**DIFFERENTIAL GEOMETRY, REVISION SHEET**

**G.P. Paternain Lent 2007**

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

**0**. Do all the problems that you could not do during term because of lack of time.

**1**. For a submanifold $X$ of $Y$, let $\iota : X \to Y$ be the inclusion map. Show that $di_x$ is the inclusion map of $T_xX$ into $T_xY$.

**2**. Let $f : X \to Y$ be a smooth map between manifolds and suppose $f$ is an immersion at $x$. Show that there exist local coordinates around $x$ and $f(x)$ such that in these coordinates $f$ can be expressed as the canonical immersion:

$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, 0, \dots, 0).$$

**3**. Let $Z$ be a submanifold of $X$ with $\dim X = k$ and $\dim Z = l$. Apply the previous problem to the inclusion $\iota : Z \to X$ to show that given $z \in Z$ there exists a local coordinate system $\{x_1, \dots, x_k\}$ defined in a neighbourhood $U$ of $z$ such $Z \cap U$ is defined by the equations $x_{l+1} = 0, \dots, x_k = 0$.

**4**. Show that every connected manifold is path-connected.

**5**. Make sure you know the proofs of the following important theorems that we saw in lectures: the preimage theorem, the degree modulo 2 theorem, the isoperimetric inequality, Gauss-Bonnet Theorem, Theorema Egregium.

**6**. Define minimal surface and explain what is meant by the Weierstrass representation. Give the Weierstrass representation of the Enneper surface.

**7**. (i) Let $\alpha : I \to \mathbb{R}^3$ be a closed regular curve parametrized by arc-length. Define curvature and torsion. Write down the Frenet formulas.

(ii) Suppose now that $\alpha$ has non-zero curvature everywhere. Let $n : I \to S^2 \subset \mathbb{R}^3$ be the curve given by the normal vector $n(s)$ to $\alpha(s)$. Let $\bar{s}$ be the arc-length of the curve $n$ on $S^2$. Show that the geodesic curvature $k_g$ of $n$ is given by

$$k_g = -\frac{d}{ds}\tan^{-1}(\tau/k)\,\frac{ds}{d\bar{s}},$$

where $k$ and $\tau$ are the curvature and torsion of $\alpha$.

**8**. Show that in a system of normal coordinates centered at $p$ (i.e. cartesian coordinates $(x, y)$ in $T_pS$ and parametrization given by $(x, y) \mapsto \exp_p(xe_1 + ye_2)$) , all the Christoffel symbols are zero at $p$.

**9**. What is the exponential map? What are geodesic polar coordinates? Find the equations of the geodesics in geodesic polar coordinates.

**10**. If $p$ is a point of a regular surface $S$, prove that

$$K(p) = \lim_{r\to 0} \frac{12(\pi r^2 - A)}{\pi r^4}$$

where $K(p)$ is the Gaussian curvature at $p$, $r$ is the radius of the geodesic circle centered at $p$ and $A$ is the area of the region bounded by the geodesic circle.

**11**. Define covariant derivative. Show that it only depends on the first fundamental form.

**12**. Compute the Gaussian curvature of the tube from Problem 6 in Example sheet 2.