

## GROUPS EXAMPLES 2

### G.P. Paternain Michaelmas 2007

The questions on this sheet are not all equally difficult and the harder ones are marked with \*'s. Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at [g.p.paternain@dpmmms.cam.ac.uk](mailto:g.p.paternain@dpmmms.cam.ac.uk).

1. Show that if a group  $G$  contains an element of order six, and an element of order ten, then  $G$  has order at least 30.
2. Show that the set  $\{1, 3, 5, 7\}$ , with multiplication modulo 8 is a group. Is this group isomorphic to  $C_4$  or  $C_2 \times C_2$ ?
3. How many subgroups of order four does the quaternion group  $\mathbb{H}_8$  have?
4. Let  $H$  be a subgroup of a group  $G$ . Show that if  $aH = bH$  then  $Ha^{-1} = Hb^{-1}$ . Use this to show that there is a bijection between the set of left cosets and the set of right cosets of  $H$ .
5. What is the order of the Möbius map  $f(z) = iz$ ? If  $h$  is any Möbius map, find the order of  $hfh^{-1}$  and its fixed points. Use this to construct a Möbius map of order four that fixes 1 and  $-1$ .
6. Show that  $\rho(t, (x, y)) = (e^t x, e^{-t} y)$ , where  $t \in \mathbb{R}$  and  $(x, y) \in \mathbb{R}^2$ , defines an action of  $(\mathbb{R}, +)$  on  $\mathbb{R}^2$ . Describe the orbits of the action and find all possible stabilizers. There is a differential equation in the plane that is satisfied by all orbits, can you find it?
7. Suppose that  $G$  acts on  $X$  and that  $y = gx$ , where  $x, y \in X$  and  $g \in G$ . Show that  $\text{Stab}(y) = g\text{Stab}(x)g^{-1}$  (hence, the stabilizers of two points in the same orbit are conjugate).
8. Let  $G$  be a finite group and let  $X$  be the set of all its subgroups. Given  $g \in G$  and  $H \in X$  show that  $(g, H) \mapsto gHg^{-1}$  defines an action of  $G$  on  $X$ . Show that the orbit of  $H$  has at most  $|G|/|H|$  elements. If  $H \neq G$ , show that there is an element in  $G$  which does not belong to any conjugate of  $H$ .
9. Let  $G$  be a finite group acting faithfully on a finite set  $X$ . Show that if  $G$  is abelian and there is only one orbit, then  $|G| = |X|$ .
10. Show that  $D_{2n}$  (the group of symmetries of the regular  $n$ -gon) has one conjugacy class of reflections if  $n$  is odd, and two conjugacy classes of reflections if  $n$  is even.
11. Let  $G$  be the group of all symmetries of the cube and let  $\ell$  be the line between two diagonally opposite vertices. Let  $H = \{g \in G : g\ell = \ell\}$ . Show that  $H$  is a subgroup of  $G$  isomorphic to  $S_3 \times C_2$ .
12. Classify all groups of order 10.
13. Let  $S^1$  denote the unit circle in  $\mathbb{C}$  and let  $S^3 = \{(w_1, w_2) \in \mathbb{C}^2 : |w_1|^2 + |w_2|^2 = 1\}$ . Show that given  $(t_1, t_2) \in S^1 \times S^1$  and  $(w_1, w_2) \in S^3$ ,  
$$((t_1, t_2), (w_1, w_2)) \mapsto (t_1 w_1, t_2 w_2)$$
defines an action of the torus  $S^1 \times S^1$  on  $S^3$ . Describe the orbits and find all stabilizers.
- 14\*. Let  $p$  be a prime. Show that every group of order  $p^2$  is abelian. [Hint: consider the action of the group on itself by conjugation.]