GROUPS EXAMPLES 4

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The questions on this sheet are not all equally difficult and the harder ones are marked with $*$'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. Write the following permutations as products of disjoint cycles and compute their order and sign:
(a) (12)(1234)(12);
(b) (123)(45)(16789)(15).

2. What is the largest possible order of an element in S_5? And in S_9? Show that every element in S_{10} of order 14 is odd.

3. Show that any subgroup of S_n which is not contained in A_n contains an equal number of odd and even permutations.

4. Let N be a normal subgroup of the orthogonal group $O(2)$. Show that if N contains a reflection in some line through the origin, then $N = O(2)$.

5. Show that S_n is generated by the two elements (12) and $(123 \dots n)$.

6. Let z_1, z_2, z_3 and z_4 be four distinct points in $\mathbb{C}_\infty$ and let $\lambda = [z_1, z_2, z_3, z_4]$ be the cross ratio of the four points. Let G be the group of Möbius maps which map the set $\{0, 1, \infty\}$ onto itself. Show that given $\sigma \in S_4$, there exists $f_\sigma \in G$ such that $f_\sigma(\lambda) = [z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)}, z_{\sigma(4)}]$.

7. Show that the map $S_4 \ni \sigma \mapsto f_{\sigma^{-1}} \in G \cong S_3$ given by the previous question is a surjective homomorphism. Find its kernel.

8. Let H be a normal subgroup of a group G and let K be a normal subgroup of H. Is it true that K must be a normal subgroup of G?

9. Let X be the set of all 2×2 real matrices with trace zero. Given $A \in SL(2, \mathbb{R})$ and $B \in X$, show that

$$(A, B) \mapsto ABA^{-1}$$

defines an action of $SL(2, \mathbb{R})$ on X. Find the orbit and stabilizer of

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Show that the set of matrices in X with zero determinant is a union of 3 orbits.

10. When do two elements in $SO(3)$ commute?

11. If A is a complex $n \times n$ matrix with entries a_{ij}, let A^* be the complex $n \times n$ matrix with entries $\bar{a}_{ji}$. The matrix A is called *unitary* if $AA^* = I$. Show that the set $U(n)$ of unitary matrices forms a group under matrix multiplication. Show that

$$SU(n) = \{A \in U(n) : \det A = 1\}$$

is a normal subgroup of $U(n)$ and that $U(n)/SU(n)$ is isomorphic to S^1. Show that $SU(2)$ contains the quaternion group $\mathbb{H}_8$ as a subgroup.

12. Show that any subgroup of A_5 has order at most 12.

13. Find the elements in S_n that commute with (12).

14*. Let G be a finite non-trivial subgroup of $SO(3)$. Let X be the set of points on the unit sphere in $\mathbb{R}^3$ which are fixed by some non-trivial rotation in G. Show that G acts on X and that the number of orbits is either 2 or 3. What is G if there are only two orbits? [With more work one can show that if there are three orbits, then G must be dihedral or the group of rotational symmetries of a Platonic solid.]