RIEMANNIAN GEOMETRY. EXAMPLES 3.

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Let Γ be a free abelian group generated by k elements. Show that the counting function $n(\lambda)$ of Γ is given by

$$n(\lambda) = \sum_{i=0}^{k} 2^i \binom{k}{i} \binom{\lambda}{i}.$$

[*Hint: the number of sequences* $(a_1, \dots, a_i)$ *of* i *positive integers such that* $\sum_{j \le i} a_j \le \lambda$ *equals* $\binom{\lambda}{i}$ for $i \ge 1$.]

2. Let $\tau : M \to N$ be a finite Riemannian covering of order k and suppose that M is compact. Show that $\mathrm{Vol}(M) = k\,\mathrm{Vol}(N)$.

3. Let M be a compact Riemannian manifold and let $\tau : \widetilde{M} \to M$ be its universal covering endowed with the pull back metric. Let Γ be the group of deck transformations and given $\varepsilon > 0$ and $p \in \widetilde{M}$ let

$$S_\varepsilon := \{\gamma \in \Gamma : \ d(p, \gamma(p)) \le 2d + \varepsilon\},$$

where d is the diameter of M.

(a) Show that S_ε is finite.

(b) Show that S_ε generates Γ and conclude that the fundamental group of a compact manifold is finitely generated.

(c) Show that if ε is small enough, then $S_\varepsilon = S_0$.

4. As in Problem 3, let M be a compact Riemannian manifold, $\tau : \widetilde{M} \to M$ its universal covering endowed with the pull back metric and Γ the group of deck transformations. Given $p \in M$, let I_p be the complement of the cut locus of p. Let $U_p \subset T_pM$ be the open set bounded by the tangential cut locus, so that $\exp_p : U_p \to I_p$ is a diffeomorphism.

Given $\widetilde{p} \in \widetilde{M}$ such that $\tau(\widetilde{p}) = p$ set

$$D := \exp_{\widetilde{p}} \circ\, d\tau_{\widetilde{p}}^{-1}(U_p).$$

(a) Show that $\gamma(D) \cap D = \emptyset$ for all $\gamma \in \Gamma$ different from the identity and that $\overline{D}$ is compact.

(b) Show that $\tau(\overline{D}) = M$ and $\widetilde{M} = \cup_{\gamma \in \Gamma} \gamma(\overline{D})$.

(c) Show that $\mathrm{Vol}(\overline{D}) = \mathrm{Vol}(D) = \mathrm{Vol}(M)$.

Such a $\overline{D}$ is called a *fundamental domain* of τ.

5. This problem will guide you through the proof of the following theorem of M. Anderson (1990): For numbers $n \in \mathbb{N}$, $k \in \mathbb{R}$, $v, d \in (0, \infty)$, let $\mathcal{M}(n, k, v, d)$ denote the class of n-dimensional compact Riemannian manifolds with

$$\mathrm{Ric} \ge k, \quad \mathrm{Vol} \ge v, \quad \mathrm{diam} \le d.$$

Theorem. For fixed n, k, v, d, there are only finitely fundamental groups among the manifolds in $\mathcal{M}(n, k, v, d)$.

For its proof we shall assume the following lemma due to Gromov (you can find a proof in Petersen's book, p. 254): Given $\widetilde{p} \in \widetilde{M}$ we can always find generators $\{\gamma_1, \dots, \gamma_m\}$ for the fundamental group $\Gamma = \pi_1(M)$ such that $d(\widetilde{p}, \gamma_i(\widetilde{p})) \leq 2\,d$ (here d is the diameter of M) and such that all relations for Γ in these generators are of the form $\gamma_i\,\gamma_j = \gamma_k$.

(a) Choose generators $\{\gamma_1, \dots, \gamma_m\}$ as in Gromov's lemma and show that the number of possible relations is bounded by 2^{m^3} and conclude that to prove the theorem we need to show that m is bounded.

(b) Consider a fundamental domain D that contains $\widetilde{p}$ as in Problem 4. Show that $\cup_i \gamma_i(D) \subset B(\widetilde{p}, 4d)$.

(c) Using that the sets $\gamma_i(D)$ are disjoint and all have the same volume show that

$$m \leq \frac{\mathrm{Vol}(B(\widetilde{p}, 4d))}{\mathrm{Vol}(D)}.$$

(d) Use volume comparison to complete the proof of the theorem.

6. Consider the lens spaces from Problem 10 in the first example sheet. Show that a lower bound on volume is really necessary for Anderson's theorem.

7. Let M be a compact Riemannian manifold, $\tau : \widetilde{M} \to M$ its universal covering endowed with the pull back metric and Γ the group of deck transformations. Let E be a compact subset of $\widetilde{M}$ for which $\widetilde{M} = \cup_{\gamma \in \Gamma} \gamma(E)$. Set

$$S_E := \{\gamma \in \Gamma : \ \gamma(E) \cap E \neq \emptyset.\}$$

(a) Show that S_E is finite and generates Γ.

(b) If we set $\nu := \inf_{\gamma \notin S_E} d(\gamma(E), E)$, show that for any given $\gamma \in \Gamma$ we have:

$$|\gamma| \leq \left[\frac{d(y, \gamma(x))}{\nu}\right] + 1,$$

for any x and y in E, where $|\gamma|$ is the norm of γ with respect to S_E.

8. Same setting as in the previous problem. Let D be a fundamental domain as in Problem 4 and let δ be its diameter.

(a) Prove that $\overline{B}(p, \delta)$ contains $\overline{D}$ for all $p \in \overline{D}$.

(b) Show that

$$n(\lambda) \geq \frac{V(p, \lambda\nu - (\nu + 2\delta))}{V(p, \delta)},$$

for all $\lambda \geq 1 + 3\delta/\nu$, where $\nu := \inf_{p \in \widetilde{M}} \nu_p$ and ν_p is given by Problem 7 when $E = \overline{B}(p, \delta)$.

We stated this lemma in lectures without proof.

(c) Using (b) and Problem 3 in the second example sheet show that if M is a closed manifold with negative sectional curvature, then $\pi_1(M)$ has exponential growth.

9. Let M be a complete n-dimensional Riemannian manifold that is isometric to Euclidean space outside some compact set $K \subset M$, i.e., the complement of K is isometric to the complement of a compact set in $\mathbb{R}^n$. If M has non-negative Ricci curvature, show that M is isometric to $\mathbb{R}^n$. (Hint: use the splitting theorem.)