DIFFERENTIAL GEOMETRY, PART III, EXAMPLES 3.

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. Most of the examples in this sheet are taken from Alexei Kovalev's example sheets. The questions are not equally difficult. Those marked with * are not always harder, but are less central to the lectured material and may be regarded as a supplement for the enthusiasts.

1. Let (M, g) be a Riemannian manifold. Recall that in lectures we defined geodesics as those curves in M which are projections of the orbits of the Hamiltonian flow of $H(x, p) = \frac{1}{2}|p|_x^2$ with respect to the canonical symplectic form of T^*M. Show that geodesics can also be characterized as curves in M whose velocity vector is parallel with respect to the Levi-Civita connection of g.

2.Suppose that $\mathbf{r} = \mathbf{r(u, v)}$ is a regular parameterization of a surface S in the affine space $\mathbb{R}^3$. There is a standard choice of a 'moving frame' (a basis of the tangent space $T_{\mathbf{r}}\mathbb{R}^3$) $\mathbf{r}_u$, $\mathbf{r}_v$, $\mathbf{n}$ at every point $\mathbf{r}$ of S, where $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v / |\mathbf{r}_u \times \mathbf{r}_v|$ is a unit normal vector to S. (Here the subscripts u and v at $\mathbf{r}$ are used to denote the respective partial derivatives.) Then there is a unique way to write the second derivatives of $\mathbf{r}$ as

$$\begin{aligned}
\mathbf{r}_{uu} &= \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + L\mathbf{n} \\
\mathbf{r}_{uv} &= \Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + M\mathbf{n} \\
\mathbf{r}_{vv} &= \Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + N\mathbf{n},
\end{aligned}$$

for some functions Γ^i_{jk}, L, M, N on S. By deducing the expressions for Γ^i_{jk} in terms of the first fundamental form of S, or otherwise, show that Γ^i_{jk} are the Christoffel symbols for the Levi–Civita connection of the metric induced on S by restriction from the ambient $\mathbb{R}^3$.

3. (i) Prove that any connection ∇ on M uniquely determines a covariant derivative on the *cotangent bundle* T^*M (still to be denoted by ∇), such that $\nabla_X : \Omega^1(M) \to \Omega^1(M)$ satisfies $X\alpha(Y) = \langle \nabla_X \alpha, Y\rangle + \langle \alpha, \nabla_X Y\rangle$. Here $\alpha \in \Omega^1(M)$, and X, Y are vector fields on M. In particular, prove that if $\alpha = \alpha_j dx^j$ in local coordinates and Γ^i_{jk} are the coefficients of ∇ on the tangent bundle then $(\nabla_X \alpha)_j = \big(\dfrac{\partial \alpha_j}{\partial x^k} - \Gamma^i_{jk}\alpha_i\big)X^k$.

Show further that if ∇ is the Levi–Civita of some metric (g_{ij}) on M then the induced connection is an orthogonal connection for the dual metric (g^{ij}) on T^*M. (It is natural to call this induced connection the Levi–Civita on T^*M).

(ii)* Recall from linear algebra that the space of all bilinear forms on a vector space V is naturally isomorphic to the space of linear maps $\mathrm{End}(V, V^*)$, from V to its dual space V^*. Using this result and given a connection ∇ on M, write out Leibnitz formula for the induced connection (still denoted by ∇) on the bundle of bilinear forms over M. Give the expression for the latter induced connection ∇ in local coordinates and show that if ∇ is the Levi–Civita of a Riemannian metric g on M then $\nabla g = 0$. (A Riemannian metric is covariant-constant, or 'parallel', with respect to its Levi–Civita connection.)

4. (Holonomy transformations) Let $\pi : E \to M$ be a vector bundle over a manifold M and A a connection on E. Show that for each smooth path $\gamma(t)$ $(0 \le t \le 1)$ in M and a vector v_0 in the fibre $E_{\gamma(0)}$ there exists unique path $\gamma^E(t)$ in E, such that $\pi \circ \gamma^E = \gamma$, the velocity vector $\dot{\gamma}^E(t)$ is *horizontal* for each t, and $\gamma^E(0) = v_0$. The vector $v_1 = \gamma^E(1)$ is sometimes called parallel transport of v_0 over γ (with respect to A). Show that the assignment of v_1 of v_0 defines a linear map from $E_{\gamma(0)}$ to $E_{\gamma(1)}$.

Now suppose the $E = TM$ endowed with the Levi–Civita connection of a Riemannian metric on M. Show that the parallel transport over a closed loop γ based at $x \in M$ defines an orthogonal linear transformation of T_xM.

5. Let (M, g) be a Riemannian manifold and $R(X,Y) \in \Gamma(\operatorname{End} TM)$ the endomorphism defined using the Riemann curvature of g and vector fields X, Y. Show that the Levi–Civita covariant derivative of $R(X,Y)$ is an endomorphism $(D_Z R)(X,Y)$ given by $(D_Z R)(X,Y) = [D_Z, R(X,Y)] - R(D_Z X, Y) - R(X, D_Z Y)$. Deduce from this a special version of the second Bianchi identity for the Levi–Civita connection

$$(*) \qquad (D_X R)(Y,Z) + (D_Y R)(Z,X) + (D_Z R)(X,Y) = 0,$$

[Hint: use the identities $R(X,Y)Z = [D_X, D_Y]Z - D_{[X,Y]}Z$ and $D_X Y - D_Y X = [X,Y]$ from the Lectures and exploit the cyclic symmetry of the expression $(*)$.]

6. For this question, note that at each point x the Riemann curvature (R_{ijkl}) of (M, g) defines a symmetric bilinear form on $\Lambda^2(T_xM)$. Show that if $\dim M = 3$ then the Riemann curvature is determined at each point of M by the Ricci curvature $\operatorname{Ric}(g)$.
[Hint: note that the map that takes $R(g)$ to $\operatorname{Ric}(g)$ is a linear map, at each point of M. A special feature of the dimension 3 is that the spaces of 1-forms and 2-forms on $\mathbb{R}^3$ have the same dimension.]

7. (i) Show that the Hodge star on $\Lambda^2(\mathbb{R}^4)^*$ determines an orthogonal decomposition $\Lambda^2(\mathbb{R}^4)^* = \Lambda^+ \oplus \Lambda^-$ into the ± 1 eigenspaces and $\dim \Lambda^+ = \dim \Lambda^- = 3$. Deduce that on every oriented 4-dimensional Riemannian manifold M there is a decomposition of 2-forms $\Omega^2(M) = \Omega^+ \oplus \Omega^-$, so that $\alpha \wedge \alpha = \pm |\alpha|_g^2 \omega_M$, for every $\alpha \in \Omega^\pm$, where ω_M is the volume form. (2-forms in the subspaces $\Omega^\pm$ are called, respectively, the self- and anti-self-dual forms on M.)

(ii) Now assume that M is a *compact* 4-dimensional oriented Riemannian manifold. Show that the expression $\int_M \alpha \wedge \beta$, for $\alpha, \beta \in \Omega^2(M)$, induces a well-defined symmetric bilinear form on the de Rham cohomology $H^2_{\mathrm{dR}}(M)$. Let $(b^+(M), b^-(M))$ denote the signature of this bilinear form. Show that $b^\pm(M) = \dim \mathcal{H}^\pm$, where $\mathcal{H}^\pm$ denotes the space of harmonic (anti-)self-dual forms on M.

8. Calculate explicitly the expression of the Laplacian for functions:

(a) on the hyperbolic plane $H = \{(x,y) \in \mathbb{R}^2 : y > 0\}$, where the metric is $g(x,y) = \dfrac{dx^2 + dy^2}{y^2}$;

(b) on the sphere S^n, in the local coordinates given by stereographic projections. (The metric on S^n is the 'round' metric induced by standard embedding in the Euclidean $\mathbb{R}^{n+1}$.)

9. Express the Laplacian on the Euclidean $\mathbb{R}^{n+1} \setminus \{0\}$ in terms of the Laplacian on the unit sphere S^n (you might like to use a result of Question 13 of Example Sheet 2).
Deduce a formula for the Laplacian for spherically-symmetric functions $f(r)$, where r denotes the polar radius on $\mathbb{R}^n$.

10. Show that the partial diferential equation $\Delta f = \phi$ for a function $f \in C^\infty(M)$ on a compact oriented Riemannian manifold (M, g), with a given $\phi \in C^\infty(M)$, has a solution if and only if $\int_M \phi\, \omega_g = 0$. (Here ω_g denotes the volume form.) Is a solution unique?

11. (i) Let g_0 be an Einstein metric $\operatorname{Ric}(g_0) = \lambda\, g_0$. Let $g_t := u(t) g_0$. Find u so that g_t evolves along the Ricci flow $\frac{\partial g_t}{\partial t} = -2 \operatorname{Ric}(g_t)$ and $u(0) = 1$. Show that the solution for $\lambda > 0$ (called shrinking solution) only exists for finite time.

(ii) Let g_t be a solution of the Ricci flow and let ω_t be the Riemannian volume form of g_t. Show that $\frac{\partial \omega_t}{\partial t} = -s_t\, \omega_t$, where s_t is the scalar curvature of g_t.