} \setminus \{-1, 0, 1\}$ , and that  $R_0$  is analytically isomorphic to the Riemann surface (constructed by gluing) associated with the complete analytic function  $(z^3 - z)^{1/2}$  over  $\mathbb{C} \setminus \{-1, 0, 1\}$ .

**16.** Let  $f(z) = \sum a_n z^n$  be a power series of radius of convergence 1, and for  $w$  in the open unit disc, set  $\rho(w)$  to be the radius of convergence for the power series expansion about  $w$  (so that  $\rho(0) = 1$ ). Show that a point  $\zeta \in C(0, 1)$  on the unit circle is regular if and only if  $\rho(\zeta/2) > \frac{1}{2}$ . Suppose furthermore that all the  $a_n$  are non-negative real numbers. If  $\zeta \in C(0, 1)$ , show that  $|f^{(r)}(\zeta/2)| \leq f^{(r)}(1/2)$  for all  $r$ , and hence that  $\rho(\zeta/2) \geq \rho(1/2)$ . Deduce that 1 is a singular point.