DYNAMICAL SYSTEMS. EXAMPLES 3.

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Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `g.p.paternain@dpmms.cam.ac.uk`. Most of the examples in this sheet are taken from the text that I am following in lectures: *Introduction to dynamical systems,* by M. Brin & G. Stuck.

1. Let T be a measure-preserving transformation of a probability space $(X, \mathcal{U}, \mu)$ and let $f \in L^1(X, \mu)$ satisfy $f(T(x)) \leq f(x)$ for a.e. $x \in X$. Show that $f(T(x)) = f(x)$ for a.e. $x \in X$.

2. Let X be a compact metric space, μ a Borel probability measure and $T : X \to X$ a continuous transformation preserving μ. Suppose that for every pair of continuous functions f and g with zero integrals,

$$\lim_{n\to\infty} \int_X f(T^n(x))\, g(x)\, d\mu = 0.$$

Prove that T is mixing.

3. Show that an isometry of a compact metric space is not mixing for any invariant Borel probability measure whose support is not a single point.

4. Describe $\mathcal{M}_T$ and $\mathcal{M}_T^e$ for the homeomorphism of the circle $T(x) = x + a \sin 2\pi x \bmod 1$, with $0 < a \leq 1/2\pi$.

5. Describe $\mathcal{M}_T$ and $\mathcal{M}_T^e$ for the homeomorphism of the torus $T(x, y) = (x, x + y) \bmod 1$.

6. Give an example of a map of the circle which is discontinuous at exactly one point and does not have Borel invariant probability measures.

7. Let X and Y be compact metric spaces and $T : X \to Y$ a continuous map. Show that T induces a natural map $\mathcal{M}(X) \to \mathcal{M}(Y)$ which is continuous in the weak* topology.

8*. Let σ be the two-sided 2-shift. Show that $\mathcal{M}_\sigma^e$ is dense in $\mathcal{M}_\sigma$ in the weak* topology.

9. A continuous map $T : X \to X$ of a compact metric space is said to be *uniquely ergodic* if there is only one invariant Borel probability measure μ. Show that if T is uniquely ergodic, then T is minimal if and only if $\mu(U) > 0$ for all non-empty open sets U.

10. Let G be a compact metrisable topological group. Show that a left translation is minimal if and only if is uniquely ergodic. Show that R_α is uniquely ergodic if α is irrational.

11. Using the Birkhoff ergodic theorem show that for almost every $x \in [0, 1)$ (with respect to Lebesgue measure) the frequency of 1's in the binary expansion of x is 1/2.

12. A subset $D \subset \mathbb{Z}$ has *positive upper density* if there are a_n and b_n in $\mathbb{Z}$ such that $b_n - a_n \to \infty$ and for some $\delta > 0$,

$$\frac{\#(D \cap [a_n, b_n])}{b_n - a_n + 1} > \delta$$

for all $n \in \mathbb{N}$.

Let $D \subset \mathbb{Z}$ have positive upper density and let ω_D be the sequence in $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$ for which $(\omega_D)_n = 1$ if $n \in D$ and $(\omega_D)_n = 0$ if $n \notin D$. Let X_D be the closure of its orbit under the shift in Σ_2.

Show that there exists a shift invariant Borel probability measure μ on X_D such that $\mu(\{\omega \in X_D : \omega_0 = 1\}) > 0$. (Hint: consider the Borel probability measure μ_n defined by

$$\int f \, d\mu_n = \frac{1}{b_n - a_n + 1} \sum_{i=a_n}^{b_n} f(\sigma^i(\omega_D))$$

for any continuous function $f \in C(X_D)$. Use compactness of $\mathcal{M}(X_D)$ in the weak* topology.)

13. (Furstenberg's proof of Szemerédi's theorem.) Furstenberg proved the following remarkable extension of the Poincaré recurrence theorem:

Furstenberg's multiple recurrence theorem. *Let T be an automorphism of a probability space $(X, \mathcal{U}, \mu)$. Then for every $n \in \mathbb{N}$ and every $A \in \mathcal{U}$ with $\mu(A) > 0$, there is $k \in \mathbb{N}$ such that*

$$\mu(A \cap T^{-k}(A) \cap T^{-2k}(A) \cap \cdots \cap T^{-nk}(A)) > 0.$$

Combine this theorem with Problem 12 to give a proof of *Szemerédi's theorem:* every subset $D \subset \mathbb{Z}$ of positive upper density contains arbitrarily long arithmetic progressions.