GROUPS EXAMPLES 3

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The questions on this sheet are not all equally difficult and the harder ones are marked with $*$'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`.

1. If H is a subgroup of a finite group G and G has twice as many elements as H, show that H is normal in G.

2. Let H be a subgroup of the cyclic group C_n. What is C_n/H?

3. Show that every subgroup of rotations in the dihedral group D_{2n} is normal.

4. Show that a subgroup H of a group G is normal if and only if it is a union of conjugacy classes.

5. We know that in an abelian group every subgroup is normal. Now, let G be a group in which every subgroup is normal, is it true that G must be abelian?

6. Show that $\mathbb{Q}/\mathbb{Z}$ is an infinite group in which every element has finite order.

7. Let G be the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$. Show that G is a subgroup of the group of invertible real matrices under multiplication. Let H be the subset of G given by those matrices with $x = z = 0$. Show that H is a normal subgroup of G and find G/H. [Use the isomorphism theorem.]

8. Consider the additive group $\mathbb{C}$ and the subgroup Γ consisting of all *Gaussian integers* $m+in$, where $m, n \in \mathbb{Z}$. By considering the map

$$x + iy \mapsto (e^{2\pi ix}, e^{2\pi iy}),$$

show that the quotient group $\mathbb{C}/\Gamma$ is isomorphic to the torus $S^1 \times S^1$.

9. Let H be a subgroup of a group G. Show that H is a normal subgroup of G if and only if there is some group K, and some homomorphism $\theta : G \to K$, whose kernel is H.

10. Let $GL(2,\mathbb{R})$ be the group of all 2×2 invertible matrices and let $SL(2,\mathbb{R})$ be the subset of $GL(2,\mathbb{R})$ consisting of matrices of determinant 1. Show that $SL(2,\mathbb{R})$ is a normal subgroup of $GL(2,\mathbb{R})$. Show that the quotient group $GL(2,\mathbb{R})/SL(2,\mathbb{R})$ is isomorphic to the multiplicative group of non-zero real numbers.

11. Let G be a finite group and $H \neq G$ a subgroup. Let k be the cardinality of the set of left cosets of H (k is sometimes called the *index* of H) and suppose that $|G|$ does not divide $k!$. Show that H contains a non-trivial normal subgroup of G. [Let G act on the set of left cosets and reinterpret the action as a homomorphism from G to the group of permutations of the set of left cosets.] Show that a group of order 28 has a normal subgroup of order 7. [Use Cauchy's theorem.]

12. Show that if a group G of order 28 has a normal subgroup of order 4, then G is abelian. [Use Question 11. You might wish to note that if H is a subgroup of order 4 and K is a subgroup of order 7, then $H \cap K = \{e\}$.]

13. Let G be a subgroup of the group of isometries of the plane. Show that the set T of translations in G is a normal subgroup of G (T is called the *translation subgroup*). [If we think of the plane as $\mathbb{C}$ you may assume that all isometries have the form $z \mapsto az + b$ or $z \mapsto a\bar{z} + b$, where a and b are complex numbers and in both cases $|a| = 1$.]

14*. A *frieze group* is a group F of isometries of $\mathbb{C}$ that leaves the real line invariant (that is, if $z \in \mathbb{C}$ has zero imaginary part and $g \in F$, then $g(z)$ also has zero imaginary part) and whose translation subgroup T is infinite cyclic. If F is a frieze group, classify F/T.