RIGIDITY PROPERTIES OF ANOSOV OPTICAL HYPERSURFACES

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ABSTRACT. We consider an optical hypersurface Σ in the cotangent bundle $\tau : T^*M \to M$ of a closed manifold M endowed with a twisted symplectic structure. We show that if the characteristic foliation of Σ is Anosov, then a smooth 1-form θ on M is exact if and only if $\tau^*\theta$ has zero integral over every closed characteristic of Σ. This result is derived from a related theorem about magnetic flows which generalizes our work in [7]. Other rigidity issues are also discussed.

1. INTRODUCTION

Let M be a closed connected n-manifold and let $\tau : T^*M \to M$ be its cotangent bundle. Given an arbitrary smooth closed 2-form Ω on M, we consider T^*M endowed with the *twisted symplectic structure*

$$\omega := -d\lambda + \tau^*\Omega,$$

where λ is the Liouville 1-form.[1]

A smooth, closed, connected, fiberwise strictly convex hypersurface $\Sigma \subset T^*M$ is called *optical*.[2] Fiberwise strict convexity means that Σ intersects each fiber T^*_xM along a hypersurface whose second fundamental form is positive definite. Denote by σ the characteristic foliation of Σ, i.e., the 1–dimensional foliation tangent to the kernel of $\omega|_{T\Sigma}$. Note that σ is orientable.

We shall say that an optical hypersurface $\Sigma \subset T^*M$ is *Anosov* (or hyperbolic) if the characteristic foliation admits a (non-vanishing) tangent vector field whose flow is Anosov. Since the flows of two such vector fields are reparametrizations of one another, the property of being Anosov is independent of the chosen vector field (cf. [1]) and is a property of Σ.

In the present paper we shall study various rigidity properties of Anosov optical hypersurfaces on cotangent bundles equipped with twisted symplectic structures. These properties are motivated by recent results that we obtained for two dimensional magnetic flows [7].

Here is one of our main results:

Theorem A. *Let $\Sigma \subset T^*M$ be an Anosov optical hypersurface, where T^*M is endowed with a twisted symplectic structure $-d\lambda + \tau^*\Omega$. Let θ be a smooth 1-form on*

[1]Hence, we use the convention that the Hamiltonian vector field X_H of a Hamiltonian H is determined by $i_{X_H}\omega = dH$.

[2]For the origins of the term *optical* see [2, Section 9].

M. Then θ is exact if and only if

$$\int_\Gamma \tau^*\theta = 0$$

for every closed characteristic Γ of σ.

If the closed 2-form Ω determines an integral class, we can introduce the notion of action spectrum as follows. Suppose $[\Omega] \in H^2(M,\mathbb{Z})$. Then there exists a principal circle bundle $\Pi : P \to M$ with Euler class $[\Omega]$. The bundle admits a connection 1-form α such that $d\alpha = -2\pi\,\Pi^*\Omega$. Let $\log \mathrm{hol}_\alpha : Z_1(M) \to \mathbb{R}/\mathbb{Z}$, be the logarithm of the holonomy of the connection α. Here, $Z_1(M)$ is the space of 1-cycles and for every 2-chain $f : \Sigma \to M$ we have

$$\log \mathrm{hol}_\alpha(\partial\Sigma) = -\int_\Sigma f^*\Omega \mod 1.$$

We define the *action* of an oriented closed characteristic Γ as:

$$A(\Gamma) := \int_\Gamma \lambda + \log \mathrm{hol}_\alpha(\tau(\Gamma)) \mod 1.$$

We call the set $\mathcal{S} \subset \mathbb{R}/\mathbb{Z}$ of values $A(\Gamma)$ as Γ ranges over all (oriented) closed characteristics, the *action spectrum* of Σ.

If Ω does not determine an integral class, but there exists $c \neq 0$ such that $[c\Omega] \in H^2(M,\mathbb{Z})$ we can still define the action spectrum by considering $R_c(\Sigma)$ and $-d\lambda + c\tau^*\Omega$, where $R_c(x,p) := (x, cp)$. The characteristic foliations of $(\Sigma, -d\lambda + \tau^*\Omega)$ and $(R_c(\Sigma), -d\lambda + c\tau^*\Omega)$ are conjugate by R_c.

Suppose now that we vary the connection 1-form α. Let α_r be a smooth 1-parameter family of connections for $r \in (-\varepsilon, \varepsilon)$ with $\alpha_0 = \alpha$ Then we can write $\alpha_r - \alpha = \Pi^*\beta_r$, where β_r are smooth 1-forms on M. The connection α_r has curvature form $-2\pi\,\Omega + d\beta_r$. If we let $\Omega_r = \Omega - \frac{1}{2\pi}d\beta_r$ we get a characteristic foliation σ^r and an action spectrum $\mathcal{S}_r$. If the characteristic foliation σ is Anosov, then for ε small enough σ^r is Anosov for all $r \in (-\varepsilon, \varepsilon)$.

Corollary 1. *Let M be a closed connected manifold and let $\Sigma \subset T^*M$ be an optical hypersurface. Let Ω be a closed integral 2-form and suppose that $(\Sigma, -d\lambda + \tau^*\Omega)$ is Anosov. If $\mathcal{S}_r = \mathcal{S}$ for all r sufficiently small, then the deformation is trivial, that is, $\alpha_r = \alpha + \Pi^* dF_r$ and $\Omega_r = \Omega$, where F_r are smooth functions on M.*

The proof of Corollary 1 is very similar to that of Theorem C in [7] and hence we omit it.

Theorem A will be a consequence of the following result. Let M be a closed connected manifold endowed with a Finsler metric F. The Legendre transform $\ell_F : TM \setminus \{0\} \to T^*M \setminus \{0\}$ associated with the Lagrangian $\frac{1}{2}F^2$ is a diffeomorphism and $\omega_0 := \ell_F^*(-d\lambda)$ defines a symplectic form on $TM \setminus \{0\}$. Now let Ω be a smooth closed 2-form on M and $\pi : TM \to M$ the canonical projection. The *magnetic flow* of the pair (F, Ω) is the Hamiltonian flow ϕ of $\frac{1}{2}F^2$ with respect to the symplectic form $\omega_0 + \pi^*\Omega$. We shall consider ϕ restricted to the unit sphere bundle $SM := F^{-1}(1)$. A curve $\gamma : \mathbb{R} \to M$ given by $\gamma(t) = \pi(\phi_t(x,v))$ will be called a *magnetic geodesic.*

Theorem B. *Let (M, F) be a closed connected Finsler manifold and Ω an arbitrary smooth closed 2-form. Suppose the magnetic flow ϕ of the pair (F, Ω) is Anosov and let $\mathbf{G}_M$ be the vector field generating ϕ.*

If $h : M \to \mathbb{R}$ is any smooth function and θ is any smooth 1-form on M such that there is a smooth function $u : SM \to \mathbb{R}$ for which $h(x) + \theta_x(v) = \mathbf{G}_M(u)$, then h is identically zero and θ is exact.

Note that by the smooth Livšic theorem [12] saying that $h(x) + \theta_x(v) = \mathbf{G}_M(u)$ is equivalent to saying that $h(x) + \theta_x(v)$ has zero integral over every closed magnetic geodesic.

Various versions of Theorem B were previously known:

(1) V. Guillemin and D. Kazhdan in [8] proved Theorem B for M a surface, $\Omega = 0$ and F a negatively curved Riemannian metric. In [9] they extended this to higher dimensional manifolds under a pointwise curvature pinching assumption and Min-Oo [14] proved it when the curvature operator is negative definite. All these results were based on Fourier analysis.

(2) A major breakthrough was obtained by C. Croke and V. Sharafutdinov [5] in which results like Theorem B were proved just assuming negative sectional curvature and in any dimension. The novel ingredient here was the *Pestov identity.*

(3) In [6], Dairbekov and Sharafutdinov proved Theorem B, just assuming that the geodesic flow of the Riemannian metric is Anosov.

(4) In [7], the authors proved Theorem B when M is a surface and F is a Riemannian metric, but Ω is arbitrary.

We now describe some applications of these results.

1.1. **Infinitesimal spectral rigidity.** Corollary 1 and the results of V. Guillemin and A. Uribe in [10] give a version of infinitesimal spectral rigidity for magnetic flows. This version was obtained in [7] for the case of surfaces. Suppose Ω is a closed integral 2-form and g a Riemannian metric. For every positive integer m, let L_m be the Hermitian line bundle with connection over M associated with Π via the character $e^{i\theta} \mapsto e^{im\theta}$ of S^1. The metric on M, together with the connection on L_m determine a Bochner-Laplace operator acting on sections of L_m. For each m, let $\{\nu_{m,j} : j = 1, 2, \dots\}$ be the spectrum of this operator. If we now vary the connection 1-form α as above we obtain eigenvalues $\nu^r_{m,j}$.

Corollary 2. *Let M be a closed connected manifold endowed with a Riemannian metric g and let Ω be an integral 2-form. Suppose the magnetic flow of the pair (g, Ω) is Anosov. If $\nu^r_{m,j}$ is independent of r for all m and j (i.e. the deformation is isospectral), then the deformation is trivial, that is, $\alpha_r = \alpha + \Pi^* dF_r$ and $\Omega_r = \Omega$, where F_r are smooth functions on M.*

Indeed, let us consider the periodic distribution

$$\Upsilon(s) = \sum_{m,j} \varphi\left(\sqrt{\nu_{m,j} + m^2} - m\sqrt{2}\right) e^{ims}$$

where φ is a Schwartz function on the real line. Theorem 6.9 in [10] asserts that the singularities of Υ are included in the set of all $s \in \mathbb{R}$ for which $s/2\pi \bmod 1 \in \mathcal{S}$. Moreover, each point of the action spectrum arises as a singularity of Υ for an appropriate choice of φ. Corollary 2 is now an immediate consequence of Corollary 1.

There is an equivalent way of formulating Corollary 2 in purely Riemannian terms using the *Kaluza-Klein metric*. Consider on P the metric g_{KK} defined uniquely by the following conditions: the restriction of $d\Pi$ to the horizontal subspace of the connection α is an isometry, vertical and horizontal subspaces are orthogonal and the vector field $\partial/\partial\theta$ tangent to the fibres has norm one. If we vary the connection α as above we obtain a 1-parameter family of Kaluza-Klein metrics g^r_{KK}, $r \in (-\varepsilon, \varepsilon)$. Consider the usual Laplacian Δ^r_{KK} of these metrics. Corollary 2 could be rephrased by saying that if the spectrum of Δ^r_{KK} remains unchanged, then the deformation is trivial. In fact, the eigenvalues $\lambda_{m,j}$ of Δ_{KK} restricted to the $(-m)$-eigenspace of $-i\partial/\partial\theta$ are related to $\nu_{m,j}$ by $\lambda_{m,j} = \nu_{m,j} + m^2$, cf. [10, Section 6].

1.2. **Regularity of the Anosov splitting.** Theorem B can be used for the study of the regularity of the Anosov splitting of magnetic flows. In fact, in dimension two this problem is completely solved in the Riemannian setting in [7] and is one of the main motivations of this paper. Here we show:

Theorem C. *Let M be a closed connected manifold endowed with a Finsler metric F and let Ω be an exact 2-form. Suppose that the magnetic flow ϕ of the pair (F, Ω) is Anosov. If the Anosov splitting of ϕ is of class C^1, then Ω must vanish, i.e., the magnetic flow is a Finsler geodesic flow.*

Theorem C was proved in [16], when F is a Riemannian metric, using Aubry-Mather theory. The proof in [16] *cannot* be extended to include arbitrary (non-reversible) Finsler metrics, since it uses the invariance of the Riemannian metric under the flip $(x, v) \mapsto (x, -v)$.

1.3. **Sketch of the proof of Theorem B.** Perhaps the most important element in the proof is the Pestov identity in our setting. This comes in two flavours. We first obtain a scalar identity (cf. Lemma 3.1 in dimension two and Lemma 4.6 in arbitrary dimension). When this identity is manipulated and integrated with respect to the Liouville measure μ of SM it gives rise to our key integral identity:

$$\begin{aligned}(1)\quad \int_{SM} \big\{ &|\mathbf{X}(\nabla^{\cdot} u)|^2 - \langle \mathbf{R}_y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle - L(Y(y), \nabla^{\cdot} u, \nabla^{\cdot} u) - \langle \nabla^{\cdot}(\mathbf{X}u), Y(\nabla^{\cdot} u)\rangle \\ &- 2\langle Y(y), \nabla^{\cdot} u\rangle^2 + \langle \nabla^{\cdot} u, Y(\nabla^{\cdot} u)\rangle + \langle \nabla_{|(\nabla^{\cdot} u)} Y(y), \nabla^{\cdot} u\rangle \big\}\, d\mu \\ &= \int_{SM} \big\{ |\nabla^{\cdot}(\mathbf{X}u)|^2 - n(\mathbf{X}u)^2 \big\}\, d\mu.\end{aligned}$$

Of course, this formula needs explaining and we shall fully do so in Sections 3 and 4, but for the purpose of this sketch it suffices to note the following points:

(1) u is a smooth positively homogeneous function of degree zero on $TM \setminus \{0\}$;
(2) $\mathbf{X}$ is a suitable vector field on $TM \setminus \{0\}$ whose restriction to SM coincides with $\mathbf{G}_M$;
(3) the various derivatives that appear in the formula are all obtained using the Chern connection of the Finsler metric and are explained in detail in Section 4;
(4) $\nabla^{\cdot} u$ vanishes if and only if u is the pull back of a function on M;
(5) inner products and norms are all taken with respect to the fundamental tensor in Finsler geometry:

$$g_{ij}(x,y) = \frac{1}{2}[F^2]_{y^i y^j}(x,y);$$

(6) $\mathbf{R}$ and L are respectively the Riemann curvature operator and the Landsberg tensor from Finsler geometry; Y is the Lorentz force associated with the magnetic field;
(7) n is the dimension of M.

We may regard the identity as a kind of "dynamical Weitzenböck formula". Suppose now that $\mathbf{G}_M(u) = h \circ \pi + \theta$ and extend u to a positively homogeneous function of degree zero on $TM \setminus \{0\}$ (still denoted by u). Then $\mathbf{X}(u) = Fh \circ \pi + \theta$ and it is not hard to see (cf. Lemma 4.4) that the right hand side of (1) is non-positive and thus

$$\int_{SM} \big\{ |\mathbf{X}(\nabla^{\cdot} u)|^2 - \langle \mathbf{R}_y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle - L(Y(y), \nabla^{\cdot} u, \nabla^{\cdot} u) - \langle \nabla^{\cdot}(\mathbf{X}u), Y(\nabla^{\cdot} u)\rangle \\ - 2\langle Y(y), \nabla^{\cdot} u\rangle^2 + \langle \nabla^{\cdot} u, Y(\nabla^{\cdot} u)\rangle + \langle \nabla_{|(\nabla^{\cdot} u)} Y(y), \nabla^{\cdot} u\rangle \big\}\, d\mu \leq 0.$$

It is at this point that we need a new ingredient. We will note that the left hand side of the last inequality is closely related to an analogue of the classical index form in Riemannian geometry. Bilinear forms of this type already appeared in [18] and were very useful for the study of derivatives of topological entropy. This time the form that we need is a sharper version of the one that appears in [18]. The key point is that the Anosov property, via the abscence of conjugate points established in [17, 15] (see [4] for a proof using the asymptotic Maslov index), will imply that when we integrate the expression inside the brackets in the last inequality along every closed magnetic geodesic the outcome should be *non-negative* and *zero* if and only if $\nabla^{\cdot} u$ vanishes along every closed magnetic geodesic. When we combine this fact with the recent *non-negative Livšic theorem* [13, 21] we deduce that $\nabla^{\cdot} u$ must vanish over every closed magnetic geodesic and thus it must be identically zero on $TM \setminus \{0\}$. This means that $u = f \circ \pi$ where f is a smooth function on M. But in this case, since $d\pi_{(x,v)}(\mathbf{G}_M) = v$ we have $\mathbf{G}_M(u) = df_x(v)$ and Theorem B follows.

A considerable part of the paper will be devoted to the proof of the integral formula (1). This necessitates the language and formalism of Finsler geometry which makes the derivation of the formula a bit cumbersome. To help the reader, we have included a brief section in which we prove the integral formula in dimension two. This easier case still shows some of the main features and it can be read independently of the other sections.

2. Theorem B implies Theorem A

Let us explain why Theorem B implies Theorem A.

Suppose $\Sigma \subset T^*M$ is an optical hypersurface which encloses an open region U in T^*M. Let $\Sigma_x := \Sigma \cap T_x^*M$ which is a strictly convex hypersurface in the vector space T_x^*M which encloses $U_x := U \cap T_x^*M$. Consider an auxiliar smooth Riemannian metric g on $\tau : T^*M \to M$, that is, for each $x \in M$, g_x is an inner product in T_x^*M. For each $x \in M$, the inner product g_x gives rise to a volume form ϖ_x in T_x^*M. Consider the barycenter of U_x, i.e.,

$$\beta_x := \frac{\int_{U_x} p\,\varpi_x}{\int_{U_x} \varpi_x}.$$

The map $x \mapsto \beta_x$ can be seen as a smooth 1-form and by strict convexity $\beta_x \in U_x$ for all $x \in M$.

Consider the map $B : T^*M \to T^*M$ given by $B(x,p) = (x, p - \beta_x)$. It is easy to check that $B^*(\lambda) = \lambda - \tau^*\beta$ and that $B^*(\tau^*\Omega) = \tau^*\Omega$. Hence if we let $\widetilde{\Omega} := \Omega + d\beta$, B is a symplectomorphism between $(T^*M, -d\lambda + \tau^*\Omega)$ and $(T^*M, -d\lambda + \tau^*\widetilde{\Omega})$. Now set $\widetilde{\Sigma} := B(\Sigma)$ and observe that $\widetilde{\Sigma}$ is optical and contains the zero section of T^*M. Also note that

$$\int_\Gamma \tau^*\theta = 0$$

for all Γ of σ if and only if

$$\int_{\widetilde{\Gamma}} \tau^*\theta = 0$$

for all $\widetilde{\Gamma}$ of $\widetilde{\sigma}$. Thus, without loss of generality, we may assume that Σ contains the zero section of T^*M. But in that case we can define a Finsler metric F on M using homogeneity and declaring that Σ corresponds to the unit cosphere bundle of F. The hypothesis in Theorem A tells us that

$$\int_\gamma \theta = 0$$

for every closed magnetic geodesic γ of (F, Ω). The smooth Livšic theorem [12] and Theorem B imply that θ must be exact.

3. Proof of Theorem B for surfaces

3.1. **Canonical coframing.** Let M be a closed oriented connected surface. A smooth *Finsler structure* on M is a smooth hypersurface $SM \subset TM$ for which the canonical projection $\pi : SM \to M$ is a surjective submersion having the property that for each $x \in M$, the π-fibre $\pi^{-1}(x) = SM \cap T_xM$ is a smooth, closed, strictly convex curve enclosing the origin $0_x \in T_xM$.

Given such a structure it is possible to define a canonical coframing $(\omega_1, \omega_2, \omega_3)$ on SM that satisfies the following structural equations (see [3, Chapter 4]):

$$d\omega_1 = -\omega_2 \wedge \omega_3, \tag{2}$$

$$d\omega_2 = -\omega_3 \wedge (\omega_1 - I\omega_2), \tag{3}$$

$$d\omega_3 = -(K\omega_1 - J\omega_3) \wedge \omega_2. \tag{4}$$

where I, K and J are smooth functions on SM. The function I is called the *main scalar* of the structure. When the Finsler structure is Riemannain, K is the Gaussian curvature.

The form ω_1 is the canonical contact form of SM whose Reeb vector field is the geodesic vector field X. The volume form $\omega_1 \wedge d\omega_1$ gives rise to the Liouville measure $d\mu$ of SM.

Consider the vector fields (X, H, V) dual to $(\omega_1, \omega_2, \omega_3)$. As a consequence of (2–4) they satisfy the commutation relations

$$[V, X] = H, \quad [H, V] = X + IH + JV, \quad [X, H] = KV. \tag{5}$$

Below we will use the following general fact. Let N be a closed oriented manifold and Θ a volume form. Let X be a vector field on N and $f : N \to \mathbb{R}$ a smooth function. Then

$$\int_N X(f)\,\Theta = -\int_N f\,L_X\Theta, \tag{6}$$

where $L_X\Theta$ is the Lie derivative of Θ along X.

Now let $\Theta := \omega_1 \wedge \omega_2 \wedge \omega_3$. Using the commutation relations we obtain:

$$L_X\Theta = 0; \tag{7}$$

$$L_H\Theta = -J\,\Theta; \tag{8}$$

$$L_V\Theta = I\,\Theta. \tag{9}$$

3.2. **Identities.** Let Ω be a 2-form on M. An important observation is this: $\pi^*\Omega = \lambda\,\omega_1 \wedge \omega_2$, where $\lambda : SM \to \mathbb{R}$ is a function such that

$$V(\lambda) = -\lambda\,I.$$

This relation is obtained using the structure equations in $d(\lambda\,\omega_1 \wedge \omega_2) = 0$. The magnetic vector field is

$$\mathbf{G}_M = X + \lambda V.$$

The brackets are now:

(10)

$$[V, \mathbf{G}_M] = H - \lambda IV, \quad [H, V] = \mathbf{G}_M + IH + (J - \lambda)V, \quad [\mathbf{G}_M, H] = \mathbb{K}V - \lambda\mathbf{G}_M - \lambda IH,$$

where $\mathbb{K} := K - H(\lambda) + \lambda^2 - \lambda J$.

Using these brackets we obtain as in [7, Lemma 3.1]:

Lemma 3.1 (The Pestov identity)**.** *For every smooth function $u : SM \to \mathbb{R}$ we have*

$$2Hu \cdot V\mathbf{G}_M u = (\mathbf{G}_M u)^2 + (Hu)^2 - \mathbb{K}(Vu)^2 + \mathbf{G}_M(Hu \cdot Vu) - H(\mathbf{G}_M u \cdot Vu) + V(\mathbf{G}_M u \cdot Hu) + \mathbf{G}_M u \cdot (IHu + JVu).$$

We omit the proof which is (once you know the formula!) a straightforward verification using the bracket relations.

Integrating Pestov's identity over SM against the Liouville measure $d\mu$ and using (6) and (7–9) we obtain:

$$2\int_{SM} Hu \cdot V\mathbf{G}_M u \, d\mu = \int_{SM} (\mathbf{G}_M u)^2 \, d\mu + \int_{SM} (Hu)^2 \, d\mu - \int_{SM} \mathbb{K}(Vu)^2 \, d\mu. \tag{11}$$

By the commutation relations, we have

$$\mathbf{G}_M V u = V\mathbf{G}_M u - Hu + \lambda I V u.$$

Therefore,

$$(\mathbf{G}_M V u)^2 = (V\mathbf{G}_M u)^2 + (Hu)^2 + \lambda^2 I^2 (Vu)^2 - 2V\mathbf{G}_M u \cdot Hu + 2V\mathbf{G}_M u \cdot \lambda IVu - 2\lambda IVu \cdot Hu.$$

Thus:

$$(\mathbf{G}_M V u)^2 = (V\mathbf{G}_M u)^2 + (Hu)^2 + \lambda^2 I^2 (Vu)^2 - 2V\mathbf{G}_M u \cdot Hu + 2\mathbf{G}_M Vu \cdot \lambda IVu - 2\lambda^2 I^2 (Vu)^2.$$

Integrating this equation and

$$2\lambda IVu \cdot \mathbf{G}_M(Vu) = \mathbf{G}_M((Vu)^2 \lambda I) - (Vu)^2 \cdot \mathbf{G}_M(\lambda I)$$

and combining the outcomes with (11) we arrive at the final integral identity:

Theorem 3.2.

$$\int_{SM} (\mathbf{G}_M V u)^2 \, d\mu - \int_{SM} Q(Vu)^2 \, d\mu = \int_{SM} (V\mathbf{G}_M u)^2 \, d\mu - \int_{SM} (\mathbf{G}_M u)^2 \, d\mu, \tag{12}$$

where $Q := \mathbb{K} - \lambda^2 I^2 - \mathbf{G}_M(\lambda I)$.

When the Finsler metric is Riemannian (i.e. $I = J = 0$), the identity (12) is exactly identity (8) in [7].

If $\mathbf{G}_M u = h(x) + \theta_x(v)$, then one can see that the right-hand side of (12) is non-positive. Indeed, since $V\mathbf{G}_M(u) = V\theta$ we have:

$$\int_{SM} (V\mathbf{G}_M u)^2 \, d\mu - \int_{SM} (\mathbf{G}_M u)^2 \, d\mu = \int_{SM} (V\theta)^2 \, d\mu - \int_{SM} \theta^2 \, d\mu - 2\int_{SM} h\theta \, d\mu - \int_{SM} h^2 \, d\mu.$$

With a bit of work one can see that the linearity of θ in v implies:

$$\int_{SM} (V\theta)^2 \, d\mu = \int_{SM} \theta^2 \, d\mu,$$

$$\int_{SM} h\theta \, d\mu = 0.$$

This will follow from Lemma 4.4, which holds in any dimension.

3.3. **Jacobi equation.** For $\zeta \in T(SM)$ write

$$d\phi_t(\zeta) = x(t)\mathbf{G}_M + y(t)H + z(t)V,$$

where $x(t)$, $y(t)$ and $z(t)$ are smooth functions. Equivalently,

$$\zeta = x(t)d\phi_{-t}(\mathbf{G}_M) + y(t)d\phi_{-t}(H) + z(t)d\phi_{-t}(V).$$

If we differentiate the last equality with respect to t we obtain:

$$0 = \dot{x}\mathbf{G}_M + \dot{y}H + y[\mathbf{G}_M, H] + \dot{z}V + z[\mathbf{G}_M, V].$$

Using the bracket relations and regrouping we have:

$$0 = (\dot{x} - \lambda y)\mathbf{G}_M + (\dot{y} - z - \lambda I y)H + (\dot{z} + y\mathbb{K} + z\lambda I)V,$$

hence

$$\begin{aligned} \dot{x} &= \lambda y; \\ \dot{y} &= z + \lambda I y; \\ \dot{z} &= -\lambda I z - \mathbb{K}y. \end{aligned}$$

From these equations we get:

$$\ddot{y} + Qy = 0. \tag{13}$$

3.4. **Index form.**

Lemma 3.3. *If ϕ is Anosov, then for every closed magnetic geodesic $\gamma : [0, T] \to M$ and every smooth function $z : [0, T] \to \mathbb{R}$ such that $z(0) = z(T)$ and $\dot{z}(0) = \dot{z}(T)$ we have*

$$\mathbb{I} := \int_0^T \left\{ \dot{z}^2 - Qz^2 \right\} dt \geq 0$$

with equality if and only if $z \equiv 0$.

Using (13) the proof of this lemma is quite similar to the proof of Lemma 3.3 in [7]. The proof of the lemma in any dimension is given in Lemma 4.10. A key ingredient is the transversality of the weak stable (or unstable) bundle of ϕ with respect to the vertical distribution, which implies the abscence of conjugate points.

3.5. **End of the proof of Theorem B for surfaces.** Set $\psi := V(u)$. The last lemma, applied to the function $z = \psi(\gamma)$, yields

$$\int_\gamma \left\{ (\mathbf{G}_M\psi)^2 - Q\psi^2 \right\} dt \geq 0 \tag{14}$$

for every closed magnetic geodesic γ. Since the flow is Anosov, the invariant measures supported on closed orbits are dense in the space of all invariant measures on SM. Therefore, the above yields

$$\int_{SM} \left\{ (\mathbf{G}_M\psi)^2 - Q\psi^2 \right\} d\mu \geq 0.$$

Combining this with the fact that the right hand side of (12) is non-positive, we find that

$$\int_{SM} \left\{ (\mathbf{G}_M \psi)^2 - Q\psi^2 \right\} \, d\mu = 0. \tag{15}$$

By the non-negative version of the Livšic theorem, proved independently by M. Pollicott and R. Sharp and by A. Lopes and P. Thieullen (see [13, 21]), we conclude from (14) and (15) that

$$\int_\gamma \left\{ (\mathbf{G}_M \psi)^2 - Q\psi^2 \right\} \, dt = 0$$

for every closed magnetic geodesic γ. Applying again Lemma 3.3, we see that ψ vanishes on all closed magnetic geodesics. Since the latter are dense in SM, the function ψ vanishes on all of SM, as required.

4. Proof of Theorem B

4.1. **Differential identities of Finsler geometry.** Henceforth M is a closed n-dimensional manifold and F is a Finsler metric on M.

Let $\pi : TM \setminus \{0\} \to M$ be the natural projection, and let $\beta^r_s M := \pi^* \tau^r_s M$ denote the bundle of semibasic tensors of degree (r, s), where $\tau^r_s M$ is the bundle of tensors of degree (r, s) over M. Sections of the bundles $\beta^r_s M$ are called semibasic tensor fields and the space of all smooth sections is denoted by $C^\infty(\beta^r_s M)$. For such a field T, the coordinate representation

$$T = (T^{i_1 \dots i_r}_{j_1 \dots j_s})(x, y)$$

holds in the domain of a standard local coordinate system (x^i, y^i) on $TM \setminus \{0\}$ associated with a local coordinate system (x^i) in M. Under a change of a local coordinate system, the components of a semibasic tensor field are transformed by the same formula as those of an ordinary tensor field on M.

Every "ordinary" tensor field on M defines a semibasic tensor field by the rule $T \mapsto T \circ \pi$, so that the space of tensor fields on M can be treated as embedded in the space of semibasic tensor fields.

Let (g_{ij}) be the fundamental tensor,

$$g_{ij}(x, y) = \frac{1}{2} [F^2]_{y^i y^j}(x, y),$$

and let (g^{ij}) be the contravariant fundamental tensor,

$$g_{ik} g^{kj} = \delta^j_i. \tag{16}$$

In the usual way, the fundamental tensor defines the inner product $\langle \cdot, \cdot \rangle$ on $\beta^1_0 M$, and we put $|U|^2 = \langle U, U \rangle$.

Let

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$$

be the spray induced by F. Here G^i are the geodesic coefficients [24, (5.7)],

$$G^i(x,y) = \frac{1}{4} g^{il} \left\{ 2\frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\} y^j y^k.$$

Let

$$T(TM \setminus \{0\}) = \mathcal{H}TM \oplus \mathcal{V}TM$$

be the decomposition of $T(TM \setminus \{0\})$ into horizontal and vertical vectors. Here

$$\mathcal{H}TM = \operatorname{span}\left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{V}TM = \operatorname{span}\left\{ \frac{\partial}{\partial y^i} \right\},$$

with

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$$

and

$$N_j^i = \frac{\partial G^i}{\partial y^j}.$$

Let

$$\nabla : C^\infty(T(TM)) \times C^\infty(\pi^* TM) \to C^\infty(\pi^* TM)$$

be the Chern connection,

$$\nabla_{\hat{X}} U = \left\{ dU^i(\hat{X}) + U^j \omega_j^i(\hat{X}) \right\} \frac{\partial}{\partial x^i},$$

where

$$\omega_j^i = \Gamma^i_{jk} dx^k$$

are the connection forms. Recall that

$$N_j^i = \Gamma^i_{jk} y^k. \tag{17}$$

Given a function $u \in C^\infty(TM \setminus \{0\})$, we put

$$u_{|k} := \frac{\delta u}{\delta x^k}, \quad u_{\cdot k} := \frac{\partial u}{\partial y^k}$$

and, given a semibasic vector field $U = (U^i) \in C^\infty(\beta_0^1 M)$, put

$$U^i_{|k} := \left(\nabla_{\frac{\delta}{\delta x^k}} U \right)^i, \quad U^i_{\cdot k} := \left(\nabla_{\frac{\partial}{\partial y^k}} U \right)^i.$$

We have

$$u_{|k} = \frac{\partial u}{\partial x^k} - \Gamma^p_{kq} y^q \frac{\partial u}{\partial y^p}, \quad u_{\cdot k} = \frac{\partial u}{\partial y^k},$$

and

$$U^i_{|k} = \frac{\partial U^i}{\partial x^k} - \Gamma^p_{kq} y^q \frac{\partial U^i}{\partial y^p} + \Gamma^i_{kp} U^p, \quad U^i_{\cdot k} = \frac{\partial U^i}{\partial y^k}.$$

In the usual way, we extend these formulas to higher order tensors:

$$T^{i_1\dots i_r}_{j_1\dots j_s|k} = \frac{\partial}{\partial x^k} T^{i_1\dots i_r}_{j_1\dots j_s} - \Gamma^p_{kq} y^q \frac{\partial}{\partial y^p} T^{i_1\dots i_r}_{j_1\dots j_s} + \sum_{m=1}^{r} \Gamma^{i_m}_{kp} T^{i_1\dots i_{m-1} p i_{m+1}\dots i_r}_{j_1\dots j_s} - \sum_{m=1}^{s} \Gamma^{p}_{kj_m} T^{i_1\dots i_r}_{j_1\dots j_{m-1} p j_{m+1}\dots j_s}$$

and

$$T^{i_1\dots i_r}_{j_1\dots j_s \cdot k} = \frac{\partial}{\partial y^k} T^{i_1\dots i_r}_{j_1\dots j_s}.$$

We define the operators

$$\nabla_| : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M), \quad \nabla_. : C^\infty(\beta^r_s M) \to C^\infty(\beta^r_{s+1} M)$$

by

$$(\nabla_| T)^{i_1\dots i_r}_{j_1\dots j_s k} = \nabla_{|k} T^{i_1\dots i_r}_{j_1\dots j_s} := T^{i_1\dots i_r}_{j_1\dots j_s|k}$$

and

$$(\nabla_. T)^{i_1\dots i_r}_{j_1\dots j_s k} = \nabla_{\cdot k} T^{i_1\dots i_r}_{j_1\dots j_s} = T^{i_1\dots i_r}_{j_1\dots j_s \cdot k}.$$

For convenience, we also define $\nabla^|$ and $\nabla^\cdot$ by

$$\nabla^{|i} = g^{ij} \nabla_{|j}, \quad \nabla^{\cdot i} = g^{ij} \nabla_{\cdot j}.$$

In the case of Riemannian manifolds, the above operators were denoted in [20, 22] by $\overset{h}{\nabla}$ and $\overset{v}{\nabla}$ respectively.

Given a function $u \in C^\infty(TM \setminus \{0\})$, note that $\nabla^\cdot u = 0$ if and only if u does not depend on y.

Equivalently, the above can be described as follows. In a natural way, the connection ∇ on $\beta^1_0 M = \pi^* TM$ defines a connection on the dual bundle $\beta^0_1 = \pi^* T^* M$, as well as connections on the tensor product bundles $\beta^r_s M$ for all r and s. Then for $T \in C^\infty(\beta^r_s M)$ we have

$$\nabla_{|k} T = \nabla_{\frac{\delta}{\delta x^k}} T, \quad \nabla_{\cdot k} T = \nabla_{\frac{\partial}{\partial y^k}} T.$$

This shows also that $\nabla_|$ and $\nabla_.$ are compatible with tensor products and contractions.

Note that

$$g_{ij\cdot k} = 2C_{ijk}, \quad g^{ij}_{\cdot k} = -2 g^{il} g^{jm} C_{lmk},$$

where

$$C_{ijk} = \frac{1}{4} [F^2]_{y^i y^j y^k}$$

is the Cartan tensor of F.

Also, note that the fundamental tensor is parallel with respect to $\nabla_|$:

$$g_{ij|k} = 0 \quad g^{ij}_{|k} = 0. \tag{18}$$

Indeed, using (5.29) of [24], we see that

$$g_{ij|k} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma^p_{kq} y^q \frac{\partial g_{ij}}{\partial y^p} - \Gamma^p_{ki} g_{pj} - \Gamma^p_{kj} g_{ip} = 2C_{ipj} N^p_k - 2\Gamma^p_{kq} y^q C_{ijp} = 0,$$

while the second identity is obtained by differentiating (16).

By [24, Lemma 5.2.1]

$$F_{|k} = 0.$$

On the other hand, for $(x, y) \in SM$

$$F_{\cdot k} = y_k = g_{kj} y^j. \tag{19}$$

Indeed, using homogeneity we have

$$F F_{y^k} = \frac{1}{2}[F^2]_{y^k} = \frac{1}{2}[F^2]_{y^k y^j} y^j = g_{kj} y^j.$$

However, $F = 1$ on SM, which gives (19).

A straightforward computation shows also that

$$y^i_{|k} = 0, \quad y^i_{\cdot k} = \delta^i_j.$$

Let P denote the Chern curvature tensor and R denote the Riemann curvature tensor (see [24, (8.12), (8.13)]):

$$P^i_{jkl} = -\frac{\partial \Gamma^i_{jk}}{\partial y^l},$$

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \frac{\partial \Gamma^i_{jk}}{\partial y^m} N^m_l - \frac{\partial \Gamma^i_{jl}}{\partial y^m} N^m_k + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml},$$

and put (see [24, p. 127])

$$P^i_{kl} = y^j P^i_{jkl},$$
$$R^i_{kl} = y^j R^i_{jkl}.$$

Note that

$$R^i_k = R^i_{kl} y^l$$

corresponds to the Riemann curvature operator

$$\mathbf{R}_y(V) = (R^i_k V^k),$$

while

$$y^k P^i_{kl} = 0. \tag{20}$$

Lemma 4.1. *If $u \in C^\infty(TM \setminus \{0\})$, then*

$$u_{\cdot l \cdot k} - u_{\cdot k \cdot l} = 0, \tag{21}$$

$$u_{|l \cdot k} - u_{\cdot k | l} = P^i_{lk} u_{\cdot i}, \tag{22}$$

$$u_{|l|k} - u_{|k|l} = R^i_{lk} u_{\cdot i}. \tag{23}$$

Proof. (21) is trivial.

Next,

$$u_{|l\cdot k} = \frac{\partial}{\partial y^k}\left(\frac{\partial u}{\partial x^l} - \Gamma^i_{lj} y^j \frac{\partial u}{\partial y^i}\right) = \frac{\partial^2 u}{\partial y^k \partial x^l} - \frac{\partial \Gamma^i_{lj}}{\partial y^k} y^j \frac{\partial u}{\partial y^i} - \Gamma^i_{lk}\frac{\partial u}{\partial y^i} - \Gamma^i_{lj} y^j \frac{\partial^2 u}{\partial y^k \partial y^i},$$

whereas

$$u_{\cdot k|l} = \left(\frac{\partial}{\partial x^l} - \Gamma^i_{lj} y^j \frac{\partial}{\partial y^i}\right) u_{\cdot k} - \Gamma^i_{lk} u_{\cdot i} = \frac{\partial^2 u}{\partial x^l \partial y^k} - \Gamma^i_{lj} y^j \frac{\partial^2 u}{\partial y^i \partial y^k} - \Gamma^i_{lk} \frac{\partial u}{\partial y^i}.$$

Taking the difference, we come to (22).

Further,

$$\begin{aligned}
u_{|l|k} &= \left(\frac{\partial}{\partial x^k} - \Gamma^m_{ks} y^s \frac{\partial}{\partial y^m}\right) u_{|l} - \Gamma^m_{kl} u_{|m} \\
&= \left(\frac{\partial}{\partial x^k} - \Gamma^m_{ks} y^s \frac{\partial}{\partial y^m}\right)\left(\frac{\partial u}{\partial x^l} - \Gamma^i_{lj} y^j \frac{\partial u}{\partial y^i}\right) - \Gamma^m_{kl}\left(\frac{\partial u}{\partial x^m} - \Gamma^i_{mj} y^j \frac{\partial u}{\partial y^i}\right) \\
&= \frac{\partial^2 u}{\partial x^k \partial x^l} - \Gamma^m_{ks} y^s \frac{\partial^2 u}{\partial y^m \partial x^l} - \frac{\partial \Gamma^i_{lj}}{\partial x^k} y^j \frac{\partial u}{\partial y^i} - \Gamma^i_{lj} y^j \frac{\partial^2 u}{\partial x^k \partial y^i} \\
&\quad + \Gamma^m_{ks} y^s \frac{\partial \Gamma^i_{lj}}{\partial y^m} y^j \frac{\partial u}{\partial y^i} + \Gamma^m_{ks} y^s \Gamma^i_{lm} \frac{\partial u}{\partial y^i} + \Gamma^m_{ks} y^s \Gamma^i_{lj} y^j \frac{\partial^2 u}{\partial y^m \partial y^i} - \Gamma^m_{kl} \frac{\partial u}{\partial x^m} + \Gamma^m_{kl} \Gamma^i_{mj} y^j \frac{\partial u}{\partial y^i}.
\end{aligned}$$

Using (17), rearranging, and appropriately renaming indices, we obtain

$$\begin{aligned}
u_{|l|k} = \frac{\partial^2 u}{\partial x^k \partial x^l} - N^m_k \frac{\partial^2 u}{\partial y^m \partial x^l} - N^i_l \frac{\partial^2 u}{\partial x^k \partial y^i} + N^m_k N^i_l \frac{\partial^2 u}{\partial y^m \partial y^i} - \Gamma^m_{kl} \frac{\partial u}{\partial x^m} \\
- \left(\frac{\partial \Gamma^i_{lj}}{\partial x^k} - \frac{\partial \Gamma^i_{lj}}{\partial y^m} N^m_k - \Gamma^m_{kj} \Gamma^i_{lm} - \Gamma^m_{kl} \Gamma^i_{mj}\right) y^j \frac{\partial u}{\partial y^i}.
\end{aligned}$$

Alternating with respect to k and l, we come to (23). □

4.2. **Integral identities of Finsler geometry.** We will derive the Gauss–Ostrogradskiĭ formulas for vertical and horizontal divergences like those for Riemannian manifolds in [22, Section 3.6]. We proceed along the lines of [22].

Given a vector field $U = (U^i) \in C^\infty(\beta^1_0 M)$, the vertical divergence and the horizontal divergence are defined by

$$\overset{v}{\operatorname{div}}\, U = U^i_{\cdot i}, \quad \overset{h}{\operatorname{div}}\, U = U^i_{|i}.$$

Let

$$\mathbf{I}(U) = g^{ij} C_{ijk} U^k$$

be the mean Cartan torsion [24, p. 108], and let

$$\mathbf{J}(U) = g^{ij} L_{kij} U^k$$

be the mean Landsberg curvature [24, p. 116]). Here L is the Landsberg tensor, related to the Chern curvature tensor as follows [24, (8.27)]:

$$L_{ijk} = -g_{im} P^m_{jk}. \tag{24}$$

Let

$$dV^{2n} = \det(g_{ij})\, dx^1 \dots dx^n dy^1 \dots dy^n$$

be the Liouville volume form on $TM \setminus \{0\}$.

Consider the following set of local forms on $TM \setminus \{0\}$

$$\overset{v}{\omega}_k = (-1)^{n+k-1} g\, dx \wedge dy^1 \wedge \cdots \wedge \widehat{dy^k} \wedge \cdots \wedge dy^n,$$

$$\begin{aligned}\overset{h}{\omega}_k = g\big[(-1)^{k-1}\, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n \wedge dy \\ + \sum_{j=1}^{n} (-1)^{n+j} \Gamma^j_{kl} y^l\, dx \wedge dy^1 \wedge \cdots \wedge \widehat{dy^j} \wedge \cdots \wedge dy^n\big],\end{aligned}$$

where $g = \det(g_{ij})$, $dx = dx^1 \wedge \cdots \wedge dx^n$, $dy = dy^1 \wedge \cdots \wedge dy^n$, and the symbol $\widehat{}$ over a factor means that this factor is omitted.

Lemma 4.2. *Given a semibasic vector field $U = (U^k)$, the set of local forms $U^k \overset{v}{\omega}_k$ defines a global differential form on $TM \setminus \{0\}$. Similarly, the set of local forms $U^k \overset{h}{\omega}_k$ defines a global differential form on $TM \setminus \{0\}$. Moreover,*

$$d(U^k \overset{v}{\omega}_k) = (\overset{v}{\operatorname{div}}\, U + 2\mathbf{I}(U))\, dV^{2n}, \tag{25}$$

$$d(U^k \overset{h}{\omega}_k) = (\overset{h}{\operatorname{div}}\, U - \mathbf{J}(U))\, dV^{2n}. \tag{26}$$

Proof.

$$d\overset{v}{\omega}_k = \frac{\partial g}{\partial y^k}\, dx \wedge dy = g^{ij} \frac{\partial g_{ij}}{\partial y^k} g\, dx \wedge dy = 2 g^{ij} C_{ijk}\, dV^{2n}.$$

Therefore,

$$d(U^k \overset{v}{\omega}_k) = \frac{\partial U^k}{\partial y^k} g\, dx \wedge dy + 2 U^k g^{ij} C_{ijk}\, dV^{2n},$$

which coincides with (25).

Next,

$$\begin{aligned} d\overset{h}{\omega}_k &= \frac{\partial g}{\partial x^k}\, dx \wedge dy - \left(\frac{\partial g}{\partial y^j} \Gamma^j_{kl} y^l + g \frac{\partial \Gamma^j_{kl}}{\partial y^j} y^l + g \Gamma^j_{kj} \right) dx \wedge dy \\ &= \left(g^{ij} \frac{\partial g_{ij}}{\partial x^k} - g^{km} \frac{\partial g_{km}}{\partial y^j} \Gamma^j_{kl} y^l - \frac{\partial \Gamma^j_{kl}}{\partial y^j} y^l - \Gamma^j_{kj} \right) g\, dx \wedge dy \\ &= \left(g^{ij} \frac{\partial g_{ij}}{\partial x^k} - 2 g^{km} C_{kmj} N^j_k - \Gamma^j_{kj} + P^j_{kj} \right) dV^{2n} = (\Gamma^j_{kj} + P^j_{kj})\, dV^{2n}. \end{aligned}$$

Here we have used the equality [24, (5.29)]

$$\frac{\partial g_{jl}}{\partial x^m} = g_{kl} \Gamma^k_{jm} + g_{kj} \Gamma^k_{lm} + 2 C_{jkl} N^k_m.$$

Consequently,

$$\begin{aligned} d(U^k \overset{h}{\omega}_k) &= \frac{\partial U^k}{\partial x^k} g\, dx \wedge dy - \frac{\partial U^k}{\partial y^j} g \Gamma^j_{kl} y^l\, dx \wedge dy + U^k (\Gamma^j_{kj} + P^j_{kj})\, dV^{2n} \\ &= \left\{ \left(\frac{\partial U^k}{\partial x^k} - \Gamma^j_{kl} y^l \frac{\partial U^k}{\partial y^j} + \Gamma^j_{kj} U^k \right) + P^j_{kj} U^k \right\} dV^{2n}, \end{aligned}$$

which coincides with (26) in view of (24) and the symmetry of the Landsberg tensor. $\square$

Let $SM = \{(x, y) \in TM \mid F(y) = 1\}$ be the unit sphere bundle. The restriction of the form $y^k \overset{v}{\omega}_k$ to SM gives rise to the Liouville measure $d\mu$ of SM.

Theorem 4.3. *Let $U \in C^\infty(\beta^1_0 M)$ be a semibasic vector field positively homogeneous of degree λ in y. Then the following Gauss–Ostrogradskiĭ formulas hold:*

$$\int_{SM} \overset{v}{\operatorname{div}}\, U \, d\mu = \int_{SM} ((\lambda + n - 1)\langle U, y\rangle - 2\mathbf{I}(U))\, d\mu, \tag{27}$$

$$\int_{SM} \overset{h}{\operatorname{div}}\, U \, d\mu = \int_{SM} \mathbf{J}(U)\, d\mu. \tag{28}$$

These formulas follow easily from (25)–(26) by integration.

Lemma 4.4. (1) *Let $\psi \in C^\infty(TM)$ be a function which depends linearly on y. Then*

$$\int_{SM} \psi \, d\mu = 0.$$

(2) *Let $\phi \in C^\infty(TM \setminus \{0\})$ be such that $\phi = \varphi_0 F + \psi$, where φ_0 is independent of y while ψ depends linearly on y. Then*

$$\int_{SM} |\nabla^{\cdot} \phi|^2 \, d\mu = \int_{SM} (\varphi_0^2 + n\psi^2)\, d\mu.$$

Proof. To prove (1) let $\psi = \Psi_k y^k$, where Ψ is a covector field on M. Put $U^i = g^{ij}\Psi_j$ and apply (27) to get

$$(n-1)\int_{SM} \psi \, d\mu = (n-1)\int_{SM} \langle U, y\rangle \, d\mu = \int_{SM} (\overset{v}{\operatorname{div}}\, U + 2\mathbf{I}(U))\, d\mu.$$

Now,

$$\overset{v}{\operatorname{div}}\, U = (g^{ij}\Psi_j)_{\cdot i} = g^{ij}_{\cdot i}\Psi_j + g^{ij}\Psi_{j\cdot i} = -2g^{il}g^{jm}C_{lmi}\Psi_j = -2\mathbf{I}(U),$$

which implies (1).

To prove (2) note that since $\nabla_{\cdot}\phi = \varphi_0 \nabla_{\cdot} F + \nabla_{\cdot}\psi$, we have

$$|\nabla_{\cdot}\phi|^2 = \varphi_0^2 |\nabla^{\cdot} F|^2 + 2\varphi_0 \langle \nabla_{\cdot} F, \nabla_{\cdot}\psi\rangle + |\nabla^{\cdot}\psi|^2.$$

Next,

$$\begin{aligned} |\nabla^{\cdot}\psi|^2 = g^{ij}\psi_{\cdot i}\psi_{\cdot j} &= (\psi g^{ij}\psi_{\cdot i})_{\cdot j} - \psi g^{ij}_{\cdot j}\psi_{\cdot i} - \psi g^{ij}\psi_{\cdot i \cdot j} \\ &= \overset{v}{\operatorname{div}}\,(\psi \nabla^{\cdot}\psi) + 2\mathbf{I}(\psi\nabla^{\cdot}\psi), \end{aligned}$$

because $\psi_{\cdot i \cdot j} = 0$.

Thus, on SM we get

$$|\nabla^{\cdot}\phi|^2 = \varphi_0^2 + 2\varphi_0\psi + \overset{v}{\operatorname{div}}\,(\psi\nabla^{\cdot}\psi) + 2\mathbf{I}(\psi\nabla^{\cdot}\psi).$$

Integrating and using (27), we receive

$$\int_{SM} |\nabla^{\cdot}\phi|^2 \, d\mu = \int_{SM} (\varphi_0^2 + 2\varphi_0\psi + n\psi\langle\nabla^{\cdot}\psi, y\rangle)\, d\mu = \int_{SM} (\varphi_0^2 + 2\varphi_0\psi + n\psi^2)\, d\mu.$$

Since by (1)

$$\int_{SM} \varphi_0 \psi \, d\mu = 0$$

the proof of (2) is complete.

□

4.3. **Identities for the magnetic flow.** Let

$$\{dx^i,\ \delta y^j = dy^j + N^j_k \, dx^k\}$$

be a local basis for $T^*(TM \setminus \{0\})$ dual to the local basis $\left\{\frac{\delta}{\delta x^i},\ \frac{\partial}{\partial y^j}\right\}$ for $T(TM \setminus \{0\})$. The Legendre transform $\ell_F : TM \setminus \{0\} \to T^*M \setminus \{0\}$ associated with the Lagrangian $\frac{1}{2}F^2$ is a diffeomorphism and $\omega_0 := \ell_F^*(-d\lambda)$ defines a symplectic form on $TM \setminus \{0\}$, where λ is the Liouville 1-form on T^*M. In local coordinates (x, y), ℓ_F is simply the map

$$(y^j) \mapsto (y_j).$$

The canonical 1-form is $\lambda = y_i dx^i$ and $\ell_F^* \lambda = g_{ij} y^j dx^i$. From this, a calculation shows that

$$\omega_0 = g_{ij} dx^i \wedge \delta y^j.$$

Let $H : TM \setminus \{0\} \to \mathbb{R}$ be defined by

$$H = \frac{1}{2}F^2.$$

The Hamiltonian flow of H with respect to ω_0
gives rise to the geodesic flow of the Finsler manifold (M, F).

Let Ω be a closed 2-form on M and consider the new symplectic form ω defined as

$$\omega_0 + \pi^* \Omega.$$

The Hamiltonian flow of H with respect to $\omega_0 + \pi^*\Omega$ gives rise to a flow $\phi_t : TM \setminus \{0\} \to TM \setminus \{0\}$, called *magnetic flow* or *twisted geodesic flow*.

The form Ω, regarded as an antisymmetric tensor field $(\Omega_{ij}) \in C^\infty(\tau^0_2 M)$, gives rise to a corresponding semibasic tensor field. We define the *Lorentz force* $Y \in C^\infty(\beta^1_1 M)$ by

$$Y^i_j(x, y) = \Omega_{jk}(x) g^{ik}(x, y). \tag{29}$$

We also define

$$Y(U) = (Y^i_j U^j).$$

Note that Y is skew symmetric with respect to g:

$$\langle Y(U), V \rangle = -\langle U, Y(V) \rangle.$$

Let $\mathbf{G}_M$ be the generator of the magnetic flow. Straightforward calculations show that

$$\mathbf{G}_M(x, y) = y^i \frac{\delta}{\delta x^i} + y^i Y^j_i \frac{\partial}{\partial y^j}. \tag{30}$$

It is easily seen that every integral curve of $\mathbf{G}_M$ is a curve of the form $t \mapsto \dot\gamma(t) \in TM$ which satisfies the equation

$$D_{\dot\gamma}\dot\gamma = Y_{\dot\gamma(t)}(\dot\gamma),$$

where the covariant derivative D is the one determined by the Chern connection. Alternatively we could write:

$$\ddot\gamma^i(t) + \Gamma^i_{jk}(\dot\gamma(t))\dot\gamma^j(t)\dot\gamma^k(t) = Y^i_j(\dot\gamma(t))\dot\gamma^j(t).$$

A curve γ, satisfying this equation, is referred to as a magnetic geodesic.

If $u \in C^\infty(TM \setminus \{0\})$, then by (30)

$$\mathbf{G}_M u(x,y) = y^i\left(\frac{\delta u}{\delta x^i} + Y^j_i\frac{\partial u}{\partial y^j}\right) = y^i(u_{|i} + Y^j_i u_{\cdot j}).$$

Since the Hamiltonian flow ϕ_t preserves the level sets of H, the magnetic flow preserves SM and the vector field $\mathbf{G}_M$ is tangent to SM.

Suppose that for a smooth function $u : SM \to \mathbb{R}$ we have

$$\mathbf{G}_M u = \varphi.$$

Extend u to a positively homogeneous function (of degree 0) on $TM \setminus \{0\}$, denoting the extension by u again.

For $(x,y) \in TM$, define

$$\mathbf{X}u = y^i(u_{|i} + FY^j_i u_{\cdot j}).$$

Then on $TM \setminus \{0\}$ we have

$$\mathbf{X}u = \phi,$$

where ϕ is the positively homogeneous extension of φ to $TM \setminus \{0\}$ of degree 1.

Given $T = (T^{i_1\dots i_r}_{j_1\dots j_s}) \in C^\infty(\beta^r_s M)$, put

$$T^{i_1\dots i_r}_{j_1\dots j_s:k} = T^{i_1\dots i_r}_{j_1\dots j_s|k} + FY^j_k T^{i_1\dots i_r}_{j_1\dots j_s\cdot j}.$$

Straightforward calculations show that for $(x,y) \in SM$

(31)
$$\begin{aligned} g_{ij:k} &= 2Y^s_k C_{ijs},\\ g^{ij}_{:k} &= -2Y^s_k g^{il}g^{jm}C_{lms},\\ y^i_{:k} &= Y^i_k. \end{aligned}$$

It is also useful to note that differentiating (29) yields

(32)
$$Y^i_{j\cdot k} = -2Y^m_j g^{il}C_{lmk} = g^{is}_{:j}g_{sk}.$$

Lemma 4.5. *If $u \in C^\infty(TM \setminus \{0\})$, then for $(x,y) \in SM$ we have*

(33)
$$u_{:l\cdot k} - u_{\cdot k:l} = \tilde P^i_{lk}u_{\cdot i},$$

(34)
$$u_{:l:k} - u_{:k:l} = \tilde R^i_{lk}u_{\cdot i},$$

with

$$\tilde{P}^i_{lk} = P^i_{lk} + Y^i_l y_k + Y^i_{l\cdot k},$$
$$\tilde{R}^i_{lk} = R^i_{lk} + (Y^i_{l|k} - Y^i_{k|l}) - (P^i_{lm}Y^m_k - P^i_{km}Y^m_l) + (Y^j_l Y^i_{k\cdot j} - Y^j_k Y^i_{l\cdot j}) + y_s(Y^s_k Y^i_l - Y^s_l Y^i_k).$$

Proof. We have

$$u_{:l\cdot k} = (u_{|l} + FY^i_l u_{\cdot i})_{\cdot k} = u_{|l\cdot k} + F_{\cdot k}Y^i_l u_{\cdot i} + FY^i_{l\cdot k}u_{\cdot i} + FY^i_l u_{\cdot i\cdot k}$$

whereas

$$u_{\cdot k:l} = u_{\cdot k|l} + FY^i_l u_{\cdot k\cdot i}.$$

Thus, for $(x,y) \in SM$

$$u_{:l\cdot k} - u_{\cdot k:l} = (u_{|l\cdot k} - u_{\cdot k|l}) + y_k Y^i_l u_{\cdot i} + Y^i_{l\cdot k}u_{\cdot i}.$$

Using (22), we come to (33).

Further,

$$u_{:l:k} = u_{:l|k} + FY^j_k u_{:l\cdot j} = (u_{|l} + FY^j_l u_{\cdot j})_{|k} + FY^j_k(u_{|l} + FY^s_l u_{\cdot s})_{\cdot j}$$
$$= u_{|l|k} + FY^j_{l|k}u_{\cdot j} + FY^j_l u_{\cdot j|k} + FY^j_k u_{|l\cdot j} + FY^j_k F_{\cdot j}Y^s_l u_{\cdot s} + F^2 Y^j_k Y^s_{l\cdot j}u_{\cdot s} + F^2 Y^j_k Y^s_l u_{\cdot s\cdot j}.$$

Thus, for $(x,y) \in SM$

$$u_{:l:k} - u_{:k:l} = (u_{|l|k} - u_{|k|l}) + (Y^j_{l|k} - Y^j_{k|l})u_{\cdot j} + Y^j_l(u_{\cdot j|k} - u_{|k\cdot j}) + Y^j_k(u_{|l\cdot j} - u_{\cdot j|l}) + (Y^j_k Y^s_l - Y^j_l Y^s_k)y_j u_{\cdot s} + (Y^j_k Y^s_{l\cdot j} - Y^j_l Y^s_{k\cdot j})u_{\cdot s} + (Y^j_k Y^s_l - Y^j_l Y^s_k)u_{\cdot s\cdot j}.$$

Using (23), (22) and renaming indices, we come to (34). □

Given $U \in C^\infty(\beta^1_0 M)$ and $u \in C^\infty(TM \setminus \{0\})$, define

$$\overset{m}{\operatorname{div}}\, U = U^i_{:i}, \quad \nabla^{:} u = (u^{:i}) = (g^{ij}u_{:j}).$$

Lemma 4.6. *The following holds on SM* (The Pestov identity):

$$2\langle \nabla^{:} u, \nabla^{\cdot}(\mathbf{X}u)\rangle = |\nabla^{:} u|^2 + \mathbf{X}(\langle \nabla^{:} u, \nabla^{\cdot} u\rangle) - \overset{m}{\operatorname{div}}\,((\mathbf{X}u)\nabla^{\cdot} u) + \overset{v}{\operatorname{div}}\,((\mathbf{X}u)\nabla^{:} u) - \langle \tilde{\mathbf{R}}_y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle + \langle Y(\nabla^{\cdot} u), \nabla^{:} u\rangle + 2\mathbf{I}((\mathbf{X}u)\nabla^{:} u) + \mathbf{J}((\mathbf{X}u)\nabla^{\cdot} u). \tag{35}$$

Proof. With the above notations, we can write

$$\mathbf{X}u = y^i u_{:i}.$$

Therefore,

$$2\langle \nabla_{\cdot}(\mathbf{X}u), \nabla_{:} u\rangle - \overset{v}{\operatorname{div}}\,((\mathbf{X}u)\nabla^{:} u) = 2g^{ij}(\mathbf{X}u)_{\cdot i}u_{:j} - ((\mathbf{X}u)g^{ij}u_{:j})_{\cdot_i} = g^{ij}(\mathbf{X}u)_{\cdot_i}u_{:j} - (\mathbf{X}u)g^{ij}_{\cdot_i}u_{:j} - (\mathbf{X}u)g^{ij}u_{:j\cdot i} = I - II - III. \tag{36}$$

We rewrite the first term on the right-hand side of (36) as follows:

$$\begin{aligned} I &= g^{ij}(y^k u_{:k})_{\cdot i} u_{:j} = g^{ij}(u_{:i} + y^k u_{:k\cdot i})u_{:j} \\ &= g^{ij}u_{:i}u_{:j} + g^{ij}y^k(u_{\cdot i:k} + (u_{:k\cdot i} - u_{\cdot i:k}))u_{:j} \\ &= |\nabla^{\cdot} u|^2 + y^k(g^{ij}u_{\cdot i}u_{:_j})_{:k} - y^k g^{ij}_{:k} u_{\cdot i}u_{:j} - y^k g^{ij}u_{\cdot i}u_{:j:k} + g^{ij}y^k \tilde{P}^m_{ki} u_{\cdot m}u_{:j}. \end{aligned}$$

Note that

$$y^k(g^{ij}u_{\cdot i}u_{:_j})_{:k} = \mathbf{X}(\langle \nabla^{\cdot} u, \nabla^{:} u\rangle),$$

that

$$\begin{aligned} g^{ij}y^k\tilde{P}^m_{ki}u_{\cdot m}u_{:j} &= g^{ij}y^k(P^m_{ki} + Y^m_k y_i + Y^m_{k\cdot i})u_{\cdot m}u_{:j} \\ &= \langle Y(y), \nabla^{\cdot} u\rangle \mathbf{X}u + y^k g^{mj}_{:k} u_{\cdot m}u_{:j} \end{aligned}$$

where we have used (20) and (32), and that

$$\begin{aligned} y^k g^{ij}u_{\cdot i}u_{:j:k} &= y^k g^{ij}u_{\cdot i}(u_{:k:j} + (u_{:j:k} - u_{:k:j})) \\ &= g^{ij}u_{\cdot_i}(y^k u_{:_k})_{:j} - g^{ij}u_{\cdot i}y^k_{:j}u_{:k} + y^k g^{ij}u_{\cdot i}\tilde{R}^m_{jk}u_{\cdot m} \\ &= \langle \nabla^{\cdot} u, \nabla^{:}(\mathbf{X}u)\rangle - \langle Y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle + \langle \tilde{\mathbf{R}}_y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle. \end{aligned}$$

Thus,

$$\begin{aligned} I = |\nabla^{:} u|^2 + \mathbf{X}(\langle \nabla^{\cdot} u, \nabla^{:} u\rangle) + \langle Y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle - \langle \tilde{\mathbf{R}}_y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle. \\ + \langle Y(y), \nabla^{\cdot} u\rangle \mathbf{X}u - \langle \nabla^{\cdot} u, \nabla^{:}(\mathbf{X}u)\rangle. \end{aligned} \tag{37}$$

We rewrite the second term on the right-hand side of (36) as

$$II = (\mathbf{X}u)g^{ij}_{\cdot i}u_{:j} = -2(\mathbf{X}u)g^{il}g^{jm}C_{lmi}u_{:j} = -2\mathbf{I}((\mathbf{X}u)\nabla^{:} u). \tag{38}$$

Finally, we rewrite the third term in (36) as

$$\begin{aligned} III &= (\mathbf{X}u)g^{ij}u_{:j\cdot i} = (\mathbf{X}u)g^{ij}(u_{\cdot i:j} + (u_{:j\cdot i} - u_{\cdot i:j})) \\ &= ((\mathbf{X}u)g^{ij}u_{\cdot i})_{:j} - (\mathbf{X}u)_{:j}g^{ij}u_{\cdot i} - (\mathbf{X}u)g^{ij}_{:j}u_{\cdot i} + (\mathbf{X}u)g^{ij}\tilde{P}^m_{ji}u_{\cdot m}. \end{aligned}$$

Note that

$$(\mathbf{X}u)g^{ij}u_{\cdot i})_{:j} = \overset{m}{\operatorname{div}}\,((\mathbf{X}u)\nabla^{\cdot} u),$$

that

$$(\mathbf{X}u)_{:j}g^{ij}u_{\cdot i} = \langle \nabla^{\cdot} u, \nabla^{:}(\mathbf{X}u)\rangle,$$

and that

$$\begin{aligned} (\mathbf{X}u)g^{ij}\tilde{P}^m_{ji}u_{\cdot m} &= (\mathbf{X}u)g^{ij}(P^m_{ji} + Y^m_j y_i + Y^m_{j\cdot i})u_{\cdot m} \\ &= -\mathbf{J}((\mathbf{X}u)\nabla^{\cdot} u) + \langle Y(y), \nabla^{\cdot} u\rangle \mathbf{X}u + (\mathbf{X}u)g^{mj}_{:j}u_{\cdot m} \end{aligned}$$

in view of (32).

Thus,

$$III = \overset{m}{\operatorname{div}}\,((\mathbf{X}u)\nabla^{\cdot} u) - \mathbf{J}((\mathbf{X}u)\nabla^{\cdot} u) + \langle Y(y), \nabla^{\cdot} u\rangle \mathbf{X}u - \langle \nabla^{\cdot} u, \nabla^{:}(\mathbf{X}u)\rangle. \tag{39}$$

Inserting (37)–(39) in (36), we come to (35). ☐

Given a semibasic vector field V, define a new semibasic vector field $\mathbf{X}V$ by

$$\mathbf{X}V^i = y^k V^i_{:k}.$$

It easy to see that if $(x, y) \in SM$ and γ is a magnetic geodesic with $\gamma(0) = x$, $\dot\gamma(0) = y$, then

$$\mathbf{X}V(x, y) = D_{\dot\gamma}(V \circ \dot\gamma)|_{t=0},$$

the covariant derivative of the field $V \circ \dot\gamma$ along γ.

Lemma 4.7. *If $u \in C^\infty(TM \setminus \{0\})$ is positively homogeneous, then*

$$|\mathbf{X}(\nabla^{\cdot} u)|^2 = |\nabla^{\cdot}\mathbf{X}u|^2 + |\nabla^{\cdot} u|^2 - 2\langle\nabla^{\cdot} u, \nabla^{\cdot}(\mathbf{X}u)\rangle + \langle Y(y), \nabla^{\cdot} u\rangle^2. \tag{40}$$

Proof. We have

$$\begin{aligned}\mathbf{X}(u^{\cdot i}) &= y^k(g^{ij}u_{\cdot j})_{:k} = y^k g^{ij}_{:k}u_{\cdot j} + y^k g^{ij}(u_{:k\cdot j} - (u_{:k\cdot j} - u_{\cdot j:k}))\\ &= y^k g^{ij}_{:k}u_{\cdot j} + g^{ij}(y^k u_{:k})_{\cdot j} - g^{ij}u_{:j} - g^{ij}y^k\tilde P^m_{kj}u_{\cdot m}.\end{aligned}$$

By (20) and (32)

$$\begin{aligned}g^{ij}y^k\tilde P^m_{kj}u_{\cdot m} &= g^{ij}y^k(P^m_{kj} + Y^m_k y_j + Y^m_{k\cdot j})u_{\cdot m}\\ &= \langle Y(y), \nabla^{\cdot} u\rangle y^i + y^k g^{mi}_{:k}u_{\cdot m}.\end{aligned}$$

Thus

$$\mathbf{X}(\nabla^{\cdot} u) = \nabla^{\cdot}(\mathbf{X}u) - \nabla^{\cdot} u - \langle Y(y), \nabla^{\cdot} u\rangle y.$$

Squaring, we receive

$$\begin{aligned}|\mathbf{X}(\nabla^{\cdot} u)|^2 &= |\nabla^{\cdot}\mathbf{X}u|^2 + |\nabla^{\cdot} u|^2 + \langle Y(y), \nabla^{\cdot} u\rangle^2\\ &\quad - 2\langle\nabla^{\cdot}(\mathbf{X}u), \nabla^{\cdot} u\rangle - 2\langle Y(y), \nabla^{\cdot} u\rangle\langle\nabla^{\cdot}(\mathbf{X}u), y\rangle + 2\langle Y(y), \nabla^{\cdot} u\rangle\langle\nabla^{\cdot} u, y\rangle\\ &= |\nabla^{\cdot}\mathbf{X}u|^2 + |\nabla^{\cdot} u|^2 + \langle Y(y), \nabla^{\cdot} u\rangle^2 - 2\langle\nabla^{\cdot} u, \nabla^{\cdot}(\mathbf{X}u)\rangle\\ &\quad - 2\langle Y(y), \nabla^{\cdot} u\rangle\mathbf{X}u + 2\langle Y(y), \nabla^{\cdot} u\rangle\mathbf{X}u,\end{aligned}$$

coming to the sought identity. □

Suppose that we have a kinetic equation on SM

$$\mathbf{G}_M u = \varphi.$$

Extending u to a positively homogeneous function on $TM\setminus\{0\}$, the extension denoted by u again, we have on $TM \setminus \{0\}$

$$\mathbf{X}u = \phi,$$

where ϕ is the positively homogeneous extension of φ of degree 1.

Combining (35) and (40), we get

$$\begin{aligned}&|\mathbf{X}(\nabla^{\cdot} u)|^2 + \mathbf{X}(\langle\nabla^{\cdot} u, \nabla^{\cdot} u\rangle) - \overset{m}{\operatorname{div}}((\mathbf{X}u)\nabla^{\cdot} u)\\ &\qquad - \langle\tilde{\mathbf{R}}_y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle + \langle Y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle - \langle Y(y), \nabla^{\cdot} u\rangle^2\\ &\qquad\qquad + 2\mathbf{I}((\mathbf{X}u)\nabla^{\cdot} u) + \mathbf{J}((\mathbf{X}u)\nabla^{\cdot} u)\\ &\qquad\qquad\qquad = |\nabla^{\cdot}(\mathbf{X}u)|^2 - \overset{v}{\operatorname{div}}((\mathbf{X}u)\nabla^{\cdot} u).\end{aligned}$$

We integrate this identity over SM against the Liouville measure, using the flow invariance of the measure and (27):

$$\begin{aligned}(41)\quad \int_{SM}|\mathbf{X}(\nabla^{\cdot}u)|^2\,d\mu - \int_{SM}\overset{m}{\operatorname{div}}\,((\mathbf{X}u)\nabla^{\cdot}u)\,d\,\mu - \int_{SM}\langle\tilde{\mathbf{R}}_y(\nabla^{\cdot}u),\nabla^{\cdot}u\rangle\,d\mu \\ + \int_{SM}\Big\{\langle Y(\nabla^{\cdot}u),\nabla^{\cdot}u\rangle - \langle Y(y),\nabla^{\cdot}u\rangle^2 + \mathbf{J}((\mathbf{X}u)\nabla^{\cdot}u)\Big\}\,d\mu \\ = \int_{SM}\Big\{|\nabla^{\cdot}(\mathbf{X}u)|^2 - n(\mathbf{X}u)^2\Big\}\,d\mu.\end{aligned}$$

Since

$$\overset{m}{\operatorname{div}}\,U = U^i_{:i} = U^i_{|i} + Y^j_i U^i_{\cdot j} = \overset{h}{\operatorname{div}}\,U + Y^j_i U^i_{\cdot j},$$

we have

$$\begin{aligned}\overset{m}{\operatorname{div}}\,((\mathbf{X}u)\nabla^{\cdot}u) &= \overset{h}{\operatorname{div}}\,(\mathbf{X}u)\nabla^{\cdot}u) + Y^j_i((\mathbf{X}u)_{\cdot j}g^{ik}u_{\cdot k} + (\mathbf{X}u)g^{ik}_{\cdot j}u_{\cdot k} + (\mathbf{X}u)g^{ik}u_{\cdot k\cdot j}) \\ &= \overset{h}{\operatorname{div}}\,(\mathbf{X}u)\nabla^{\cdot}u) + \langle\nabla^{\cdot}(\mathbf{X}u), Y(\nabla^{\cdot}u)\rangle,\end{aligned}$$

because by the symmetry argument

$$Y^j_i g^{ik}_{\cdot j} = -2Y^j_i g^{il}C_{lmj}g^{km} = 0$$

and

$$Y^j_i g^{ik}u_{\cdot k\cdot j} = 0.$$

Using also (28), we hence have

$$\int_{SM}\overset{m}{\operatorname{div}}\,((\mathbf{X}u)\nabla^{\cdot}u)\,d\mu = \int_{SM}\big\{\mathbf{J}((\mathbf{X}u)\nabla^{\cdot}u) + \langle\nabla^{\cdot}(\mathbf{X}u), Y(\nabla^{\cdot}u)\rangle\big\}\,d\mu.$$

Next,

$$\begin{aligned}\langle\tilde{\mathbf{R}}_y(\nabla^{\cdot}u),\nabla^{\cdot}u\rangle = \big\{R^i_{kl} + (Y^i_{k|l} - Y^i_{l|k}) - (P^i_{km}Y^m_l - P^i_{lm}Y^m_k) \\ + (Y^j_kY^i_{l\cdot j} - Y^j_lY^i_{k\cdot j}) + y_s(Y^s_lY^i_k - Y^s_kY^i_l)\big\}y^lu^{\cdot k}u_{\cdot i}.\end{aligned}$$

Now,

$$R^i_{kl}y^lu^{\cdot k}u_{\cdot i} = \langle\mathbf{R}_y(\nabla^{\cdot}u),\nabla^{\cdot}u\rangle,$$

$$\begin{aligned}(Y^i_{k|l} - Y^i_{l|k})y^lu^{\cdot k}u_{\cdot i} &= \langle(\nabla_{|y}Y)(\nabla^{\cdot}u),\nabla^{\cdot}u\rangle - \langle\nabla_{|(\nabla^{\cdot}u)}Y(y),\nabla^{\cdot}u\rangle \\ &= -\langle\nabla_{|(\nabla^{\cdot}u)}Y(y),\nabla^{\cdot}u\rangle\end{aligned}$$

by skew symmetry of Y and parallelism of the fundamental tensor with respect to $\nabla_|$,

$$(P^i_{km}Y^m_l - P^i_{lm}Y^m_k)y^lu^{\cdot k}u_{\cdot i} = P^i_{km}Y^m_ly^lu^{\cdot k}u_{\cdot i} = -L(Y(y),\nabla^{\cdot}u,\nabla^{\cdot}u)$$

in view of (20) and (24),

$$(Y^j_kY^i_{l\cdot j} - Y^j_lY^i_{k\cdot j})y^lu^{\cdot k}u_{\cdot i} = -2Y^j_kY^r_lg^{is}C_{srj}y^lu^{\cdot k}u_{\cdot i} + 2Y^j_lY^r_kg^{is}C_{srj}y^lu^{\cdot k}u_{\cdot i} = 0$$

by the symmetry of C, and

$$y_s(Y_l^s Y_k^i - Y_k^s Y_l^i)y^l u^{\cdot k} u_{\cdot i} = \langle Y(y), y\rangle\langle Y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle - \langle Y(\nabla^{\cdot} u), y\rangle\langle Y(y), \nabla^{\cdot} u\rangle$$
$$= \langle Y(y), \nabla^{\cdot} u\rangle^2$$

again by the skew symmetry of Y.

Now, (41) takes the form of equation (1) in the Introduction. That is, we have proved:

Theorem 4.8.

$$\int_{SM} \{|\mathbf{X}(\nabla^{\cdot} u)|^2 - \langle \mathbf{R}_y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle - L(Y(y), \nabla^{\cdot} u, \nabla^{\cdot} u) - \langle \nabla^{\cdot}(\mathbf{X}u), Y(\nabla^{\cdot} u)\rangle$$
$$- 2\langle Y(y), \nabla^{\cdot} u\rangle^2 + \langle \nabla^{:} u, Y(\nabla^{\cdot} u)\rangle + \langle \nabla_{|(\nabla^{\cdot} u)} Y(y), \nabla^{\cdot} u\rangle\} \, d\mu$$
$$= \int_{SM} \{|\nabla^{\cdot}(\mathbf{X}u)|^2 - n(\mathbf{X}u)^2\} \, d\mu. \tag{42}$$

Remark 4.9. The identity (42) is exactly identity (12) when $n = 2$. If $\phi \in C^\infty(TM \setminus \{0\})$ is homogeneous of degree 1 and $n = 2$, then chasing definitions we have:

$$|\nabla^{\cdot} \phi|^2 = \phi^2 + (V\phi)^2.$$

Thus the right hand side of (42) becomes

$$\int_{SM} \{|\nabla^{\cdot}(\mathbf{X}u)|^2 - 2(\mathbf{X}u)^2\} \, d\mu = \int_{SM} \{(\mathbf{G}_M u)^2 + (V\mathbf{G}_M u)^2\} \, d\mu - 2\int_{SM} (\mathbf{G}_M u)^2 \, d\mu$$

which is exactly the right hand side of (12). We leave to the keen reader the task of fully verifying that the left hand sides also coincide. When the Finsler metric is Riemannian (i.e. $I = J = 0$) and $n = 2$ it is quite easy to check that (for points in SM):

$$|\mathbf{X}(\nabla^{\cdot} u)|^2 = (\mathbf{G}_M V u)^2 + \lambda^2 (Vu)^2,$$
$$\langle \mathbf{R}_y(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle = (Vu)^2 K,$$
$$\langle \nabla^{\cdot}(\mathbf{X}u), Y(\nabla^{\cdot} u)\rangle = -\lambda \mathbf{G}_M u \cdot Vu,$$
$$\langle Y(y), \nabla^{\cdot} u\rangle = \lambda V u,$$
$$\langle \nabla^{:} u, Y(\nabla^{\cdot} u)\rangle = -\lambda \mathbf{G}_M u \cdot Vu$$
$$\langle \nabla_{|(\nabla^{\cdot} u)} Y(y), \nabla^{\cdot} u\rangle = (Vu)^2 H(\lambda).$$

Inserting these relations into the left hand side of (42) we see that we get exactly the left hand side of (12).

4.4. **Jacobi equation.** Let us derive a Jacobi equation. The calculations below mimic those in the proof of [24, Lemma 6.1.1].

Let $\phi_t : TM \setminus \{0\} \to TM \setminus \{0\}$ be the magnetic flow. Take a curve $Z : (-\varepsilon, \varepsilon) \to TM \setminus \{0\}$ with $Z(0) = v$ and $Z'(0) = \xi$, and consider the variation $H(s, t) =$

$\pi(\phi_t(Z(s)))$. Set

$$T = \frac{\partial H}{\partial t}, \quad U = \frac{\partial H}{\partial s}.$$

Each $c_s(t) = H(s,t)$ is a magnetic geodesic; therefore,

$$\frac{\partial^2 H^i}{\partial t^2} + 2G^i\Big(\frac{\partial H}{\partial t}\Big) = Y^i_j\Big(\frac{\partial H}{\partial t}\Big)\frac{\partial H^j}{\partial t},$$

or

$$\frac{\partial T^i}{\partial t} + 2G^i(T) = Y^i_j(T)T^j. \tag{43}$$

Since

$$\frac{\partial T^i}{\partial s} = \frac{\partial}{\partial s}\Big(\frac{\partial H^i}{\partial t}\Big) = \frac{\partial}{\partial t}\Big(\frac{\partial H^i}{\partial s}\Big) = \frac{\partial U^i}{\partial t},$$

differentiating (43) with respect to s yields

$$\begin{aligned}\frac{\partial^2 U^i}{\partial t^2} &= -2U^k\frac{\partial G^i}{\partial x^k}(T) - 2\frac{\partial U^l}{\partial t}\frac{\partial G^i}{\partial y^l}(T)\\ &\quad + \Big(U^k\frac{\partial Y^i_j}{\partial x^k}(T) + \frac{\partial U^l}{\partial t}\frac{\partial Y^i_j}{\partial y^l}(T)\Big)T^j + Y^i_j(T)\frac{\partial U^j}{\partial t}.\end{aligned}$$

Note that

$$\begin{aligned}\frac{\partial}{\partial s}\big[G^i(T)\big] &= U^k\frac{\partial G^i}{\partial x^k}(T) + \frac{\partial U^l}{\partial t}\frac{\partial G^i}{\partial y^l}(T),\\ \frac{\partial}{\partial t}\left[\frac{\partial G^i}{\partial y^l}(T)\right] &= T^k\frac{\partial^2 G^i}{\partial x^k \partial y^l} + \frac{\partial T^k}{\partial t}\frac{\partial^2 G^i}{\partial y^l y^k}(T)\\ &= T^k\frac{\partial^2 G^i}{\partial x^k \partial y^l} + \big(-2G^k(T) + Y^k_m(T)T^m\big)\frac{\partial^2 G^i}{\partial y^l \partial y^k}(T).\end{aligned}$$

Hence,

$$
\begin{aligned}
D_T D_T(U^i) &= D_T\left(\frac{\partial U^i}{\partial t} + U^l \frac{\partial G^i}{\partial y^l}(T)\right) \\
&= \frac{\partial}{\partial t}\left(\frac{\partial U^i}{\partial t} + U^l \frac{\partial G^i}{\partial y^l}(T)\right) + \left(\frac{\partial U^k}{\partial t} + U^l \frac{\partial G^k}{\partial y^l}(T)\right)\frac{\partial G^i}{\partial y^k}(T) \\
&= \frac{\partial^2 U^i}{\partial t^2} + \frac{\partial U^l}{\partial t}\frac{\partial G^i}{\partial y^l} + U^l \frac{\partial}{\partial t}\left[\frac{\partial G^i}{\partial y^l}\right] + \frac{\partial U^k}{\partial t}\frac{\partial G^i}{\partial y^k} + U^l \frac{\partial G^k}{\partial y^l}\frac{\partial G^i}{\partial y^k} \\
&= -2U^k \frac{\partial G^i}{\partial x^k} - 2\frac{\partial U^l}{\partial t}\frac{\partial G^i}{\partial y^l} + \Big(U^k \frac{\partial Y^i_j}{\partial x^k} + \frac{\partial U^l}{\partial t}\frac{\partial Y^i_j}{\partial y^l}\Big)T^j + Y^i_j \frac{\partial U^j}{\partial t} \\
&\quad + \frac{\partial U^l}{\partial t}\frac{\partial G^i}{\partial y^l} + U^l\left[T^k \frac{\partial^2 G^i}{\partial x^k \partial y^l} + \big(-2G^k + Y^k_m T^m\big)\frac{\partial^2 G^i}{\partial y^l \partial y^k}\right] \\
&\quad + \frac{\partial U^k}{\partial t}\frac{\partial G^i}{\partial y^k} + U^l \frac{\partial G^k}{\partial y^l}\frac{\partial G^i}{\partial y^k} \\
&= -U^k\left(2\frac{\partial G^i}{\partial x^k} - T^j \frac{\partial^2 G^i}{\partial x^j \partial y_k} + 2G^j \frac{\partial^2 G^i}{\partial y^j y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}\right) \\
&\quad + \Big(U^k \frac{\partial Y^i_j}{\partial x^k} + \frac{\partial U^l}{\partial t}\frac{\partial Y^i_j}{\partial y^l}\Big)T^j + Y^i_j \frac{\partial U^j}{\partial t} + U^l Y^k_m T^m \frac{\partial^2 G^i}{\partial y^l \partial y^k}.
\end{aligned}
$$

Using the identities

$$R^i_k(T) = 2\frac{\partial G^i}{\partial x^k} - T^j \frac{\partial^2 G^i}{\partial x^j \partial y_k} + 2G^j \frac{\partial^2 G^i}{\partial y^j y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k},$$

$$\frac{\partial U^i}{\partial t} = D_T U^i - N^i_l U^l,$$

$$\frac{\partial Y^i_j}{\partial x^k} = Y^i_{j|k} + N^p_k \frac{\partial Y^i_j}{\partial y^p} - \Gamma^i_{kp} Y^p_j + \Gamma^p_{kj} Y^i_p,$$

$$\frac{\partial^2 G^i}{\partial y^l \partial y^k} = \Gamma^i_{jk} + L^i_{jk},$$

$$Y(D_T U) = Y^i_j \frac{\partial U^j}{\partial t} + Y^i_j \Gamma^j_{kl} T^l U^k,$$

we find that

$$D_T D_T(U) = -\mathbf{R}_T(U) + Y(D_T U) + (\nabla_{|U} Y)(T) + (\nabla_{\cdot D_T U} Y)(T) + \mathbf{L}(U, Y(T)),$$

which is the Jacobi equation for the magnetic flow of a Finsler metric. Here $\mathbf{L}(U,V)$ is defined by $\langle \mathbf{L}(U,V), W\rangle = L(U,V,W)$.

4.5. **Index form.** Let γ be a closed unit speed magnetic geodesic. Let $\mathcal{A}$ and $\mathcal{C}$ be the operators on smooth vector fields along γ defined by

$$\begin{aligned}\mathcal{A}(Z) &= \ddot{Z} + \mathbf{R}_{\dot\gamma}(Z) - Y(\dot{Z}) - (\nabla_{|Z}Y)(\dot\gamma) - (\nabla_{\cdot\dot{Z}}Y)(\dot\gamma) - \mathbf{L}(Z, Y(\dot\gamma)) \\ &= \ddot{Z} + \mathcal{C}(Z) - (\nabla_{\cdot\dot{Z}}Y)(\dot\gamma) - \mathbf{L}(Z, Y(\dot\gamma)),\end{aligned} \tag{44}$$

where

$$\mathcal{C}(Z) := \mathbf{R}_{\dot\gamma}(Z) - Y(\dot{Z}) - (\nabla_{|Z}Y)(\dot\gamma). \tag{45}$$

If J is a magnetic Jacobi field, then

$$\mathcal{A}(J) = 0. \tag{46}$$

Let Λ denote the $\mathbb{R}$-vector space of smooth vector fields $Z : [0,T] \to TM$ along γ, such that $Z(0) = Z(T)$ and $\dot{Z}(0) = \dot{Z}(T)$. Let $\mathbb{I}$ denote the quadratic form $\mathbb{I} : \Lambda \to \mathbb{R}$ defined by

$$\mathbb{I}(Z,Z) = -\int_0^T \{\langle \mathcal{A}(Z), Z\rangle + \langle Y(\dot\gamma), Z\rangle^2\}\, dt. \tag{47}$$

Observe that

$$\mathbb{I}(Z,Z) = \int_0^T \left\{|\dot{Z}|^2 - \langle \mathcal{C}(Z), Z\rangle - L(Y(\dot\gamma), Z, Z) - \langle Y(\dot\gamma), Z\rangle^2\right\} dt. \tag{48}$$

Indeed,

$$\begin{aligned}\mathbf{X}(\langle U, V\rangle) &= y^k (g_{ij}U^iV^j)_{:k} = y^k(g_{ij:k}U^iV^j + g_{ij}U^i_{:k}V^j + g_{ij}U^iV^j_{:k}) \\ &= -y^k g_{sj} Y^s_{k\cdot i} U^i V^j + \langle \mathbf{X}U, V\rangle + \langle U, \mathbf{X}V\rangle \\ &= -\langle (\nabla_{\cdot U}Y)(y), V\rangle + \langle \mathbf{X}U, V\rangle + \langle U, \mathbf{X}V\rangle,\end{aligned}$$

where we have used the equality $g_{ij:k} = -g_{sj}Y^s_{k\cdot i}$ following from (31) and (32).

This implies

$$\langle \ddot{Z}, Z\rangle = D_{\dot\gamma}(\langle \dot{Z}, Z\rangle) - |\dot{Z}|^2 + \langle (\nabla_{\cdot\dot{Z}}Y)(\dot\gamma), Z\rangle,$$

whence (48) is straightforward.

Lemma 4.10 (Index Lemma). *Suppose the magnetic flow ϕ_t is Anosov and let γ be a closed magnetic geodesic with period T. If Z is orthogonal to $\dot\gamma$, then*

$$\mathbb{I}(Z,Z) \geq 0,$$

with equality if and only if Z vanishes.

Proof. Let E denote the weak stable or unstable subbundle of ϕ_t. It is well known (cf. [17, 15], see [4] for a proof using the asymptotic Maslov index) that the following transversality property holds:

$$E(v) \cap \operatorname{Ker} d_v\pi = \{0\},$$

for every $v \in SM$, where $\pi : SM \to M$ is the canonical projection. Consider the splitting into horizontal and vertical subbundles described in Subsection 4.1. With respect to this splitting the transversality property can be restated as follows: for each $v \in SM$, there exists a map $S_v : T_{\pi(v)}M \to T_{\pi(v)}M$ so that its graph is $E(v)$; moreover the correspondence $v \to S_v$ is continuous.

If $\xi \in E(v)$, then $J_\xi(t) = d\pi \circ d\phi_t(\xi)$ satisfies the Jacobi equation (46). Since for all $t \in \mathbb{R}$,

$$d\pi_{\dot\gamma(t)}\big|_{E(\dot\gamma(t))} : E(\dot\gamma(t)) \to T_{\dot\gamma(t)}M$$

is an isomorphism, there exists a basis $\{\xi_1, \dots, \xi_n\}$ of $E(v)$ such that $\{J_{\xi_1}(t), \dots, J_{\xi_n}(t)\}$ is a basis of $T_{\dot\gamma(t)}M$ for all $t \in \mathbb{R}$. Without loss of generality we may assume that $\xi_1 = (v, S(v))$ and $J_{\xi_1} = \dot\gamma$.

Let us set for brevity $J_i = J_{\xi_i}$. Then if Z is an element of Λ we can write

$$Z(t) = \sum_{i=1}^{n} f_i(t) J_i(t),$$

for some smooth functions $f_1, \dots, f_n$ and thus,

$$\mathbb{I}(Z,Z) = -\sum_{i,j} \int_0^T \langle \mathcal{A}(f_i J_i), f_j J_j \rangle \, dt - \int_0^T \langle Y(\dot\gamma), Z \rangle^2 \, dt. \tag{49}$$

An easy computation shows that

$$\mathcal{A}(f_i J_i) = \ddot f_i J_i + 2\dot f_i \dot J_i - \dot f_i Y(J_i) - \dot f_i (\nabla_{\cdot J_i} Y)(\dot\gamma) + f_i \mathcal{A}(J_i).$$

Indeed,

$$D_{\dot\gamma} D_{\dot\gamma}(f_i J_i) = \ddot f_i J_i + 2\dot f_i \dot J_i + f_i \ddot J_i,$$
$$\mathbf{R}_{\dot\gamma}(f_i J_i) = f_i \mathbf{R}_{\dot\gamma}(J_i),$$
$$Y(D_{\dot\gamma}(f_i J_i)) = \dot f_i Y(J_i) + f_i Y(\dot J_i),$$
$$(\nabla_{|f_i J_i} Y)(\dot\gamma) = f_i (\nabla_{|J_i} Y)(\dot\gamma),$$
$$(\nabla_{\cdot D_{\dot\gamma}(f_i J_i)} Y)(\dot\gamma) = \dot f_i (\nabla_{\cdot J_i} Y)(\dot\gamma) + f_i (\nabla_{\cdot \dot J_i} Y)(\dot\gamma),$$
$$\mathbf{L}(f_i J_i, Y(\dot\gamma)) = f_i \mathbf{L}(J_i, Y(\dot\gamma)).$$

Since J_i satisfies equation (46), it follows that $\mathcal{A}(J_i) = 0$ and hence

$$\langle \mathcal{A}(f_i J_i), J_j \rangle = \ddot f_i \langle J_i, J_j \rangle + 2\dot f_i \langle \dot J_i, J_j \rangle - \dot f_i \langle Y(J_i), J_j \rangle - \dot f_i \langle (\nabla_{\cdot J_i} Y)(\dot\gamma), J_j \rangle.$$

Observe that since E is a Lagrangian subspace,

$$\langle J_i, \dot J_j \rangle - \langle \dot J_i, J_j \rangle + \langle Y(J_i), J_j \rangle = 0,$$

and then

$$\langle \mathcal{A}(f_i J_i), J_j \rangle = \frac{d}{dt}(\dot f_i \langle J_i, J_j \rangle).$$

Now we can write

$$\int_0^T \langle \mathcal{A}(f_i J_i), f_j J_j \rangle \, dt = \langle \dot f_i J_i, f_j J_j \rangle \Big|_0^T - \int_0^T \langle \dot f_i J_i, \dot f_j J_j \rangle \, dt.$$

Combining the last equality with (49) we obtain

$$\mathbb{I}(Z,Z) = \int_0^T \left| \sum_{i=1}^{n} \dot f_i J_i \right|^2 dt - \left\langle \sum_{i=1}^{n} \dot f_i J_i, Z \right\rangle \Bigg|_0^T - \int_0^T \langle Y(\dot\gamma), Z \rangle^2 \, dt.$$

But $\dot{Z}(0)=\dot{Z}(T)$ and $\dot{Z}=\sum_{i=1}^{n}\dot{f}_iJ_i+\sum_{i=1}^{n}f_i\dot{J}_i$, therefore

$$\Big\langle\sum_{i=1}^{n}\dot{f}_iJ_i,Z\Big\rangle\Big|_0^T=-\Big\langle\sum_{i=1}^{n}f_i\dot{J}_i,Z\Big\rangle\Big|_0^T.$$

Note that $\dot{J}_i(t)=S_{\dot{\gamma}(t)}J_i(t)$, hence

$$\sum_{i=1}^{n}f_i\dot{J}_i=S\left(\sum_{i=1}^{n}f_iJ_i\right)=S(Z),$$

which implies

$$\Big\langle\sum_{i=1}^{n}f_i\dot{J}_i,Z\Big\rangle\Big|_0^T=\Big\langle S(Z),Z\Big\rangle\Big|_0^T=0.$$

Then

$$\mathbb{I}(V,V)=\int_0^T\Big|\sum_{i=1}^{n}\dot{f}_iJ_i\Big|^2\,dt-\int_0^T\langle Y(\dot{\gamma}),Z\rangle^2\,dt. \tag{50}$$

Now let

$$W:=\sum_{i=2}^{n}\dot{f}_iJ_i.$$

Since $J_1=\dot{\gamma}$ we have:

$$\left\langle\sum_{i=1}^{n}\dot{f}_iJ_i,\sum_{i=1}^{n}\dot{f}_iJ_i\right\rangle=\langle\dot{f}_1\dot{\gamma}+W,\dot{f}_1\dot{\gamma}+W\rangle=\dot{f}_1^2+2\dot{f}_1\langle\dot{\gamma},W\rangle+\langle W,W\rangle.$$

Differentiating $\langle Z,\dot{\gamma}\rangle=0$ we get

$$\langle\dot{Z},\dot{\gamma}\rangle+\langle Z,Y(\dot{\gamma})\rangle=0.$$

But

$$\langle\dot{Z},\dot{\gamma}\rangle=\left\langle\sum_{i=1}^{n}\dot{f}_iJ_i,\dot{\gamma}\right\rangle=\dot{f}_1+\langle W,\dot{\gamma}\rangle$$

since $\langle\dot{J}_i,\dot{\gamma}\rangle=0$ for all i. Therefore

$$\langle Y(\dot{\gamma}),Z\rangle^2=\dot{f}_1^2+2\dot{f}_1\langle W,\dot{\gamma}\rangle+\langle W,\dot{\gamma}\rangle^2.$$

Thus

$$\left\langle\sum_{i=1}^{n}\dot{f}_iJ_i,\sum_{i=1}^{n}\dot{f}_iJ_i\right\rangle-\langle Y(\dot{\gamma}),Z\rangle^2=\langle W,W\rangle-\langle W,\dot{\gamma}\rangle^2.$$

If we let $W^\perp$ be the orthogonal projection of W to $\dot{\gamma}^\perp$, the last equation and (50) give:

$$\mathbb{I}(Z,Z)=\int_0^T\|W^\perp\|^2\,dt\geq 0$$

with equality if and only if $W^\perp$ vanishes identically. But if $W^\perp$ vanishes, then

$$-\langle W,\dot\gamma\rangle\dot\gamma+\sum_{i=2}^{n}\dot f_i J_i=0$$

which implies that the functions f_i are constant for $i\geq 2$. Thus Z is of the form $f_1\dot\gamma+J$ where J is a magnetic Jacobi field. If we let $J^\perp$ be the orthogonal projection of J to $\dot\gamma^\perp$, then $Z=J^\perp$. Now write

$$J=x\dot\gamma+J^\perp$$

A simple calculation shows that $\mathcal{A}(x\dot\gamma)=D_{\dot\gamma}(\dot x\dot\gamma)$ with $\dot x=\langle J,Y(\dot\gamma)\rangle=\langle J^\perp,Y(\dot\gamma)\rangle$. Hence

$$0=\mathcal{A}(J)=\mathcal{A}(J^\perp)+D_{\dot\gamma}\left(\langle J^\perp,Y(\dot\gamma)\rangle\dot\gamma\right).$$

The fact that $J^\perp$ satisfies this second order differential equation together with $J^\perp(0)=J^\perp(0)$ and $\dot J^\perp(T)=\dot J^\perp(T)$ implies that $J^\perp$ is periodic with period T. Hence $\dot x$ is also a periodic function of period T which implies that $\|J\|$ grows at most linearly with t. However, since the closed orbits of ϕ_t are hyperbolic the only Jacobi fields with that type of growth are those given by constant multiples of $\dot\gamma$. Since Z is orthogonal to $\dot\gamma$, Z must vanish. □

4.6. **End of the proof of Theorem B.** Define

$$\tilde{\mathcal{C}}(V)=\mathbf{R}_y(V)-Y(\mathbf{X}V)-(\nabla_{|V}Y)(y).$$

Then the following holds:

$$\begin{aligned}\langle\tilde{\mathcal{C}}(\nabla^{\cdot}u),\nabla^{\cdot}u\rangle&=\langle\mathbf{R}_y(\nabla^{\cdot}u),\nabla^{\cdot}u\rangle+\langle\mathbf{X}(\nabla^{\cdot}u),Y(\nabla^{\cdot}u)\rangle-\langle(\nabla_{|(\nabla^{\cdot}u)}Y)(y),\nabla^{\cdot}u\rangle\\&=\langle\mathbf{R}_y(\nabla^{\cdot}u),\nabla^{\cdot}u\rangle+\langle\nabla^{\cdot}(\mathbf{X}u)-\nabla^{\cdot}u-\langle Y(y),\nabla^{\cdot}u\rangle y,Y(\nabla^{\cdot}u)\rangle\\&\quad-\langle(\nabla_{(\nabla^{\cdot}u)}Y)(y),\nabla^{\cdot}u\rangle\\&=\langle\mathbf{R}_y(\nabla^{\cdot}u),\nabla^{\cdot}u\rangle+\langle\nabla^{\cdot}(\mathbf{X}u),Y(\nabla^{\cdot}u)\rangle-\langle\nabla^{\cdot}u,Y(\nabla^{\cdot}u)\rangle\\&\quad+\langle Y(y),\nabla^{\cdot}u\rangle^2-\langle(\nabla_{|(\nabla^{\cdot}u)}Y)(y),\nabla^{\cdot}u\rangle.\end{aligned}$$

Suppose $\mathbf{G}_M u=h\circ\pi+\theta$. From (42) and Lemma 4.4 we infer that

$$\int_{SM}\left\{|\mathbf{X}\nabla^{\cdot}u|^2-\langle\tilde{\mathcal{C}}(\nabla^{\cdot}u),\nabla^{\cdot}u\rangle-L(Y(y),\nabla^{\cdot}u,\nabla^{\cdot}u)-\langle Y(y),\nabla^{\cdot}u\rangle^2\right\}d\mu\leq 0. \tag{51}$$

Given a closed unit-speed magnetic geodesic $\gamma:[0,T]\to M$ consider the smooth vector field $Z:[0,T]\to TM$ along γ given by $Z:=\nabla^{\cdot}u(\gamma,\dot\gamma)$. Note that Z is orthogonal to $\dot\gamma$ because u is homogeneous of degree zero.

The Index Lemma 4.10 tells us that

$$\int_0^T\left\{|\dot Z|^2-\langle\mathcal{C}(Z),Z\rangle-L(Y(\dot\gamma),Z,Z)-\langle Y(\dot\gamma),Z\rangle^2\right\}dt\geq 0 \tag{52}$$

for every closed magnetic geodesic γ.

Since the flow is Anosov, the invariant measures supported on closed orbits are dense in the space of all invariant measures on SM. Therefore, the above yields

$$\int_{SM} \left\{ |\mathbf{X}\nabla^{\cdot} u|^2 - \langle \tilde{\mathcal{C}}(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle - L(Y(y), \nabla^{\cdot} u, \nabla^{\cdot} u) - \langle Y(y), \nabla^{\cdot} u\rangle^2 \right\} d\mu \geq 0.$$

Combining this (51), we find that

$$\int_{SM} \left\{ |\mathbf{X}\nabla^{\cdot} u|^2 - \langle \tilde{\mathcal{C}}(\nabla^{\cdot} u), \nabla^{\cdot} u\rangle - L(Y(y), \nabla^{\cdot} u, \nabla^{\cdot} u) - \langle Y(y), \nabla^{\cdot} u\rangle^2 \right\} d\mu = 0. \tag{53}$$

By the non-negative version of the Livšic theorem, proved independently by M. Pollicott and R. Sharp and by A. Lopes and P. Thieullen (see [13, 21]), we conclude from (52) and (53) that

$$\int_0^T \left\{ |\dot{Z}|^2 - \langle \mathcal{C}(Z), Z\rangle - L(Y(\dot{\gamma}), Z, Z) - \langle Y(\dot{\gamma}), Z\rangle^2 \right\} dt = 0$$

for every closed magnetic geodesic γ. Applying again the Index Lemma 4.10, we see that $\nabla^{\cdot} u$ vanishes on all closed magnetic geodesics. Since the latter are dense in SM, the function $\nabla^{\cdot} u$ vanishes on all of SM. This means that $u = f\circ\pi$ where f is a smooth function on M. But in this case, since $d\pi_{(x,v)}(\mathbf{G}_M) = v$ we have $\mathbf{G}_M(u) = df_x(v)$ and Theorem B follows.

5. Proof of Theorem C

Suppose the magnetic flow ϕ of the pair (F, Ω) has an Anosov splitting

$$E^s \oplus E^u \oplus \mathbb{R}\mathbf{G}_M$$

of class C^1 and suppose also that Ω is exact. Let τ denote the one-form that vanishes on $E^s \oplus E^u$ and takes the value one on the vector field $\mathbf{G}_M$. If the splitting is of class C^1 then τ is also of class C^1 and $d\tau$ is a continuous 2-form invariant under the magnetic flow. U. Hamenstädt showed in [11], for the geodesic flow case, that any continuous invariant exact 2-form must be a constant multiple of the symplectic form provided that the splitting is of class C^1. Hamenstädt's proof carries over to the case of magnetic flows *without* major changes, provided that Ω is an exact form $d\theta$ (see the appendix of [16]). Recall from the introduction that the symplectic form on $TM\setminus\{0\}$ is given by $\omega_0 + \pi^*\Omega$, where $\omega_0 = \ell_F^*(-d\lambda)$ (ℓ_F is the Legendre transform of $F^2/2$ and λ is the Liouville 1-form of T^*M). It follows that there exists a constant c such that:

$$d\tau = c(\omega_0 + \pi^*\Omega),$$

and thus

$$d(\tau + c\ell_F^*\lambda - c\pi^*\theta) = 0.$$

Let us write

$$\varphi := \tau + c\ell_F^*\lambda - c\pi^*\theta.$$

Then φ is a smooth *closed* 1-form. Since on SM $\ell_F^*\lambda(\mathbf{G}_M) = 1$[3] we obtain

$$\varphi(\mathbf{G}_M)(x, v) = 1 + c - c\theta_x(v). \tag{54}$$

[3]Using the expressions in Subsection 4.3 we see that $\ell_F^*\lambda(\mathbf{G}_M) = g_{ij}y^iy^j$.

It is well known that the map $\pi^* : H^1(M,\mathbb{R}) \to H^1(SM,\mathbb{R})$ is an isomorphism (provided that M is not diffeomorphic to a 2-torus). Therefore there exist a *closed* smooth 1-form δ in M and a smooth function $u : SM \to \mathbb{R}$ such that

$$\varphi = \pi^*\delta + du.$$

Hence equation (54) gives:

$$\mathbf{G}_M(u) + \delta_x(v) = 1 + c - c\theta_x(v). \tag{55}$$

Integrating the last equality with respect to the (normalized) Liouville measure μ and using that the magnetic flow leaves μ invariant we have

$$0 = 1 + c - c\int_{SM} \theta\, d\mu - \int_{SM} \delta\, d\mu.$$

By Lemma 4.4

$$\int_{SM} \theta\, d\mu = \int_{SM} \delta\, d\mu = 0$$

and thus $c = -1$. Replacing in (55) we finally obtain

$$\delta_x(v) + \mathbf{G}_M(u)(x,v) = \theta_x(v). \tag{56}$$

We can now apply Theorem B to conclude that θ is a closed form, i.e., Ω vanishes identically.[4]

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[4]Alternatively, we could have applied Theorem B directly to equation (55) to conclude that $c = -1$ and θ is exact.

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