

# Boundary rigidity for Lagrangian submanifolds, non-removable intersections, and Aubry–Mather theory

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February 19, 2003

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\*Supported by the United States–Israel Binational Science Foundation grant 1999086  
†Supported by a Heisenberg grant of the Deutsche Forschungsgemeinschaft

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# 1 Introduction and results

In the present note we continue a theme which goes back to Arnold's seminal survey "First steps in symplectic topology" [Arn]. A hypersurface in a cotangent bundle is called optical if it bounds a fiberwise strictly convex domain. Likewise, a Lagrangian submanifold is called optical if it lies in an optical hypersurface; a particularly important class of examples is given by invariant tori in classical mechanics. Arnold suggested to look at optical Lagrangian submanifolds from the symplectic topology point of view. Arnold's suggestion inspired a number of results in this direction (see, e.g., [BP2, BP3]).

In this paper, we go a step further and establish a *boundary rigidity* phenomenon which, roughly speaking, can be formulated as follows. Certain Lagrangian submanifolds lying in an optical hypersurface cannot be deformed into the domain bounded by that hypersurface. Furthermore, even when boundary rigidity fails, we often find another phenomenon called *non-removable intersection*: the intersection between the deformed Lagrangian submanifold and the hypersurface contains some distinguished, dynamically relevant set. This observation links the theory of symplectic intersections with modern aspects of dynamical systems.

Finally, we discuss Lagrangian submanifolds lying in the open domain bounded by some optical hypersurface. Although these submanifolds cannot be interpreted as invariant sets anymore, they still appear in a number of interesting situations in geometry and dynamics.

## 1.1 Preliminaries and basic notations

Let  $\theta : T^*X \rightarrow X$  be the cotangent bundle of a closed manifold  $X$ , equipped with the canonical symplectic form  $\omega = d\lambda$  where  $\lambda$  is the Liouville 1-form. We write  $\mathcal{O}$  for the zero section, and denote by  $\mathcal{L}$  the class of all Lagrangian submanifolds of  $T^*X$  which are Lagrangian isotopic to  $\mathcal{O}$ . Given  $\Lambda \in \mathcal{L}$ , the natural projection  $\theta|_{\Lambda} : \Lambda \rightarrow X$  induces an isomorphism between the cohomology groups  $H^1(X, \mathbb{R})$  and  $H^1(\Lambda, \mathbb{R})$ . The preimage of  $[\lambda|_{\Lambda}]$  under this isomorphism is called the *Liouville class* of  $\Lambda$  and is denoted by  $a_{\Lambda} \in H^1(X, \mathbb{R})$ . We say that a Lagrangian submanifold  $\Lambda \in \mathcal{L}$  is *exact* if  $a_{\Lambda} = 0$  and denote by  $\mathcal{L}_0$  the class of all exact Lagrangian submanifolds in  $\mathcal{L}$ .

A smooth, closed, fiberwise strictly convex hypersurface  $\Sigma \subset T^*X$  is called *optical*. Fiberwise strict convexity means that  $\Sigma$  intersects each fiber  $T_x^*X$  along a hypersurface whose second fundamental form is positive definite. Denote by  $\sigma$  the characteristic foliation of  $\Sigma$ , i.e., the 1-dimensional foliation tangent to the kernel of  $\omega|_{T\Sigma}$ . Note that  $\sigma$  is orientable and tangent to each Lagrangian submanifold contained in  $\Sigma$ .

An orientable 1-dimensional foliation on a closed manifold is called *conservative* if it admits a non-vanishing tangent vector field whose flow preserves a measure which is absolutely continuous with respect to some (and hence any) Riemannian measure on that manifold.

Let  $\Lambda \in \mathcal{L}$  be a Lagrangian submanifold lying in an optical hypersurface  $\Sigma$ . Assume, in addition, that the restriction  $\sigma|_{\Lambda}$  of the characteristic foliation is conservative. In this case, one can show that  $\Lambda$  is a *section* of the cotangent bundle; this, a multidimensional version of the Birkhoff second theorem, was established in [BP2]. The assumption on the conservativity of  $\sigma|_{\Lambda}$  can

be somewhat relaxed, but it is still unknown whether it can be omitted completely. Interestingly enough, the same assumption appears in a crucial way in the following, seemingly different context.

## 1.2 Boundary rigidity

Suppose  $\Sigma$  is a hypersurface bounding a domain  $U_\Sigma$  and containing some Lagrangian submanifold  $\Lambda$ . Can one push  $\Lambda$  inside  $U_\Sigma$  by an exact Lagrangian deformation (i.e., a Lagrangian deformation preserving the Liouville class)? We will present situations, both for the convex and nonconvex case, where this is impossible. Sometimes, one cannot even move  $\Lambda$  at all—a phenomenon we call *boundary rigidity*.

### 1.2.1 The convex case

Given an optical hypersurface  $\Sigma$ , we denote by  $U_\Sigma$  the closed domain bounded by  $\Sigma$ .

**Theorem 1.1.** *Let  $\Lambda \in \mathcal{L}$  be a Lagrangian submanifold lying in an optical hypersurface  $\Sigma$ . Assume that the restriction  $\sigma|_\Lambda$  of the characteristic foliation is conservative. Let  $K \in \mathcal{L}$  be any Lagrangian submanifold<sup>1</sup> lying in  $U_\Sigma$  with the same Liouville class  $a_K = a_\Lambda$ . Then  $K = \Lambda$ .*

In particular,  $\Lambda$  cannot be deformed inside  $U_\Sigma$  by an exact Lagrangian isotopy, i.e., by a Lagrangian isotopy that preserves the Liouville class.

Theorem 1.1 is proved in Section 2 below. As the following example shows, the assumption about the dynamics of the characteristic foliation cannot be omitted.

*Example 1.2.* Consider  $\Sigma = \{H = 1\} \subset T^*\mathbb{T}^2$  where

$$H(x_1, x_2, y_1, y_2) = (y_1 - \sin x_1)^2 + (y_2 - \cos x_1)^2. \quad (1)$$

Then  $\Sigma$  contains the zero section  $\Lambda = \mathcal{O}$ . However, the restriction  $\sigma|_\mathcal{O}$  of the characteristic foliation is a Reeb foliation with exactly two limit cycles and, therefore, not conservative. We claim that  $\mathcal{O}$  is not boundary rigid either. Indeed, the exact Lagrangian torus  $K = \text{graph}(df)$  with

$$f(x_1, x_2) = -\cos x_1$$

does lie in  $U_\Sigma$ . It is worth mentioning that  $K$  intersects  $\Sigma$  precisely at the two limit cycles of the characteristic foliation. As we will see in Section 1.3.2, this is no coincidence.

### 1.2.2 The nonconvex case

In this section, we let  $\Sigma \subset T^*X$  be a smooth closed hypersurface which need *not* be strictly convex. Denote by  $\sigma$  the characteristic foliation of  $\Sigma$ .

We introduce the following class  $\mathcal{L}'$  of closed connected Lagrangian submanifolds of  $T^*X$ . We say that  $\Lambda \in \mathcal{L}'$  if there exists a closed 1-form  $\alpha$  on  $X$  such

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<sup>1</sup>We denote Lagrangian submanifolds by Greek letters, so this is a capital  $\kappa$  and not a capital  $k$  ...

that the restriction of  $\lambda - \theta^* \alpha$  to  $\Lambda$  is exact, where  $\theta : T^* X \rightarrow X$  is the natural projection. We call  $a_\Lambda := [\alpha] \in H^1(X, \mathbb{R})$  the Liouville class of  $\Lambda$ . Clearly,  $\mathcal{L} \subset \mathcal{L}'$ , and the new notion of Liouville class agrees with the one given in Section 1.1 for Lagrangian submanifolds in  $\mathcal{L}$ .

Recall that a 1-dimensional foliation on a closed manifold is called *minimal* if its leaves are everywhere dense. The following result is an immediate consequence of Theorem 1.5 in Section 1.3.1.

**Theorem 1.3.** *Let  $\Lambda \in \mathcal{L}$  be a Lagrangian submanifold lying in  $\Sigma$ . Suppose that the restriction  $\sigma|_\Lambda$  of the characteristic foliation is minimal. Let  $K \in \mathcal{L}'$  be any Lagrangian submanifold lying in  $U_\Sigma$  with the same Liouville class  $a_K = a_\Lambda$ . Then  $K = \Lambda$ .*

Comparing Theorem 1.3 to Theorem 1.1, we see that the first is applicable to a wider class of hypersurfaces (it does not require strict convexity); on the other hand, the dynamical assumption on the characteristic foliation is more restrictive.

*Example 1.4.* Consider the case when  $\Sigma$  is the unit sphere bundle of the Euclidean metric on the torus  $\mathbb{T}^n$ , i.e.,  $\Sigma = \{|p| = 1\}$ . Theorem 1.1 yields boundary rigidity for all flat Lagrangian tori  $\{p = v\}$  lying in  $\Sigma$ , while Theorem 1.3 gives boundary rigidity only when the coordinates of  $v$  are rationally independent.

### 1.3 Non-removable intersections

Let  $\Sigma \subset T^* X$  be a hypersurface bounding a closed domain  $U_\Sigma$ . Even if boundary rigidity, as explained in Section 1.2, may fail for a given Lagrangian submanifold  $\Lambda$ , we will see that in many cases the intersection  $\Lambda \cap \Sigma$  cannot be empty. We call this phenomenon *non-removable intersection*. In fact, the intersection will always contain an invariant set of the characteristic foliation of  $\Sigma$ .

#### 1.3.1 The nonconvex case

In this section, we let  $\Sigma \subset T^* X$  be a smooth closed hypersurface which need *not* be strictly convex.

**Theorem 1.5.** *Let  $\Lambda \in \mathcal{L}$  be a Lagrangian submanifold lying in  $\Sigma$ . Let  $K \in \mathcal{L}'$  be any Lagrangian submanifold lying in  $U_\Sigma$  with the same Liouville class  $a_K = a_\Lambda$ . Then the intersection  $K \cap \Lambda$  contains a compact invariant set of the characteristic foliation  $\sigma|_\Lambda$ .*

Let us present the short and illustrative proof right here.

*Proof.* Let  $v$  be a non-vanishing vector field on  $\Sigma$  tangent to the characteristic foliation  $\sigma$ . Arguing by contradiction, we suppose that the intersection  $K \cap \Lambda$  contains no compact invariant set of  $\sigma$ . Then, by a theorem of Sullivan [Sul, LS2], there exists a smooth function  $h : \Sigma \rightarrow \mathbb{R}$  such that  $dh(x) \cdot v > 0$  for every  $x \in K \cap \Lambda$ . Extend  $h$  to a function  $H$  defined near  $\Sigma$ , and denote by  $X_H$  the corresponding Hamiltonian vector field. Note that  $X_H$  is transverse to  $\Sigma$  at each point in  $K \cap \Lambda$ . Changing, if necessary, the sign of  $v$  we can achieve that at these points  $X_H$  is pointing inside the domain  $U_\Sigma$ .

Let  $\phi_t$  be the Hamiltonian flow of  $X_H$ , defined in a neighborhood of  $\Sigma$ . Since  $K \subset U_\Sigma$  and  $\Lambda \subset \Sigma$ , it follows that  $\phi_t(K) \cap \Lambda = \emptyset$ , provided  $t > 0$  is sufficiently

small. But the Lagrangian submanifolds  $\phi_t(K)$  and  $\Lambda$  have equal Liouville classes, and a Lagrangian intersection result due to Gromov[Gro1]—saying that two Lagrangian submanifolds  $\Lambda \in \mathcal{L}$  and  $K \in \mathcal{L}'$  with equal Liouville classes must intersect—gives the desired contradiction.  $\square$

*Remark 1.6.* Note that Theorem 1.5 immediately implies Theorem 1.3 about boundary rigidity. Thus, boundary rigidity is a particular case of non-removable intersections.

In the following, it will be convenient to use the language of symplectic shapes introduced by Sikorav [Sik2, Sik3] and Eliashberg [Eli]. The *shape* of a subset  $U \subset T^*X$  is defined as

$$\text{sh}(U) := \{a_\Lambda \in H^1(X, \mathbb{R}) \mid \Lambda \in \mathcal{L}, \Lambda \subset U\}. \quad (2)$$

Gromov's theorem [Gro1] implies Sikorav's elegant reformulation [Sik2] of Arnold's Lagrangian intersection conjecture proved in [LS1, Hof, Gro1, Che]: shapes of disjoint subsets of  $T^*X$  are disjoint. Therefore, every Lagrangian submanifold  $\Lambda \in \mathcal{L}$  whose Liouville class lies on the boundary  $\partial\text{sh}(U_\Sigma)$  must intersect  $\Sigma$ . Applying Sullivan's theorem exactly as in the proof of Theorem 1.5, one can refine this observation as follows.

**Proposition 1.7.** *Suppose  $\Lambda \in \mathcal{L}$  is a Lagrangian submanifold such that  $\Lambda \subset U_\Sigma$  and  $a_\Lambda \in \partial\text{sh}(U_\Sigma)$ . Then the intersection set  $\Lambda \cap \Sigma$  contains a compact invariant set of the characteristic foliation of  $\Sigma$ .*

It would be interesting to understand, first, the dependence of the intersection set  $K \cap \Sigma$  on the Liouville class  $a_K$  of  $K$ , and, secondly, the dynamical meaning of the invariant set in the non-removable intersection. In fact, for the convex case (i.e., when  $\Sigma$  is optical) the latter can be done using Aubry–Mather theory.

### 1.3.2 The convex case

Suppose  $\Sigma \subset T^*X$  is an optical hypersurface containing a Lagrangian submanifold  $\Lambda \in \mathcal{L}_0$ . As mentioned before, this setting allows a more detailed description of non-removable intersections. Instead of going into technicalities here, let us just illustrate this by an example.

*Example 1.8.* Recall Example 1.2 where  $\Sigma = H^{-1}(1)$  with  $H$  given by (1). The characteristic foliation has exactly two limit cycles, and we constructed a particular Lagrangian submanifold  $K$  such that  $K \cap \Sigma$  consists of those two cycles. In fact, this is no coincidence, as the following result shows.

**Theorem 1.9.** *Let  $\Lambda \in \mathcal{L}_0$  be any Lagrangian submanifold with  $\Lambda \subset U_\Sigma$ . Then  $\Lambda \cap \Sigma$  contains the two limit cycles of the characteristic foliation of  $\Sigma$ .*

Precise results are stated and proved in Section 5. The methods we use are based on Aubry–Mather theory, which will be explained in Section 3.

## 1.4 Symplectic shapes of open fiberwise convex domains

In this section, we focus on Lagrangian submanifolds lying in open, fiberwise convex subsets of some cotangent bundle. Recall the definition of the symplectic

shape for a subset  $U \subset T^*X$  given in (2). In addition, we also define the *sectional shape*

$$\text{sh}_0(U) \subset \text{sh}(U)$$

to be the collection of all  $a \in H^1(X, \mathbb{R})$  such that  $U$  contains a Lagrangian section of  $T^*X$  with Liouville class  $a$  (or, in other words, the graph of a closed 1-form representing the cohomology class  $a$ ). In contrast to the symplectic shape,  $\text{sh}_0(U)$  is, in general, *not* preserved by Hamiltonian diffeomorphisms of  $T^*X$  and, hence, does not belong to the purely symplectic realm. However, it naturally arises and plays a significant role in a number of interesting situations. In fact, this antithesis was the starting point of our present research. It is resolved in a way by the following theorem, which states that for open fiberwise convex sets  $U \subset T^*X$  both notions coincide.

**Theorem 1.10.** *Let  $U \subset T^*X$  be an open fiberwise convex subset. Then every class  $a \in \text{sh}(U)$  can be represented by a Lagrangian section of the cotangent bundle. In other words,*

$$\text{sh}_0(U) = \text{sh}(U).$$

By taking convex combinations, the following is a direct consequence of Theorem 1.10.

**Corollary 1.11.** *The shape of an open fiberwise convex subset of  $T^*X$  is an open convex subset of  $H^1(X, \mathbb{R})$ .*

Note that the shape of an open subset is always open (this follows immediately from Weinstein's Lagrangian neighborhood theorem), so the main statement here is about convexity. The proof of Theorem 1.10 is given in Section 6.

*Example 1.12.* Take a Riemannian metric  $g$  on  $X$  and consider the open unit ball bundle

$$U = \{(x, p) \in T^*X \mid |p|_g < 1\}.$$

There exists a remarkable norm on  $H^1(X, \mathbb{R})$ , called *Gromov–Federer stable norm*. Let us illustrate the corresponding dual norm  $\|A\|$  for a homology class  $A \in H_1(X, \mathbb{Z})$ . Write  $\ell(A)$  for the minimal length of a closed geodesic representing  $A$ . Then

$$\|A\| = \lim_{k \rightarrow \infty} \frac{\ell(kA)}{k}.$$

Gromov showed [Gro2] that the open unit ball of the stable norm coincides with the sectional shape of  $U$ . In view of Theorem 1.10, this is equal to  $\text{sh}(U)$ . Thus, for the Riemannian case, Theorem 1.10 leads to a geometric description of the symplectic shape of a Riemannian unit ball bundle and, vice versa, to a symplectic characterization of the unit stable norm ball.

*Example 1.13.* Let  $H : T^*X \rightarrow \mathbb{R}$  be a fiberwise strictly convex Hamiltonian function. Assume that  $H$  has superlinear growth. Define the function  $\alpha : H^1(X, \mathbb{R}) \rightarrow \mathbb{R}$  by

$$\alpha(a) := \inf\{h \in \mathbb{R} \mid a \in \text{sh}_0(\{H < h\})\}.$$

This function is known as the convex conjugate of the *Mather minimal action* [Mat1]; it was intensively studied in the past decade. Again, Theorem 1.10

translates the variational definition of Mather's minimal action into symplectic language.

As an illustration, consider the value  $\min \alpha$ . It is called *Mañé's strict critical value*. It plays an important role when one studies the dependence of the dynamics in the energy levels  $\{H = h\}$  on the energy value  $h$ . This problem is still far from being solved completely, even in a basic model of the magnetic field on a closed manifold  $X$  [BuPa, PS]. It was proved in [CIPP1] that for  $h > \min \alpha$ , the dynamics in the energy level  $\{H = h\}$  can be seen as a time reparametrization of an appropriate Finsler flow on  $X$ . Mañé's critical value will appear in a crucial way in Sections 3–5 below.

## 2 Graph selectors of Lagrangian submanifolds and boundary rigidity

The main symplectic ingredient of our approach to proving Theorem 1.1 is supplied by the following theorem which was outlined by Sikorav (in a talk held in Chaperon's seminar) and proven by Chaperon (in the framework of generating functions) and Oh (via Floer homology).

**Theorem 2.1 (Sikorav, Chaperon [Cha], Oh [Oh]).** *Let  $\Lambda \subset T^*X$  be a Lagrangian submanifold in  $\mathcal{L}_0$ . Then there exists a Lipschitz continuous function  $\Phi : X \rightarrow \mathbb{R}$ , which is smooth on an open set  $X_0 \subset X$  of full measure, such that*

$$(x, d\Phi(x)) \in \Lambda$$

for every  $x \in X_0$ . Moreover, if  $d\Phi(x) = 0$  for all  $x \in X_0$  then  $\Lambda$  coincides with the zero section  $\mathcal{O}$ .

We call the function  $\Phi$  a *graph selector* of the Lagrangian submanifold  $\Lambda$ . In order to explain this terminology, consider  $\Lambda$  as a multi-valued section of the cotangent bundle. Then the differential  $d\Phi(x)$  selects a single value of this section over the set  $X_0$  in a smooth way.

In the following two sections, we will prove Theorem 2.1 by using generating functions quadratic at infinity, a powerful tool of symplectic topology in cotangent bundles. Although this proof of Theorem 2.1 is well known to experts, we were unable to locate it in the literature.

In the final Section 2.3, we apply Theorem 2.1 in order to prove Theorem 1.1.

### 2.1 Generating functions quadratic at infinity

Let  $X$  be a closed manifold, and  $E$  a finite dimensional real vector space. Denote by  $\mathcal{O}_E$  the zero section of  $T^*E$  and set

$$V := T^*X \times \mathcal{O}_E \subset T^*X \times T^*E = T^*(X \times E).$$

**Definition 2.2.** *A smooth function  $S : X \times E \rightarrow \mathbb{R}$  is called a generating function quadratic at infinity (gfqi) if*

$$S(x, \xi) = Q_x(\xi)$$

outside a compact subset of  $X \times E$ , where  $Q_x$  is a smooth family of nondegenerate quadratic forms on  $E$ , and  $\text{graph}(dS)$  is transversal to  $V$  in  $T^*(X \times E)$ .

In particular,  $W := \text{graph}(dS) \cap V$  is a smooth closed submanifold of  $V$  of the same dimension as  $X$ . Let  $\chi : V \rightarrow T^*X$  be the natural projection. One can show that the restriction of  $\chi$  to  $W$  is a Lagrangian immersion (see [AGV, Sect. 19]). If  $\chi|_W$  is an embedding then

$$\Lambda := \chi(W)$$

is a Lagrangian submanifold of  $T^*X$ . In this case we say that  $S$  is a gfqi of  $\Lambda$ ; this means that

$$\Lambda = \{(x, d_x S(x, \xi)) \mid x \in X, \xi \in E, d_\xi S(x, \xi) = 0\}. \quad (3)$$

Regarding the existence of a gfqi of a given Lagrangian submanifold  $\Lambda$ , it is known that every  $\Lambda \in \mathcal{L}_0$  admits a gfqi [Sik1, Che]. We do not know any existence result for Lagrangian submanifolds  $\Lambda \notin \mathcal{L}_0$ .

## 2.2 The graph selector—proof of Theorem 2.1

Let  $S : X \times E \rightarrow \mathbb{R}$  be a gfqi of a Lagrangian submanifold  $\Lambda \in \mathcal{L}$ . The graph selector is defined by a suitable minimax procedure which we are going to describe now.

Fix a scalar product on  $E$ . Let  $B_x : E \rightarrow E$  be a self-adjoint operator so that  $Q_x(\xi) = (B_x \xi, \xi)$ . Denote by  $E_x^-$  the subspace of  $E$  generated by all eigenvectors of  $B_x$  with negative eigenvalues. Set

$$E_x^a = \{\xi \in E \mid S_x(\xi) \leq a\}$$

where  $a \in \mathbb{R}$  and  $S_x(\cdot) := S(x, \cdot)$ . Take  $N > 0$  so that  $S(x, \xi) = Q_x(\xi)$  whenever  $|Q_x(\xi)| \geq N$ . The quadratic forms  $Q_x$ ,  $x \in X$  have the same index which we denote by  $m$ . The homology group  $H_m(E_x^N, E_x^{-N}; \mathbb{Z}_2)$  is isomorphic to  $\mathbb{Z}_2$ , and the generator, say  $A_x$ , is represented by the  $m$ -dimensional disc in  $E_x^-$  whose boundary lies in  $\{Q_x(\xi) = -N\}$ . For  $a \in [-N, N]$ , consider the natural morphism

$$I_{a,x} : H_m(E_x^a, E_x^{-N}; \mathbb{Z}_2) \rightarrow H_m(E_x^N, E_x^{-N}; \mathbb{Z}_2).$$

**Definition 2.3.** *The function  $\Phi : X \rightarrow \mathbb{R}$  defined by*

$$\Phi(x) := \inf \{a \mid A_x \in \text{Image}(I_{a,x})\}$$

*is called the graph selector of  $\Lambda$  associated to the gfqi  $S$ .*

We claim that  $\Phi$  has the properties stated in Theorem 2.1. Clearly,  $\Phi(x)$  is a critical value of  $S_x$ . Consider the subset  $X_0 \subset X$  consisting of all those  $x$  for which  $S_x$  is a Morse function whose critical points have pairwise distinct critical values.

In a neighborhood  $U$  of any point of  $X_0$  there exists a smooth function  $\varphi : U \rightarrow E$  such that  $\varphi(x)$  is a critical point of  $S_x$  and  $\Phi(x) = S(x, \varphi(x))$ . Differentiating with respect to  $x$  and taking into account that  $d_\xi S(x, \varphi(x)) = 0$  we get that  $d\Phi(x) = d_x S(x, \varphi(x))$ . Thus, in view of (3),  $(x, d\Phi(x)) \in \Lambda$  for all  $x \in X_0$ , so  $\Phi$  is indeed a selector.

**Proposition 2.4.**  *$X_0$  is an open subset of  $X$  of full measure.*

*Proof.* Let  $\theta : T^*X \rightarrow X$  be the natural projection. A simple local analysis shows that  $S_x$  is Morse if and only if  $x$  is a regular value of  $\theta|_{\Lambda}$  (see [AGV], section 21.2). Denote the set of such  $x \in X$  by  $X_1$ . It is an open subset of  $X$ , and by Sard's Theorem it has full measure.

Let  $U \subset X_1$  be a sufficiently small open subset. The critical points of  $S_x$  depend smoothly on  $x \in U$ . Denote them by  $\varphi_1(x), \dots, \varphi_d(x)$ , and put

$$a_{ij}(x) = S(x, \varphi_i(x)) - S(x, \varphi_j(x))$$

for  $i \neq j$ . Note that

$$da_{ij}(x) = d_x S(x, \varphi_i(x)) - d_x S(x, \varphi_j(x)) \neq 0$$

since the map

$$\chi|_W : W \rightarrow T^*X, (x, \xi) \mapsto (x, d_x S(x, \xi))$$

is an embedding. Therefore the sets  $\Gamma_{ij} = \{x \in U \mid a_{ij}(x) = 0\}$  are smooth hypersurfaces. But, by definition of  $X_0$ , we have

$$X_0 \cap U = U \setminus \bigcup_{i \neq j} \Gamma_{ij},$$

so  $X_0 \cap U$  is an open subset of full measure in  $X \cap U$ .  $\square$

**Proposition 2.5.** *If  $d\Phi(x) = 0$  for all  $x \in X_0$  the submanifold  $\Lambda$  coincides with the zero section of  $T^*X$ .*

*Proof.* Identify  $X$  with the zero section of  $T^*X$ . Since  $X_0$  has full measure, its closure equals  $X$ . Hence  $\Lambda$  contains  $X$  since  $d\Phi(x) = 0$  for  $x \in X_0$ , and thus  $\Lambda = X$ .  $\square$

**Proposition 2.6.**  *$\Phi$  is a Lipschitz function on  $X$ .*

*Proof.* Since  $X$  is compact it suffices to prove this locally. Let  $U \subset X$  be a sufficiently small open subset. There exists a smooth family of linear automorphisms  $F_x : E \rightarrow E$ ,  $x \in U$ , and a quadratic form  $Q$  on  $E$ , so that  $Q_x \circ F_x = Q$  for all  $x \in U$ . It is easy to see that the function  $S'(x, \xi) := S(x, F_x \xi)$  is again a gfqi of  $\Lambda$  over  $U$ , whose graph selector coincides with  $\Phi|_U$ . In what follows we work with  $S'$  instead of  $S$ , because the functions  $S'_x$ ,  $x \in U$ , equal the same quadratic form  $Q$  outside a compact subset of  $E$ . Therefore there exists a positive constant  $C$  such that for all  $x, y \in U$  and  $\xi \in E$  we have

$$|S'(x, \xi) - S'(y, \xi)| \leq C|x - y|. \quad (4)$$

Fix  $\epsilon > 0$  and  $x \in U$ , and set

$$a(y) := \Phi(x) + \epsilon + C|x - y|,$$

for all  $y \in U$ . It follows from inequality (4) that  $E_x^{a(x)} \subset E_y^{a(y)}$  for all  $y \in U$ . By definition, the pair  $(E_x^{a(x)}, E^{-N})$  contains a relative cycle representing the class  $A_x$ . Therefore, the same holds for the pair  $(E_y^{a(y)}, E^{-N})$ . We get that  $\Phi(y) \leq a(y)$ , which yields

$$\Phi(y) - \Phi(x) \leq C|x - y| + \epsilon.$$

Since the last inequality is valid for each  $\epsilon > 0$  we have

$$\Phi(y) - \Phi(x) \leq C|x - y|.$$

Finally, interchanging  $x$  and  $y$  we get that  $\Phi$  is Lipschitz continuous.  $\square$

Thus, the function  $\Phi$  satisfies all requirements of a graph selector. This finishes the proof of Theorem 2.1.  $\square$

### 2.3 Proof of Theorem 1.1

Let  $\Lambda \in \mathcal{L}$  be a Lagrangian submanifold lying in some optical hypersurface  $\Sigma$ , and assume that the restriction  $\sigma|_{\Lambda}$  of the characteristic foliation is conservative. Let  $K \in \mathcal{L}$  be any Lagrangian submanifold lying in  $U_{\Sigma}$  with the same Liouville class. We want to prove that  $K = \Lambda$ .

By the multidimensional Birkhoff theorem [BP2],  $\Lambda$  is a Lagrangian section, i.e.,  $\Lambda = \text{graph}(\alpha)$  for some closed 1-form  $\alpha$ . By applying the symplectic shift  $(x, p) \mapsto (x, p - \alpha(x))$  we may assume that  $\Lambda = \mathcal{O}$  is the zero section. Note that the transformed hypersurface remains optical.

Suppose now there is another Lagrangian submanifold  $K \subset U_{\Sigma}$ , obtained from  $\Lambda$  by an exact Lagrangian deformation. Let  $\Phi : X \rightarrow \mathbb{R}$  be a graph selector of  $K$  so that  $(x, d\Phi(x)) \in K$  for all  $x \in X_0$ , where  $X_0 \subset X$  is a set of full measure as in Theorem 2.1.

Pick a smooth Hamiltonian function  $H : T^*X \rightarrow \mathbb{R}$  which is fiberwise strictly convex such that  $\Sigma$  is a regular level set of  $H$ . Since  $\Lambda = \mathcal{O}$  the vector  $\frac{\partial H}{\partial p}(x, 0)$  gives the outer normal direction to the hypersurface  $\Sigma \cap T_x^*X \subset T_x^*X$ . Because  $K \subset U_{\Sigma}$ , we have

$$d\Phi(x) \cdot \frac{\partial H}{\partial p}(x, 0) < 0 \quad (5)$$

in local canonical coordinates  $(x, p)$  for all  $x \in X_0$  with  $d\Phi(x) \neq 0$ .

Let  $v$  be a non-singular vector field on  $\Lambda$  which is tangent to the characteristic foliation, and whose flow  $\psi_s$  preserves a measure  $\mu$  which is absolutely continuous with respect to some Riemannian measure. Then the Hamiltonian differential equations for  $H$  show that  $v$  is collinear to the vector field  $\frac{\partial H}{\partial p}(x, 0)$  on  $\Lambda$ . In view of (5), we may assume that

$$d\Phi(x) \cdot v(x) < 0 \quad (6)$$

for all  $x \in X_0$  with  $d\Phi(x) \neq 0$ .

On the other hand, we claim that

$$\int_{X_0} d\Phi(x) \cdot v(x) d\mu(x) = 0. \quad (7)$$

Note that the theorem is an immediate consequence of (7). Indeed, combining (7) with (6) we see that  $d\Phi$  must vanish on  $X_0$ , and hence

$$K = \mathcal{O} = \Lambda$$

in view of Theorem 2.1.

It remains to prove formula (7). Since the function  $\Phi$  is Lipschitz continuous, the function  $s \mapsto \Phi(\psi_s x) - \Phi(x)$  on  $[0, 1]$  is also Lipschitz continuous for every  $x \in X$ . By Rademacher's theorem, it is differentiable almost everywhere with

$$\Phi(\psi_1 x) - \Phi(x) = \int_0^1 \frac{d}{ds} \Phi(\psi_s x) ds.$$

Since the flow  $\psi_s$  preserves the measure  $\mu$  we have

$$0 = \int_X [\Phi(\psi_1 x) - \Phi(x)] d\mu(x) = \int_X \int_0^1 \frac{d}{ds} \Phi(\psi_s x) ds d\mu(x).$$

Since  $X_0$  has full measure with respect to  $\mu$  and since  $\psi_s$  preserves  $\mu$ , we have

$$\begin{aligned} 0 &= \int_0^1 \int_X \frac{d}{ds} \Phi(\psi_s x) d\mu(x) ds \\ &= \int_0^1 \int_{\psi_s^{-1}(X_0)} d\Phi(\psi_s x) \cdot v(\psi_s x) d\mu(x) ds \\ &= \int_0^1 \int_{X_0} d\Phi(x) \cdot v(x) d\mu(x) ds \\ &= \int_{X_0} d\Phi(x) \cdot v(x) d\mu(x). \end{aligned}$$

This proves (7) and finishes the proof of the theorem.  $\square$

### 3 Brief summary of Aubry–Mather theory

In this section, we give a brief overview about what is known as Aubry–Mather theory. We refer the reader to the books [CI, Fa1] for various preliminaries related to the material presented here.

#### 3.1 Mañé's critical value

Let  $X$  be a closed connected smooth manifold and let  $L : TX \rightarrow \mathbb{R}$  be a smooth, fiberwise convex, superlinear Lagrangian<sup>2</sup>. This means that  $L$  restricted to each  $T_x X$  has positive definite Hessian and for some Riemannian metric we have

$$\lim_{|v| \rightarrow \infty} \frac{L(x, v)}{|v|} = \infty$$

uniformly on  $x \in X$ . Let  $H : T^* X \rightarrow \mathbb{R}$  be the Hamiltonian associated to  $L$  and

$$\ell : TX \rightarrow T^* X$$

be the Legendre transform  $\ell : (x, v) \mapsto \frac{\partial L(x, v)}{\partial v}$ . Since  $X$  is compact, the extremals of  $L$  give rise to a complete flow  $\phi_t$  on  $TX$ , called the Euler–Lagrange flow of the Lagrangian. Using the Legendre transform we can push forward  $\phi_t$  to obtain another flow  $\phi_t^*$  on  $T^* X$  which is the Hamiltonian flow of  $H$  with

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<sup>2</sup>We always distinguish between the term “Lagrangian” (i.e., Lagrangian function) and “Lagrangian submanifold”.

respect to the canonical symplectic structure of  $T^*X$ . The energy of  $L$  is the function  $E : TX \rightarrow \mathbb{R}$  given by

$$E(x, v) := \frac{\partial L}{\partial v}(x, v) \cdot v - L(x, v) = H(\ell(x, v)).$$

The energy  $E$  is a first integral of the Euler–Lagrange flow  $\phi_t$ .

Recall that the  $L$ –action of an absolutely continuous<sup>3</sup> curve  $\gamma : [a, b] \rightarrow X$  is defined by

$$A_L(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t)) dt.$$

Given two points  $x_1, x_2 \in X$  and some  $T > 0$  denote by  $\mathcal{C}_T(x_1, x_2)$  the set of absolutely continuous curves  $\gamma : [0, T] \rightarrow M$  with  $\gamma(0) = x_1$  and  $\gamma(T) = x_2$ . For each  $k \in \mathbb{R}$ , we define

$$\Phi_k(x_1, x_2; T) := \inf\{A_{L+k}(\gamma) \mid \gamma \in \mathcal{C}_T(x_1, x_2)\}.$$

The *action potential*  $\Phi_k : X \times X \rightarrow \mathbb{R} \cup \{-\infty\}$  of  $L$  is defined by

$$\Phi_k(x_1, x_2) := \inf_{T>0} \Phi_k(x_1, x_2; T).$$

**Definition 3.1 (Mañé).** *The critical value of  $L$  is the real number*

$$c = c(L) := \inf\{k \in \mathbb{R} \mid \Phi_k(x, x) > -\infty \text{ for some } x \in X\}.$$

Note that actually  $\Phi_k(x, x) > -\infty$  for all  $x \in X$ . Since  $L$  is convex and superlinear, and  $X$  is compact, such a number exists. It singles out the energy level in which relevant globally action–minimizing orbits and/or measures live [D, Ma1, CDI, Fa1]. Their study has a long history that goes back M. Morse and G.A. Hedlund; recently, there has been a great deal of activity on this subject, cf. [BP1, Ba1, Ba2, BuPa, D, Fa1, Ma1, Ma2, Mat1, Mat2, PS, Sib].

The critical value can be characterized in a variety of ways [Ma1, CDI, CIPP1, CIPP2]. Each of these characterizations gives a new insight into geometry and dynamics. Let us explain first the relation of the critical value with Mather’s theory of minimizing measures [Mat1].

Let  $\mathcal{P}(L)$  be the set of Borel probability measures on  $TX$  that have compact support and are invariant under the Euler–Lagrange flow  $\phi_t$ . Mañé [Ma1, CDI] showed that the critical value can be described in terms of measures as

$$c(L) = -\min \left\{ \int L d\mu \mid \mu \in \mathcal{P}(L) \right\}. \quad (8)$$

We will say that  $\mu \in \mathcal{P}(L)$  is a *minimizing measure* if  $\mu$  realizes the minimum in (8). The *Mather set* in  $TX$  is defined as

$$\tilde{\mathcal{M}} := \overline{\bigcup_{\mu} \text{supp}(\mu)},$$

where  $\text{supp}(\mu)$  is the support of the measure  $\mu$ , the bar denotes the closure of a set, and the union is taken over all minimizing measures. Mather’s Lipschitz

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<sup>3</sup>A curve  $\gamma : [a, b] \rightarrow X$  is called absolutely continuous if for every  $\epsilon > 0$  there exists  $\delta > 0$  so that for each finite collection of pairwise disjoint open intervals  $(s_i, t_i)$  in  $[a, b]$  of total length  $< \delta$  one has  $\sum_{i=1}^N \text{dist}(\gamma(t_i), \gamma(s_i)) < \epsilon$ . Here  $\text{dist}$  is any Riemannian distance on  $X$ .

graph theorem [Mat1] asserts that  $\tilde{\mathcal{M}}$  is a Lipschitz graph with respect to the canonical projection  $\tau : TX \rightarrow X$ . We call  $\mathcal{M} := \tau(\tilde{\mathcal{M}}) \subset X$  the *projected Mather set*. It is known that  $\tilde{\mathcal{M}}$  is contained in the energy level  $E^{-1}(c)$  [D]. We define the *Mather set*  $\tilde{\mathcal{M}}^*$  in  $T^*X$  as the image of  $\tilde{\mathcal{M}}$  under the Legendre transform.

It turns out the critical value  $c(L)$  can be recovered purely from the Hamiltonian as the following result obtained in [CIPP1] (and also independently by Fathi) shows. Namely, we have

$$c = c(H) = \inf_{u \in C^\infty(X, \mathbb{R})} \max_{x \in X} H(x, du(x)). \quad (9)$$

In fact, Theorem 1.10 gives a new, more geometric way of looking at this quantity (cf. Example 1.13). It implies that

$$c = \inf_{\Lambda \in \mathcal{L}_0} \max_{(x, p) \in \Lambda} H(x, p) \quad (10)$$

where, as usual,  $\mathcal{L}_0$  denotes the class of exact Lagrangian submanifolds in  $\mathcal{L}$ .

### 3.2 Weak KAM solutions and Peierls barrier

Given a continuous function  $u : X \rightarrow \mathbb{R}$ , we write

$$u \prec L + c$$

whenever  $u(x) - u(y) \leq \Phi_c(y, x)$  for all  $x, y \in X$ . Here,  $\Phi_c$  is the action potential for the critical value  $c$ .

*Remark 3.2.* The condition  $u \prec L + c$  is actually equivalent to  $u$  being Lipschitz and  $H(x, du(x)) \leq c$  for almost every  $x \in X$  [Fa1, Thm. 4.2.10 & Lemma 4.2.11]. Recall that by Rademacher's theorem, Lipschitz functions are differentiable almost everywhere.

We say that a continuous function  $u_+ : X \rightarrow \mathbb{R}$  is a *positive weak KAM solution* if  $u_+$  satisfies the following two conditions:

1.  $u_+ \prec L + c$ ;
2. for all  $x \in X$ , there exists a absolutely continuous curve  $\gamma_+^x : [0, \infty) \rightarrow X$  such that  $\gamma_+^x(0) = x$  and

$$u_+(\gamma_+^x(t)) - u_+(x) = \int_0^t (L + c)(\gamma_+^x(s), \dot{\gamma}_+^x(s)) ds$$

for all  $t \geq 0$ .

Similarly we say that a continuous function  $u_- : X \rightarrow \mathbb{R}$  is a *negative weak KAM solution* if  $u_-$  satisfies the following two conditions:

1.  $u_- \prec L + c$ ;
2. for all  $x \in X$ , there exists an absolutely continuous curve  $\gamma_-^x : (-\infty, 0] \rightarrow X$  such that  $\gamma_-^x(0) = x$  and

$$u_-(x) - u_-(\gamma_-^x(-t)) = \int_{-t}^0 (L + c)(\gamma_-^x(s), \dot{\gamma}_-^x(s)) ds$$

for all  $t \geq 0$ .

Fathi's weak KAM theorem asserts that positive and negative weak KAM solutions always exist [Fa1]. At any point  $x$  of differentiability of a weak KAM solution  $u$ , Conditions 1 and 2 imply that

$$H(x, du(x)) = c.$$

In fact, the points  $x$  of differentiability of  $u_+$  (resp.  $u_-$ ) are precisely those for which the curve  $\gamma_+^x$  (resp.  $\gamma_-^x$ ) is unique.

We denote by  $\mathcal{S}_\pm$  the set of all positive (respectively, negative) weak KAM solutions. A pair of functions  $(u_-, u_+)$  is said to be *conjugate* if  $u_\pm \in \mathcal{S}_\pm$  and  $u_- = u_+$  on the projected Mather set  $\mathcal{M}$ . We will need the following result.

**Theorem 3.3 ([Fa1, Thm 5.1.2]).** *If  $u : X \rightarrow \mathbb{R}$  is a function such that  $u \prec L + c$ , then there exists a unique pair of conjugate functions  $(u_-, u_+)$  such that  $u_+ \leq u \leq u_-$ .*

The *Peierls barrier* [Mat2] is the function  $h : X \times X \rightarrow \mathbb{R}$  defined by

$$h(x, y) := \liminf_{T \rightarrow \infty} \Phi_c(x, y; T).$$

The function  $h$  is Lipschitz (cf. [Fa1, Corollary 5.3.3]) and, obviously, satisfies  $h(x, y) \geq \Phi_c(x, y)$ . The Peierls barrier can be recovered from the weak KAM solutions or the action potential (cf. [CIPP2, Prop. 13]). Corollary 5.3.7 in [Fa1] gives

$$h(x, y) = \max_{(u_-, u_+)} (u_-(y) - u_+(x)) \tag{11}$$

where the maximum is taken over all pairs  $(u_-, u_+)$  of conjugate functions.

### 3.3 The Aubry set

By definition, two conjugate functions  $u_\pm$  coincide on the projected Mather set  $\mathcal{M}$ . It turns out, however, that in general there is a bigger set, called Aubry set, with this property. Namely, setting

$$\mathcal{I}_{(u_-, u_+)} := \{x \in X \mid u_-(x) = u_+(x)\},$$

we can define

$$\mathcal{A} := \bigcap_{(u_-, u_+)} \mathcal{I}_{(u_-, u_+)}$$

where the intersection is taken over all pairs of conjugate functions. This set is the *projected Aubry set*. Clearly, it contains the projected Mather set  $\mathcal{M}$ .

In order to define the Aubry set in  $T^*X$ , we note that the functions  $u_-$  and  $u_+$  are differentiable at every point  $x \in \mathcal{I}_{(u_-, u_+)}$  with the same derivative. Moreover, the map  $\mathcal{I}_{(u_-, u_+)} \ni x \mapsto du_-(x) = du_+(x) \in T^*X$  is Lipschitz continuous. This was proved by Fathi [Fa1, Thm. 5.2.2]. That map defines a set

$$\tilde{\mathcal{I}}_{(u_-, u_+)} \subset T^*X$$

that projects injectively onto  $\mathcal{I}_{(u_-, u_+)}$  and contains the Mather set. The *Aubry set* in  $T^*X$  is defined as

$$\tilde{\mathcal{A}}^* := \bigcap_{(u_-, u_+)} \tilde{\mathcal{I}}_{(u_-, u_+)},$$

where, again, the intersection is taken over all pairs  $(u_-, u_+)$  of conjugate functions. It turns out that  $\mathcal{A} = \theta(\tilde{\mathcal{A}}^*)$ . As usual, we denote the preimage of  $\tilde{\mathcal{A}}^*$  under the Legendre transform by  $\tilde{\mathcal{A}}$  and call it the Aubry set in  $TX$ . The sets  $\tilde{\mathcal{M}}$  and  $\tilde{\mathcal{A}}$  are compact and invariant under the Euler–Lagrange flow  $\phi_t$ .

It turns out that the Aubry set consists of a distinguished kind of orbits. To make this precise, we say that an absolutely continuous curve  $\gamma : [a, b] \rightarrow X$  is *semistatic* if

$$A_{L+c}(\gamma|_{[s,t]}) = \Phi_c(\gamma(s), \gamma(t))$$

for all  $a \leq s \leq t \leq b$ . Semistatic curves are solutions of the Euler–Lagrange equation because of their minimizing properties. Also it is not hard to check that semistatic curves have energy precisely  $c$  [Ma1, CDI]. We say that an absolutely continuous curve  $\gamma : [a, b] \rightarrow X$  is *static* if it is semistatic and

$$\Phi_c(\gamma(s), \gamma(t)) + \Phi_c(\gamma(t), \gamma(s)) = 0$$

for all  $a \leq s \leq t \leq b$ . The notions of static and semistatic curves are closely related to Mather’s notions of  $c$ –minimal trajectories and regular  $c$ –minimal trajectories [Mat2].

**Proposition 3.4.** *The Aubry set  $\tilde{\mathcal{A}}$  consists precisely of those orbits whose projections to  $X$  are static curves.*

This is well known to experts; nevertheless, we include its proof for the sake of completeness.

*Proof.* Take  $(x, v) \in \tilde{\mathcal{A}}$ . We show that  $\gamma(t) = \tau(\phi_t(x, v))$  is a static curve. By the definition of  $\tilde{\mathcal{A}}$  and Theorem 5.2.2 in [Fa1], we have for any pair  $(u_-, u_+)$  of conjugate functions that

$$u_+(\gamma(t)) - u_-(\gamma(s)) = u_+(\gamma(t)) - u_+(\gamma(s)) = A_{L+c}(\gamma|_{[s,t]})$$

for all  $s \leq t$ . Using (11) we can choose a pair  $(u_-, u_+)$  of conjugate functions for which the Peierls barrier  $h$  satisfies

$$h(\gamma(t), \gamma(s)) = u_-(\gamma(s)) - u_+(\gamma(t)).$$

Therefore, we can estimate

$$A_{L+c}(\gamma|_{[s,t]}) + \Phi_c(\gamma(t), \gamma(s)) \leq u_+(\gamma(t)) - u_-(\gamma(s)) + h(\gamma(t), \gamma(s)) = 0. \quad (12)$$

It is easy to show that  $\Phi_c$  satisfies the triangle inequality

$$\Phi_c(x, y) \leq \Phi_c(x, z) + \Phi_c(z, y)$$

for all  $x, y, z \in X$ , as well as  $\Phi_c(x, x) = 0$  for all  $x \in X$ . Hence we have

$$\begin{aligned} 0 &= \Phi_c(\gamma(s), \gamma(s)) \\ &\leq \Phi_c(\gamma(s), \gamma(t)) + \Phi_c(\gamma(t), \gamma(s)) \\ &\leq A_{L+c}(\gamma|_{[s,t]}) + \Phi_c(\gamma(t), \gamma(s)) \\ &\leq 0 \end{aligned}$$

in view of (12). This implies that  $\gamma$  is a static curve.

Suppose now that  $\gamma : \mathbb{R} \rightarrow X$  is a static curve. Then  $\gamma$  is a semistatic curve with energy  $c$  and given  $s < t$  and  $\epsilon > 0$ , there exists a curve  $\bar{\gamma}$  connecting  $\gamma(t)$  to  $\gamma(s)$  such that

$$A_{L+c}(\gamma|_{[s,t]}) + A_{L+c}(\bar{\gamma}) \leq \epsilon.$$

Looking at the loop formed by  $\gamma|_{[s,t]}$  and  $\bar{\gamma}$ , we conclude that  $h(\gamma(t), \gamma(t)) \leq 0$ . But  $h(x, x) \geq \Phi_c(x, x) = 0$  for all  $x \in X$ , and hence  $h(\gamma(t), \gamma(t)) = 0$ . It follows from (11) that  $\gamma(t) \in \mathcal{A}$ , and thus  $(\gamma(t), \dot{\gamma}(t)) \in \tilde{\mathcal{A}}$ , as we wanted to prove.  $\square$

## 4 Minimizing optical hypersurfaces

In this section, we show that many of the concepts that we presented in Section 3 do not really depend on the Lagrangian (or the Hamiltonian), but can rather be formulated in the more general framework of optical hypersurfaces.

Let  $\theta : T^*X \rightarrow X$  be the cotangent bundle of a closed manifold  $X$ , equipped with the canonical symplectic form  $\omega = d\lambda$ , where  $\lambda$  is the Liouville 1-form. Let  $\Sigma \subset T^*X$  be an optical hypersurface. Denote by  $\sigma$  its characteristic foliation. Recall that  $\sigma$  is orientable and we choose the orientation defined by the Hamiltonian vector field of any Hamiltonian function which is fiberwise strictly convex and has  $\Sigma$  as a regular level set. Denote by  $U_\Sigma$  the closed domain bounded by  $\Sigma$ .

**Definition 4.1.** *An optical hypersurface  $\Sigma$  is minimizing if the interior of  $U_\Sigma$  does not contain a Lagrangian submanifold from  $\mathcal{L}_0$ , but any open neighborhood of  $U_\Sigma$  does.*

*Remark 4.2.* 1. Theorem 1.10 ensures that we may replace “Lagrangian submanifold from  $\mathcal{L}_0$ ” by “Lagrangian section from  $\mathcal{L}_0$ ”, and obtain precisely the same concept.

2. Suppose  $\Sigma$  is a minimizing optical hypersurface, and pick any convex superlinear Hamiltonian  $H$  that has  $\Sigma$  as a regular level set  $H^{-1}(h)$ . Then, in view of (10),  $h = c(H)$  is the Mañé critical value of  $H$ .
3. If an optical hypersurface  $\Sigma$  contains a Lagrangian submanifold  $\Lambda \in \mathcal{L}_0$  (e.g., an exact Lagrangian section) then  $\Sigma$  is minimizing. Indeed, by Gromov’s theorem [Gro1], any two Lagrangian submanifolds  $\Lambda, K \in \mathcal{L}_0$  must intersect, so the interior of  $U_\Sigma$  cannot contain an element of  $\mathcal{L}_0$ .

The converse is certainly not true. However, Fathi explained [Fa2] that if  $\Sigma$  is minimizing and every open neighborhood of  $\Sigma$  contains a  $C^\infty$  exact Lagrangian section, then  $\Sigma$  contains a  $C^1$  exact Lagrangian section. On the other hand, Fathi and Siconolfi [FS] recently proved that there always exists a  $C^1$  function  $f : X \rightarrow \mathbb{R}$  such that, first,  $(x, df(x)) \in U_\Sigma$  for all  $x \in X$  and, secondly,  $(x, df(x)) \in \Sigma$  if and only if  $x \in \mathcal{A}$ .

The last remark prompts the following question.

**Question 4.3.** *Suppose an optical hypersurface  $\Sigma$  contains an exact Lagrangian submanifold  $\Lambda \notin \mathcal{L}_0$ . Is  $\Sigma$  minimizing?*

*Remark 4.4.* The answer is “Yes” if  $K$  admits a generating function quadratic at infinity (see Definition 2.2). Indeed, in this case, Theorem 2.1 guarantees the existence of a graph selector, and Theorem 1.10 shows that every neighborhood of  $U_\Sigma$  does contain a Lagrangian submanifold (even a section) from  $\mathcal{L}_0$ .

In the following, we are going to replace the concept of minimizing measure for a convex Lagrangian  $L$  by a notion that depends only on the foliation  $\sigma$  of an energy surface, and not on the particular choice of  $L$ . The appropriate notion is that of *foliation cycle* introduced by D. Sullivan in [Sul]. We briefly review these ideas.

Let  $M$  be a closed  $n$ -dimensional manifold and let  $\Omega_p$  be the real vector space of smooth  $p$ -forms on  $M$ . This vector space has a natural topology which makes it a locally convex linear space. A continuous linear functional  $f : \Omega_p \rightarrow \mathbb{R}$  is called a  $p$ -current. Let  $\mathcal{D}_p = (\Omega_p)^*$  be the real vector space of all  $p$ -currents. With a natural topology,  $\mathcal{D}_p$  also becomes a locally convex linear space. Given a  $p$ -current  $f$ , we define its boundary  $\partial f$  as the  $(p-1)$ -current such that  $\partial f(\omega) = f(d\omega)$  for all  $\omega \in \Omega_{p-1}$ . Currents with zero boundary are called cycles.

Among the set of all 1-currents, Sullivan considers a distinguished subset that he calls *foliation currents*. This subset is defined as follows. Given  $x \in M$ , let  $\delta_x : \Omega_1 \rightarrow \mathbb{R}$  be the Dirac 1-current defined by  $\delta_x(\omega) := \omega_x(V(x))$ . By definition, foliation currents are the elements of the closed convex cone in  $\mathcal{D}_1$  generated by all the Dirac currents. A *foliation cycle* is a foliation current whose boundary is zero.

Suppose now that  $V$  is a non-vanishing vector field on  $M$ . Then  $V$  defines a map  $\mu \mapsto f_{V,\mu}$  from measures to 1-currents, given by

$$f_{V,\mu}(\omega) := \int_M \omega(V) d\mu.$$

Sullivan [Sul, Prop. II.24] shows that this map yields continuous bijections between

1. nonnegative measures on  $M$  and foliation currents;
2. measures on  $M$ , invariant under the flow of  $V$ , and foliation cycles.

In our setting,  $M$  is a minimizing optical hypersurface  $\Sigma \subset T^*X$ . Pick some fiberwise convex, superlinear Hamiltonian  $H$  such that  $\Sigma = H^{-1}(h)$  is a regular level set, and let  $L$  be the corresponding Lagrangian. In view of Remark 4.2, we have  $h = c$ . The following simple observation allows us to translate the notion of minimizing measure into the language of foliation cycles of the characteristic foliation. Namely, if  $(x, v)$  is a point in the critical energy level  $E^{-1}(c) \subset TX$  then

$$L(x, v) + c = \lambda(d\ell(V(x, v))), \quad (13)$$

where  $\lambda$  is the Liouville form,  $V$  the Euler–Lagrange vector field, and  $\ell$  the Legendre transform. Now, by (8), an invariant measure  $\mu$  is minimizing if  $\int_{TX} (L + c) d\mu = 0$ . We also know from [D] that minimizing measures have their support contained in the energy level  $E^{-1}(c)$ . Hence, the correct translation of the notion of minimizing measures into the language of foliation cycles is the following.

**Definition 4.5.** *Let  $\Sigma$  be a minimizing optical hypersurface in  $T^*X$ , and  $\sigma$  its characteristic foliation. A foliation cycle  $f$  of  $\sigma$  is called minimizing if, and only if,  $f(\lambda) = 0$ .*

In other words, minimizing foliation cycles are precisely those which can be represented by measures  $\ell_*\mu$  on  $T^*X$ , where  $\mu$  is some minimizing measure for some Hamiltonian  $H$  with  $\Sigma = H^{-1}(h)$ . Observe also that, if we have two Hamiltonians  $H_1, H_2$  with the same regular level set  $\Sigma$ , and two minimizing measures  $\mu_1, \mu_2$  of  $H_1, H_2$  representing the same foliation cycle  $f$ , then the supports  $\mu_1$  and  $\mu_2$  will coincide. Hence it makes sense to talk about the support of a foliation cycle  $f$  of  $\sigma$ .

Now, the *Mather set* of  $\Sigma$  is defined as the closure of the union of the supports of all minimizing foliation cycles; it coincides with the Mather set  $\tilde{\mathcal{M}}^*$  in  $T^*X$  of any convex superlinear Hamiltonian  $H$  having  $\Sigma$  as regular level set.

In order to go further and define the Aubry set of  $\Sigma$ , we first have to explain what a weak KAM solution should be in our setting. Given a point  $(x, p) \in \Sigma$ , let  $\Gamma^\pm(x, p)$  be the oriented positive (respectively, negative) half of the leaf  $\Gamma_{(x, p)}$  of  $\sigma$  through  $(x, p)$ .

**Definition 4.6.** *Let  $\Sigma$  be a minimizing optical hypersurface in  $T^*X$ . A function  $u_+ : X \rightarrow \mathbb{R}$  is called a positive weak KAM solution of  $\Sigma$  if the following two conditions hold:*

1.  $u_+$  is Lipschitz, and  $(x, du_+(x)) \in U_\Sigma$  for almost every  $x \in X$ ;
2. for every  $x \in X$ , there exists  $(x, p) \in \Sigma$  such that, if  $(y, p')$  is any point in  $\Gamma_{(x, p)}^+$ , then

$$u_+(y) - u_+(x) = \int_{\Gamma_{(x, p)}^+(y, p')} \lambda,$$

where  $\Gamma_{(x, p)}^+(y, p')$  is the oriented part of the leaf between  $(x, p)$  and  $(y, p')$ .

Similarly, a function  $u_- : X \rightarrow \mathbb{R}$  is called a negative weak KAM solution of  $\Sigma$  if the following two conditions hold:

1.  $u_-$  is Lipschitz, and  $(x, du_-(x)) \in U_\Sigma$  for almost every  $x \in X$ ;
2. for every  $x \in X$ , there exists  $(x, p) \in \Sigma$  such that, if  $(y, p')$  is any point in  $\Gamma_{(x, p)}^-$ , then

$$u_-(x) - u_-(y) = \int_{\Gamma_{(x, p)}^-(y, p')} \lambda,$$

where  $\Gamma_{(x, p)}^-(y, p')$  is the oriented part of the leaf between  $(y, p')$  and  $(x, p)$ .

Again, (13) shows that the sets  $\mathcal{S}_\pm = \mathcal{S}_\pm(\Sigma)$  of positive (respectively, negative) weak KAM solutions depend only on  $\Sigma$  and not on the particular choice of  $H$  (or  $L$ ). Setting

$$\mathcal{I}_{(u_-, u_+)} := \{x \in X \mid u_-(x) = u_+(x)\}$$

for a pair of conjugate functions, we see as before that the functions  $u_\pm$  are differentiable on  $\mathcal{I}_{(u_-, u_+)}$  with the same derivative. Therefore, the map  $x \mapsto du_-(x) = du_+(x)$  defines a set  $\tilde{\mathcal{I}}_{(u_-, u_+)}$  in  $T^*X$  that contains the Mather set of  $\Sigma$ . The *Aubry set* of  $\Sigma$  in  $T^*X$  is then given by

$$\tilde{\mathcal{A}}^* = \tilde{\mathcal{A}}^*(\Sigma) = \bigcap_{(u_-, u_+)} \tilde{\mathcal{I}}_{(u_-, u_+)},$$

where the intersection is taken over all pairs  $(u_-, u_+)$  of conjugate functions.

Having defined the Aubry set, one would now like to study the dynamics on it and single out a certain dynamically relevant set inside the Aubry set. For this, we need the following general definition.

**Definition 4.7.** *Let  $\phi_t$  be a continuous flow on a compact metric space  $(X, d)$ . Given  $\epsilon > 0$  and  $T > 0$ , a strong  $(\epsilon, T)$ -chain joining  $x$  and  $y$  in  $X$  is a finite sequence  $\{(x_i, t_i)\}_{i=1}^n \subset X \times \mathbb{R}$  such that  $x_1 = x, x_{n+1} = y, t_i > T$  for all  $i$ , and  $\sum_{i=1}^n d(\phi_{t_i}(x_i), x_{i+1}) < \epsilon$ .*

*A point  $x \in X$  is said to be strongly chain recurrent if for all  $\epsilon > 0$  and  $T > 0$ , there exists a strong  $(\epsilon, T)$ -chain that begins and ends in  $x$ .*

We denote by  $\mathcal{R}$  the set of all strong chain recurrent points. It contains the nonwandering set<sup>4</sup>, but it is easy to give examples showing that it could be strictly larger. The notion of strong chain recurrence strengthens the usual notion of chain recurrence where one requires only  $d(\phi_{t_i}(x_i), x_{i+1}) < \epsilon$  for every single  $i$ . Strong chain recurrence was probably first considered by R. Easton in [Eas].

Given a smooth orientable 1-dimensional foliation  $\sigma$  on a closed manifold, the strong chain recurrent set of  $\sigma$  is the strong chain recurrent set of the flow of any non-vanishing vector field  $V$  tangent to  $\sigma$ . In the case where  $\sigma$  is the characteristic foliation of a hypersurface  $\Sigma \subset T^*X$ , we denote by  $\mathcal{R}^*(\sigma) \subset \Sigma$  the strong chain recurrent set in  $T^*X$ , and by  $\mathcal{R}(\sigma) \subset TX$  its preimage under the Legendre transform.

**Theorem 4.8.** *Let  $\Sigma$  be a minimizing hypersurface in  $T^*X$ , and let  $K \subset \Sigma$  be an exact Lagrangian submanifold (not necessarily in  $\mathcal{L}$ ). Then*

$$\mathcal{R}^*(\sigma|_K) \subset \tilde{\mathcal{A}}^*(\Sigma).$$

*In particular,  $\mathcal{R}^*(\sigma|_K)$  is a Lipschitz graph over  $X$ .*

*Proof.* Choose a convex superlinear Hamiltonian  $H$  which has  $\Sigma$  as a regular level set and let  $L$  be its associated Lagrangian. For the proof, we will work on  $TX$ . Endow  $TX$  and  $X$  with auxiliary Riemannian distances  $d_{TX}$  and  $d_X$  so that the natural projection  $\tau : TX \rightarrow X$  does not increase the distances. Consider  $(x, v) \in \mathcal{R}$  and let  $\gamma(t) = \tau(\phi_t(x, v))$ . In view of Proposition 3.4 it suffices to show that the curve  $\gamma$  is static. This will imply that  $\mathcal{R} \subset \tilde{\mathcal{A}}$ .

Take  $s \leq t$  and let  $\xi := \phi_s(x, v)$  and  $\eta := \phi_t(x, v)$ . We claim that, for any  $\epsilon > 0$ , there exists a strong  $(\epsilon, 1)$ -chain that goes from  $\eta$  to  $\xi$ . To see this, let us start with a strong  $(\delta, T)$ -chain from  $x_1 = \eta$  to  $x_{n+1} = \eta$  where  $T > 1$  is large compared to  $t - s$ , and replace  $x_{n+1}$  by  $\phi_{t_n - (t-s)}(x_n)$ . If  $\delta > 0$  is chosen sufficiently small, the point  $\phi_{t_n - (t-s)}(x_n)$  lies in an  $\epsilon$ -neighborhood of  $\xi$ , and we obtain a strong  $(\epsilon, 1)$ -chain from  $\eta$  to  $\xi$ . Let us call this chain  $\{(\eta_i, t_i)\}_{i=1}^n$  with  $\eta_1 = \eta, \eta_{n+1} = \xi, t_i > 1$  and  $\sum_{i=1}^n d_{TX}(\phi_{t_i}(\eta_i), \eta_{i+1}) < \epsilon$ . Set  $p_i := \tau(\eta_i)$  and  $q_i := \tau(\phi_{t_i}(\eta_i))$ .

Recall that  $\Phi_c$  satisfies

$$\Phi_c(x, y) \leq \Phi_c(x, z) + \Phi_c(z, y)$$

<sup>4</sup>A point  $x \in X$  is nonwandering if, and only if, for every neighborhood  $U$  of  $x$  there exists a  $T > 1$  such that  $\phi_T(U) \cap U \neq \emptyset$ ; this implies that there are also arbitrarily large  $T$  with that property.

and  $\Phi_c(x, x) = 0$  for all  $x, y, z \in X$ . Hence

$$\Phi_c(p_1, p_{n+1}) \leq \Phi_c(p_1, q_1) + \Phi_c(q_1, p_2) + \cdots + \Phi_c(p_n, q_n) + \Phi_c(q_n, p_{n+1}).$$

Given  $p$  and  $q$  in  $X$ , let  $\gamma : [0, d_X(p, q)] \rightarrow X$  be a unit speed minimizing geodesic connecting  $p$  to  $q$ . We have

$$\Phi_c(p, q) \leq \int_0^{d_X(p, q)} (L + c)(\gamma(t), \dot{\gamma}(t)) dt \leq \kappa_1 d_X(p, q)$$

where  $\kappa_1 := \max\{|(L + c)(x, v)| \mid (x, v) \in TX \text{ and } |v| = 1\}$ . Thus

$$\sum_i \Phi_c(q_i, p_{i+1}) \leq \kappa_1 \sum_i d_X(q_i, p_{i+1}) \leq \kappa_1 \epsilon. \quad (14)$$

Using (13) and the fact that  $K$  is exact, we have

$$\Phi_c(p_i, q_i) \leq A_{L+c}(\tau \circ \phi_{t_i}(\eta_i)|_{[0, t_i]}) = g(\phi_{t_i}(\eta_i)) - g(\eta_i), \quad (15)$$

where  $g : \ell^{-1}(K) \rightarrow \mathbb{R}$  is a smooth function such that  $d(g \circ \ell^{-1}|_K) = \lambda|_K$ . Combining (14) and (15), we obtain

$$\Phi_c(p_1, p_{n+1}) \leq \sum_i \Phi_c(p_i, q_i) + \Phi_c(q_i, p_{i+1}) \leq \kappa_1 \epsilon + \kappa_2 \epsilon + g(\xi) - g(\eta),$$

where  $\kappa_2$  is a Lipschitz constant for  $g$ . On the other hand,

$$\Phi_c(p_{n+1}, p_1) \leq A_{L+c}(\gamma|_{[s, t]}) = g(\eta) - g(\xi).$$

Therefore

$$0 = \Phi_c(p_1, p_1) \leq \Phi_c(p_1, p_{n+1}) + \Phi_c(p_{n+1}, p_1) \leq (\kappa_1 + \kappa_2) \epsilon.$$

Since  $\epsilon$  is arbitrary, we have

$$\Phi_c(p_1, p_{n+1}) + \Phi_c(p_{n+1}, p_1) = 0.$$

Using the triangle inequality for  $\Phi_c$  as in the proof of Proposition 3.4, we see that  $\gamma$  is a static curve.  $\square$

## 5 Non-removable intersections in the convex case

Let  $\Sigma \subset T^*X$  be an optical hypersurface bounding a domain  $U_\Sigma$ , and  $\tilde{\mathcal{A}}^* = \tilde{\mathcal{A}}^*(\Sigma)$  its Aubry set in  $T^*X$ . The following theorem is the main result that combines non-removable intersections with Aubry–Mather theory. Its statement was pointed out to us by A. Fathi, who also explained how results from Nonsmooth Analysis could be used to simplify its proof.

**Theorem 5.1.** *Let  $\Sigma$  be a minimizing optical hypersurface such that  $U_\Sigma$  contains a Lagrangian submanifold  $\Lambda \in \mathcal{L}_0$ . Then*

$$\tilde{\mathcal{A}}^* \subset \Lambda \cap \Sigma.$$

*Proof.* Let  $u : X \rightarrow \mathbb{R}$  be a graph selector associated to  $\Lambda$  (see Theorem 2.1). The function  $u$  is Lipschitz and  $(x, du(x)) \in \Lambda$  for almost every  $x \in X$ . By Remark 3.2 and Theorem 3.3, there exists a pair of conjugate functions  $(u_-, u_+)$  with  $u_+ \leq u \leq u_-$ . At any point  $x \in \mathcal{I}_{(u_-, u_+)}$ , the three functions are differentiable with the same derivative. Hence  $du(x)$  exists for each  $x \in \mathcal{I}_{(u_-, u_+)}$  and satisfies  $(x, du(x)) \in \Sigma$ .

We will now show that for every  $x \in \mathcal{I}_{(u_-, u_+)}$  we have  $(x, du(x)) \in \Lambda$ . This is the main difficulty since, a priori, we only know that this is true for almost every  $x$ . For each  $x \in X$ , let  $C_x(\Lambda)$  denote the convex hull of  $\Lambda \cap T_x^*X$ . Since  $\Lambda \cap T_x^*X$  is compact,  $C_x(\Lambda)$  is also compact by Carathéodory's theorem. Let  $C(\Lambda) = \bigcup_{x \in X} C_x(\Lambda)$ . Since  $C(\Lambda)$  is compact, a result from Nonsmooth Analysis (cf. [Cla, pp. 62–63] and [FM, Prop. 8.4]) ensures that for any point  $x$  of differentiability of  $u$  we have  $(x, du(x)) \in C(\Lambda)$ . But since  $(x, du(x)) \in \Sigma$ ,  $\Sigma \cap T_x^*X$  is strictly convex and  $\Lambda \subset U_\Sigma$ , the point  $(x, du(x))$  is an extreme point of  $C_x(\Lambda)$ . But any extreme point in the convex hull must belong to  $\Lambda \cap T_x^*X$  and thus  $(x, du(x)) \in \Lambda$ .

Since  $\tilde{\mathcal{A}}$  is contained in  $\tilde{\mathcal{I}}_{(u_-, u_+)}$ , the theorem follows.  $\square$

This result can be applied to boundary rigidity. The following result is a generalization of Theorem 1.1.

**Theorem 5.2.** *Let  $\Lambda \in \mathcal{L}$  be a Lagrangian submanifold lying in an optical hypersurface  $\Sigma$ . Assume that  $\sigma|_\Lambda$  is strongly chain recurrent. Let  $K \in \mathcal{L}$  be any Lagrangian submanifold lying in  $U_\Sigma$  with the same Liouville class as  $\Lambda$ . Then  $K = \Lambda$ .*

*Proof.* Since the multidimensional Birkhoff theorem is valid if  $\sigma|_\Lambda$  is chain recurrent [BP2, Prop. 1.2], we may, as in Section 2.3, apply a symplectic shift and assume that  $\Lambda = \mathcal{O} \subset T^*X$ . The shifted hypersurface obtained from  $\Sigma$  is minimizing since it contains  $\mathcal{O}$  (see Remark 4.2). We then know from Theorem 4.8 that  $\mathcal{O} \subset \tilde{\mathcal{A}}^*$ . Since the natural projection  $\theta|_{\tilde{\mathcal{A}}^*} : \tilde{\mathcal{A}}^* \rightarrow \mathcal{A}$  is a homeomorphism [Fa1, Prop. 5.2.8], we have  $\tilde{\mathcal{A}}^* = \mathcal{O}$ .

Now pick a graph selector  $\Phi$  of  $K$ ; see Theorem 2.1. As in the proof of Theorem 5.1, it will be differentiable at every point in  $\theta(\tilde{\mathcal{A}}^*) = X$  with zero derivative, i.e.,  $K$  coincides with the zero section  $\mathcal{O} = \Lambda$ .  $\square$

*Proof of Theorem 1.9.* Recall that we deal with the zero section  $\mathcal{O}$  of  $T^*\mathbb{T}^2$  lying inside the optical hypersurface

$$\Sigma = \{(y_1 - \sin x_1)^2 + (y_2 - \cos x_1)^2 = 1\}.$$

The restriction  $\sigma|_{\mathcal{O}}$  of the characteristic foliation is a Reeb foliation; see Figure 1. Denote by  $Z$  the union of the two limit cycles. Note that  $Z$  is the strong chain recurrent set of  $\sigma|_{\mathcal{O}}$ , and so, by Theorem 4.8, we have  $Z \subset \tilde{\mathcal{A}}^*$ . Since  $\Sigma$  contains the zero section  $\mathcal{O}$  it is minimizing in view of Remark 4.2. Applying Theorem 5.1, we see that  $Z \subset \Lambda \cap \Sigma$  for any Lagrangian submanifold  $\Lambda \in \mathcal{L}_0$  lying in  $U_\Sigma$ . This completes the proof.  $\square$

As a by-product of the proof, we get the following explicit description of both the Aubry and the Mather sets of  $\Sigma$  in this situation:

$$\tilde{\mathcal{A}}^* = \tilde{\mathcal{M}}^* = Z.$$

Figure 1: The dynamics on  $\mathcal{O}$  in Theorem 1.9 (left) and Example 5.3 (right)

Indeed, we have seen in Example 1.2 that the graph of  $df$  with  $f(x_1, x_2) = -\cos x_1$  intersects  $\Sigma$  exactly along  $Z$ . Hence, by Theorem 5.1, we obtain  $\tilde{\mathcal{A}}^* \subset Z$ . Together with the opposite inclusion established in the proof above, this yields  $Z = \tilde{\mathcal{A}}^*$ . Furthermore, each of the limit cycles in  $Z$  is a foliation cycle. It vanishes on the Liouville form since  $\lambda|_{\mathcal{O}} = 0$ . Hence  $\tilde{\mathcal{M}}^* = Z$ .

*Example 5.3.* Let  $f : S^1 \rightarrow S^1$  be a diffeomorphism with only two fixed points such that the fixed points are neither attractors nor repellers. Let  $X$  be the unit norm vector field on  $\mathbb{T}^2$  obtained by suspending  $f$ . Write

$$X(x_1, x_2) = (a_1(x_1, x_2), a_2(x_1, x_2))$$

and let  $H$  be the Hamiltonian

$$H(x_1, x_2, y_1, y_2) = (y_1 - a_1(x_1, x_2))^2 + (y_2 - a_2(x_1, x_2))^2.$$

Consider  $\Sigma = \{H = 1\} \subset T^*\mathbb{T}^2$ . Since  $\Sigma$  contains the zero section  $\mathcal{O}$  it is minimizing in view of Remark 4.2. If we identify  $\mathcal{O}$  with  $\mathbb{T}^2$  then  $X$  is tangent to the characteristic foliation  $\sigma|_{\mathcal{O}}$ . Note that  $\sigma|_{\mathcal{O}}$  is strongly chain recurrent, hence  $\mathcal{O}$  is boundary rigid by Theorem 5.2.

In this example, one can also describe the Aubry and the Mather sets of  $\Sigma$  explicitly. Indeed, Theorems 4.8 and 5.1 yield  $\tilde{\mathcal{A}}^* \subset \mathcal{O}$  and  $\mathcal{O} \subset \tilde{\mathcal{A}}^*$ , respectively, so  $\tilde{\mathcal{A}}^* = \mathcal{O}$ . The Mather set is strictly contained in the Aubry set. Indeed, the support of any invariant measure of the vector field  $X$  lies in the union  $Z$  of the limit cycles, each of which vanishes on the Liouville form; hence  $\tilde{\mathcal{M}}^* = Z$ .

## 6 Constructing Lagrangian sections—proof of Theorem 1.10

Suppose  $U \subset T^*X$  be an open, fiberwise convex set. We want to prove that every class  $a \in \text{sh}(U)$  can be represented by a Lagrangian *section* of the cotangent bundle. Indeed, this is an immediate consequence of the following theorem (a more general version of which was proved independently in [FM, Appendix]). Let us denote the fiberwise convex hull of a set  $S \subset T^*X$  by  $\text{conv}(S)$ .

**Theorem 6.1.** *Given a Lagrangian submanifold  $\Lambda \in \mathcal{L}$ , the fiberwise convex hull  $\text{conv}(W)$  of any neighborhood  $W$  of  $\Lambda$  contains a Lagrangian section  $\Lambda_0 \in \mathcal{L}$  with  $a_{\Lambda_0} = a_\Lambda$ .*

*Proof.* We may assume that  $\Lambda$  is an exact Lagrangian submanifold, simply by applying the symplectic shift  $(x, y) \mapsto (x, y - \alpha(x))$  where  $\alpha$  is the closed 1-form on  $X$  representing the Liouville class  $a_\Lambda$ .

Let  $\Phi : X \rightarrow \mathbb{R}$  be a graph selector associated to  $\Lambda$  as described in Theorem 2.1; namely,  $\Phi$  is Lipschitz continuous, smooth on an open subset  $X_0 \subset X$  of full measure, and satisfies

$$\text{graph}(d\Phi|_{X_0}) \subset \Lambda. \quad (16)$$

The proof of Theorem 6.1 is divided into two steps.

*Smoothing:* We are going to regularize the Lipschitz function  $\Phi$  by a convolution argument, similar to the proof of Prop. 7 in [CIPP1]. For this, we embed  $X$  into some Euclidean space  $\mathbb{R}^N$ . Denote by  $V_r$  the  $r$ -neighborhood of  $X$  in  $\mathbb{R}^N$  where  $r > 0$  is chosen small enough so that the orthogonal projection  $\pi : V_r \rightarrow X$  is well defined. We extend  $\Phi : X \rightarrow \mathbb{R}$  to a function  $\bar{\Phi} : V_r \rightarrow \mathbb{R}$  by setting

$$\bar{\Phi} := \Phi \circ \pi.$$

For each  $s \in (0, r/2)$ , we pick a smooth cut-off function  $u : [0, \infty) \rightarrow [0, \infty)$  with support in  $[0, s]$  such that  $u$  is constant near 0 and satisfies

$$\int_{\mathbb{R}^N} u(|z|) dz = 1.$$

Define the function  $\bar{\Psi} : V_s \rightarrow \mathbb{R}$  as the convolution

$$\bar{\Psi}(z) := (\bar{\Phi} * u)(z) = \int_{\mathbb{R}^N} \bar{\Phi}(y) u(|z - y|) dy.$$

Since  $\bar{\Phi}$  is Lipschitz continuous, it is differentiable almost everywhere and weakly differentiable. Therefore,  $\bar{\Psi}$  is a smooth function on  $V_s$  with

$$\begin{aligned} d\bar{\Psi}(z) &= \int_{\mathbb{R}^N} \bar{\Phi}(y) d_z u(|z - y|) dy \\ &= - \int_{\mathbb{R}^N} \bar{\Phi}(y) d_y u(|z - y|) dy \\ &= \int_{\mathbb{R}^N} d\bar{\Phi}(y) u(|z - y|) dy. \end{aligned}$$

Denote by

$$\Psi := \bar{\Psi}|_X$$

the restriction of  $\bar{\Psi}$  to  $X$ , and let  $B_s(x) \subset V_s \subset \mathbb{R}^N$  be the open ball of radius  $s$  centered at  $x \in X$ . Because  $X_0$  has full measure in  $X$ , we conclude that

$$d\Psi(x) = \int_{\pi^{-1}(X_0) \cap B_s(x)} d\bar{\Phi}(y)|_{T_x X} u(|x - y|) dy. \quad (17)$$

Note that, for this formula to make sense, we identify each  $T_y \mathbb{R}^N$  (where  $y \in \mathbb{R}^N$ ) with  $\mathbb{R}^N$ , and each  $T_x X$  (where  $x \in X$ ) with a linear subspace of  $\mathbb{R}^N$ .

*Analysing formula (17):* For each  $x \in X$ , we write

$$P_x : T_x \mathbb{R}^N \cong \mathbb{R}^N \rightarrow T_x X$$

for the orthogonal projection. Write  $|\cdot|$  for the Euclidean norm on  $\mathbb{R}^N$  and  $|\cdot|^*$  for the dual norm on  $(\mathbb{R}^N)^*$ . Introduce a distance function on  $T^* X$  by setting

$$\text{dist}((x, \xi), (y, \eta)) := |x - y| + |\xi \circ P_x - \eta \circ P_y|^*. \quad (18)$$

For  $x \in X$ , we define the set

$$\mathcal{G}_s(x) := \{(x, d\bar{\Phi}(y)|_{T_x X}) \mid y \in \pi^{-1}(X_0) \cap B_s(x)\} \subset T^* X.$$

For a subset  $Z \subset T^* X$ , we denote by  $W_\epsilon(Z)$  the  $\epsilon$ -neighborhood of  $Z$  with respect to the distance defined in (18).

**Lemma 6.2.** *For every  $\epsilon > 0$  there is an  $s > 0$  such that*

$$\mathcal{G}_s(x) \subset W_{\epsilon/2}(\text{graph}(d\Phi|_{X_0}))$$

for each  $x \in X$ .

*Proof.* Pick any point

$$\eta_1 = (x, d\bar{\Phi}(y)|_{T_x X}) \in \mathcal{G}_s(x)$$

with  $x \in X$  and  $y \in \pi^{-1}(X_0) \cap B_s(x)$ . We will show that the distance between  $\eta_1$  and

$$\eta_2 := (\pi(y), d\Phi(\pi(y))) \in \text{graph}(d\Phi|_{X_0})$$

becomes as small as we wish when  $s \rightarrow 0$  uniformly in  $x$  and  $y$ . Indeed, denote by  $c > 0$  the Lipschitz constant of  $\Phi$  with respect to the induced distance on  $X \subset \mathbb{R}^N$ . Let  $Q_y$  be the differential of the projection  $\pi$  at  $y$ ; we consider  $Q_y$  as an endomorphism of  $\mathbb{R}^N$ . Finally, write  $\|\cdot\|$  for the operator norm on  $\text{End}(\mathbb{R}^N)$ . Now we can estimate

$$\begin{aligned} \text{dist}(\eta_1, \eta_2) &= |x - \pi(y)| + |d\bar{\Phi}(y)|_{T_x X} \circ P_x - d\Phi(\pi(y)) \circ P_{\pi(y)}|^* \\ &= |x - \pi(y)| + |d\Phi(\pi(y)) \circ Q_y \circ P_x - d\Phi(\pi(y)) \circ P_{\pi(y)}|^* \\ &\leq |x - y| + |y - \pi(y)| + c\|Q_y \circ P_x - P_{\pi(y)}\|. \end{aligned}$$

Note that  $|x - y| + |y - \pi(y)| \leq 2s \rightarrow 0$  as  $s \rightarrow 0$ . It remains to handle the term  $\|Q_y \circ P_x - P_{\pi(y)}\|$ . Using that  $\|P_x\| = \|P_{\pi(y)}\| = 1$  we obtain

$$\begin{aligned} \|Q_y \circ P_x - P_{\pi(y)}\| &= \|Q_y \circ P_x - P_{\pi(y)} \circ P_x + P_{\pi(y)} \circ P_x - P_{\pi(y)} \circ P_{\pi(y)}\| \\ &\leq \|Q_y - P_{\pi(y)}\| + \|P_x - P_{\pi(y)}\| \rightarrow 0 \end{aligned}$$

as  $s \rightarrow 0$ , and the convergence is uniform in  $x \in X$  and  $y \in B_s(x)$ . This finishes the proof of Lemma 6.2.  $\square$

Now we can readily prove Theorem 6.1. Namely, given any  $\epsilon > 0$ , we choose  $s$  as in Lemma 6.2. Then (17), Lemma 6.2, and (16) imply that

$$(x, d\Psi(x)) \in \text{conv}(W_{\epsilon/2}(\mathcal{G}_s(x))) \subset \text{conv}(W_\epsilon(\text{graph}(d\Phi|_{X_0}))) \subset \text{conv}(W_\epsilon(\Lambda))$$

for each  $x \in X$ . Thus the Lagrangian section  $\Lambda_0 := \text{graph}(d\Psi)$  satisfies

$$\Lambda_0 \subset \text{conv}(W_\epsilon(\Lambda)).$$

Since  $\epsilon > 0$  was arbitrary the proof of Theorem 6.1 is completed.  $\square$

## 7 Boundary rigidity in general symplectic manifolds

The boundary rigidity phenomenon can be naturally formulated in the following more general context. Let  $(M, \omega)$  be a compact symplectic manifold with non-empty boundary, and let  $\Lambda \subset \partial M$  be a closed Lagrangian submanifold. Denote by  $\mathcal{L}_0$  the space of all Lagrangian submanifolds of  $M$  which are exact Lagrangian isotopic to  $\Lambda$  (see for instance [Pol] for the definition of exact Lagrangian isotopies in symplectic manifolds). We say that  $\Lambda$  is *boundary rigid* if  $\mathcal{L}_0 = \{\Lambda\}$ , and *weakly boundary rigid* if every  $K \in \mathcal{L}_0$  is contained in  $\partial M$ .

Theorem 1.1 already provides a class of examples of boundary rigid Lagrangian submanifolds. It would be interesting to investigate boundary rigidity in other symplectic manifolds as well.

*Example 7.1 (A toy example).* Let  $M = D^2$  be the 2-disc endowed with some area form. Then the circle  $\Lambda = \partial D^2$  is boundary rigid. Indeed, every circle  $K \in \mathcal{L}_0$  must enclose the same area as  $\Lambda$ , and hence  $K = \Lambda$ .

Can one generalize this example to higher dimensions? For instance, let  $M$  be the Euclidean ball

$$\{p_1^2 + q_1^2 + p_2^2 + q_2^2 \leq 2\}$$

in the standard symplectic vector space  $\mathbb{R}^4$ . Consider the split torus

$$\Lambda := \{p_1^2 + q_1^2 = 1, p_2^2 + q_2^2 = 1\} \subset \partial M. \quad (19)$$

One can show that  $\Lambda$  admits a nontrivial exact Lagrangian isotopy inside  $\partial M$  and hence is *not* boundary rigid.

**Question 7.2.** *Is  $\Lambda$  weakly boundary rigid? What happens with Lagrangian tori contained in general ellipsoids in  $\mathbb{R}^{2n}$ ?*

Further, it would be interesting to extend the study of non-removable intersections, both between Lagrangian submanifolds, and between a Lagrangian submanifold and a hypersurface (see Sections 1.3 and 5), to more general symplectic manifolds.

An interesting playground for this problem is given by tori in ellipsoids as in Question 7.2. For instance, let  $\Lambda$  be the split torus given by (19), and  $K$  any exact Lagrangian deformation of  $\Lambda$  which lies in the closed ball  $M$  of radius  $\sqrt{2}$ . Just recently, Y. Eliashberg outlined a beautiful argument based on symplectic field theory which suggests that  $K$  must intersect the boundary of the ball; later, F. Schlenk proposed a simpler approach using symplectic capacities, based on [Vit]. Applying Sullivan's theorem as in the proof of Theorem 1.5, we conclude that  $K \cap \partial M$  must contain a closed orbit of the characteristic foliation of  $\partial M$ .

We refer to [Bir] for further discussion on symplectic intersections.

*Acknowledgement.* Our interest in the boundary rigidity phenomenon and, in particular, non-removable intersections was motivated by numerous fruitful discussions with Paul Biran. Discussions with Albert Fathi were very helpful; in particular, he explained to us how to use results from Nonsmooth Analysis in the proof of Theorem 5.1. We are also grateful to Misha Bialy and Yasha Eliashberg for useful conversations.

Parts of this paper were written during joint visits to Tel Aviv, Cambridge, and Zürich. These visits were supported by the Hermann Minkowski Center for Geometry at Tel Aviv University, the University of Cambridge, and the Forschungsinstitut für Mathematik at ETH Zürich. We thank these institutions for their support, as well as Marc Burger, Kostya Khanin, Dietmar Salamon, and Edi Zehnder for their warm hospitality.

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