

GROUPS EXAMPLES 3

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The questions on this sheet are not all equally difficult and the harder ones are marked with *'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at g.p.paternain@dpmmms.cam.ac.uk.

1. If H is a subgroup of a finite group G and G has twice as many elements as H , show that H is normal in G .
2. Let H be a subgroup of the cyclic group C_n . What is C_n/H ?
3. Show that every subgroup of rotations in the dihedral group D_{2n} is normal.
4. Show that a subgroup H of a group G is normal if and only if it is a union of conjugacy classes.
5. We know that in an abelian group every subgroup is normal. Now, let G be a group in which every subgroup is normal, is it true that G must be abelian?
6. Show that \mathbb{Q}/\mathbb{Z} is an infinite group in which every element has finite order.
7. Let G be the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix},$$

with $x, y, z \in \mathbb{R}$. Show that G is a subgroup of the group of invertible real matrices under multiplication. Let H be the subset of G given by those matrices with $x = z = 0$. Show that H is a normal subgroup of G and find G/H . [Use the isomorphism theorem.]

8. Consider the additive group \mathbb{C} and the subgroup Γ consisting of all *Gaussian integers* $m + in$, where $m, n \in \mathbb{Z}$. By considering the map

$$x + iy \mapsto (e^{2\pi ix}, e^{2\pi iy}),$$

show that the quotient group \mathbb{C}/Γ is isomorphic to the torus $S^1 \times S^1$.

9. Let G be a finite group and $H \neq G$ a subgroup. Let k be the cardinality of the set of left cosets of H (k is sometimes called the *index* of H) and suppose that $|G|$ does not divide $k!$. Show that H contains a non-trivial normal subgroup of G . [Let G act on the set of left cosets.] Show that a group of order 28 has a normal subgroup of order 7.
10. Show that if a group G of order 28 has a normal subgroup of order 4, then G is abelian.
11. Let H be a subgroup of a group G . Show that H is a normal subgroup of G if and only if there is some group K , and some homomorphism $\theta : G \rightarrow K$, whose kernel is H .
12. Let $GL(2, \mathbb{R})$ be the group of all 2×2 invertible matrices and let $SL(2, \mathbb{R})$ be the subset of $GL(2, \mathbb{R})$ consisting of matrices of determinant 1. Show that $SL(2, \mathbb{R})$ is a normal subgroup of $GL(2, \mathbb{R})$. Show that the quotient group $GL(2, \mathbb{R})/SL(2, \mathbb{R})$ is isomorphic to the multiplicative group of non-zero real numbers.
13. Let G be a subgroup of the group of isometries of the plane. Show that the set T of translations in G is a normal subgroup of G (T is called the *translation subgroup*). [If we think of the plane as \mathbb{C} you may assume that all isometries have the form $z \mapsto az + b$ or $z \mapsto a\bar{z} + b$, where a and b are complex numbers and in both cases $|a| = 1$.]
- 14*. A *frieze group* is a group F of isometries of \mathbb{C} that leaves the real line invariant and whose translation subgroup T is infinite cyclic. If F is a frieze group, classify F/T .