ON THE GROWTH RATE OF CONTRACTIBLE CLOSED GEODESICS ON REDUCIBLE MANIFOLDS

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Abstract. We prove exponential growth rate of contractible closed geodesics for an arbitrary bumpy metric on manifolds of the form $X_1 \# X_2$ where the fundamental group of X_1 has a subgroup of finite index at least 3 and X_2 is simply connected and not a homotopy sphere.

1. Introduction

Let M be a closed manifold and g a smooth Riemannian metric on M. Given $t \geq 0$, define $N_g(t)$ to be the number of geometrically distinct closed geodesics of g with length $\leq t$. Similarly, let $N_g^0(t)$ be the number of geometrically distinct *contractible* closed geodesics of g with length $\leq t$.

If g is a *bumpy metric*, i.e., all the closed geodesics of g are non-degenerate, then $N_g(t)$ is finite for each t and we can ask the following basic question: what is the behaviour of $N_g(t)$ as t tends to infinity? Similarly if g is a *0-bumpy metric*, i.e., all the *contractible* closed geodesics of g are non-degenerate, then $N_g^0(t)$ is finite for each t and we can also wonder about its growth.

Let $\Lambda(M)$ be the space of piecewise differentiable closed curves $c : \mathbb{R}/\mathbb{Z} \to M$, endowed with the compact-open topology.

Proposition A. *Let M be a closed manifold and let $X \subset M$ be a simply connected submanifold, possibly with boundary. Consider the inclusion map $\iota : X \to M$ and let R_i be the rank of the map induced in the loop space homology for some field of coefficients k_p, p prime or zero. Then, for any 0-bumpy metric g on M there exist constants $\alpha = \alpha(g) > 0$ and $\beta = \beta(g) > 0$ such that*

$$N_g^0(t) \geq \alpha \max_{i \leq \beta t} R_i$$

for all t sufficiently large.

This proposition is a generalization of a theorem of M. Gromov [11], later improved by W. Ballmann and W. Ziller [6], who proved the proposition when M is simply connected and $X = M$ (we remark that in Gromov's theorem it is essential to assume that M is simply connected). In fact, the methods in [22] show that the proposition is still true if we replace X by an arbitrary finite simply connected CW-complex K and ι by any continuous map $f : K \to M$.

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The result has the following interesting consequence:

Theorem B. *Let M be a closed manifold of dimension $n \geq 3$. Suppose that M can be decomposed as $X_1 \# X_2$, where $\pi_1(X_1)$ has a subgroup of finite index ≥ 3 and X_2 is simply connected. Then, for any 0-bumpy metric g on M, $N_g^0(t)$ grows exponentially with t unless X_2 is a homotopy sphere.*

The novelty in Theorem B lies in the fact that M is allowed to have any fundamental group as long as it has subgroups of finite index ≥ 3. Results of this kind, when M is simply connected were obtained by P. Lambrechts in [18] and in fact, we will rely quite heavily on his Hochschild homology computations.

We note that there are finitely presented infinite groups without proper subgroups of finite index. An example is given by the Higman 4-group, see [23]. We do not know if Theorem B is still true for such groups.

A celebrated theorem of D. Gromoll and W. Meyer [10] asserts that any Riemannian metric on a closed simply connected manifold M has infinitely many geometrically distinct closed geodesics if the Betti numbers of $\Lambda(M)$ are unbounded for some field of coefficients. Their methods combined with our proof of Theorem B yield the existence of infinitely many geometrically distinct *contractible* closed geodesics for *any* metric on a manifold as in Theorem B. We do not know if the theorem is still true if we drop the bumpy condition and we just assume $N_g^0(t)$ finite for all t.

There are several papers establishing lower bounds for the growth of $N_g(t)$ in the presence of fundamental group [4, 5, 7, 8, 15]. These bounds are not exponential (e.g. $t/\log t$) and do not give information about *contractible* closed geodesics, but they do hold for *any* Riemannian metric.

The present note is a spin-off of our investigations on the topological entropy $\mathrm{h_{top}}(g)$ of the geodesic flow [20, 21]. Except in the case of surfaces, there is basically no relationship between positivity of $\mathrm{h_{top}}(g)$ and exponential growth for $N_g(t)$. M. Herman gave in [14] an example of a minimal real analytic diffeomorphism on a closed 4-manifold with positive topological entropy. However, if the geodesic flow of g has a horseshoe, then the number of *hyperbolic* closed geodesics of g grows exponentially. It has been conjectured that generically in the C^k topology, $2 \leq k \leq \infty$, the geodesic flow of g has a horseshoe, but this has only been proved for surfaces [9, 17].

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2. A brief review of Morse theory of the loop space

Let $\Lambda = \Lambda(M)$ be the space of piecewise differentiable closed curves $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$, endowed with the compact-open topology, i.e. the topology induced by the metric $\rho(c, c') = \max_{t \in S^1} d(c(t), c'(t))$. The energy functional $E : \Lambda \to \mathbb{R}$ is defined by $E(c) = \frac{1}{2} \int_0^1 \langle \dot{c}(t), \dot{c}(t) \rangle \, dt$. The critical points of E are the closed geodesics and the point curves. If $\ell(c)$ denotes the length of c, then the Cauchy-Schwarz inequality implies that $\ell(c)^2 \leq 2\, E(c)$ with equality if and only if c is parametrized proportional to arc length.

There is a natural action of S^1 on Λ. Given $s \in S^1$ and $c \in \Lambda$, we can define sc by $sc(t) := c(t+s)$. If c does not reduce to a point, then the isotropy group of c is a finite subgroup of S^1 isomorphic to $\mathbb{Z}_k$ for some $k \geq 1$. In this case, k is called the *multiplicity* of c and we can write $c = d^k$, where $d^k(t) = d(kt)$. The curve c is called the *k-th iterate* of d.

If c is a closed geodesic, then the entire orbit of c under the S^1-action is a set of critical points of E. A closed geodesic c is said to be *non-degenerate* if the orbit of c is a non-degenerate critical submanifold of Λ. Equivalently, c is non-degenerate if and only if there is no periodic Jacobi field along c orthogonal to $\dot{c}$. In more dynamical terms, c is non-degenerate if and only if the linearized Poincaré map of the orbit of the geodesic flow corresponding to c does not have 1 as an eigenvalue.

A metric is said to be *bumpy* (resp. *0-bumpy*) if all closed (resp. contractible) geodesics are non-degenerate. The bumpy metric theorem asserts that the set of C^r bumpy metrics is a residual subset of the set of all C^r metrics endowed with the C^r topology for all $2 \leq r \leq \infty$. The bumpy metric theorem is traditionally attributed to R. Abraham [1], but see also Anosov [2] and Klingenberg and Takens [16]. Obviously bumpy implies 0-bumpy.

3. Proof of Proposition A

We will need the following version of a result due to M. Gromov [11] for manifolds with non-empty boundary, see also [12, p. 102], [13, 3, 19]. In all these references the manifold is assumed to have empty boundary. In the case of the pointed loop space $\Omega(X)$ we have written down a proof of the theorem for compact manifolds with non-empty boundary in [21]. For completeness we will show in the appendix that the details work as well for the free loop space $\Lambda(X)$.

Theorem 3.1. *Given a metric g on a simply connected compact manifold X (possibly with boundary), there exists a constant $C = C(g) > 0$ such that any element in $H_i(\Lambda(X), k_p)$ can be represented by a cycle whose image lies in $\Lambda^{Ci}(X)$.*

Let us prove now Proposition A. Let $\Lambda_0 \subset \Lambda$ denote the subspace of contractible closed loops. If all the contractible closed geodesics are non-degenerate, then given $a > 0$, there exist only finitely many such closed geodesics $c_1, \ldots, c_r$ with energy precisely a. There also exists an $\varepsilon > 0$ such that there is no contractible closed geodesic c with $E(c) \neq a$ but $a - \varepsilon \leq E(c) \leq a + \varepsilon$. It follows from Morse theory that $\Lambda_0^{a+\varepsilon}$ is homotopy equivalent to $\Lambda_0^{a-\varepsilon}$ with cells e_i, $\bar{e}_i$ attached, where $\dim e_i = \dim \bar{e}_i - 1 =$ index of c_i. It follows from this that (Λ_0^a, M) is homotopy equivalent to a relative CW-complex, where the number of k-cells attached to M is equal to the number of critical circles $S^1 c$ of E with index of c equal to k or $k-1$ and $0 < E(c) \leq a$. Observe that the iterates of a closed geodesic may give rise to different critical circles of E of index k or $k-1$.

Let $b_i(t) := \dim H_i(\Lambda_0^t, M)$. It follows from the above that the number of prime and iterate contractible closed geodesics of index i or $i-1$ and length $\leq t$ is $\geq b_i(t)$.

We will need the following lemma proved in [6] that gives control of the contribution of a closed geodesic c and its iterates to $b_i(t)$. The lemma is not properly stated as

such in [6], but the statement can be found at the end of the proof of their main theorem.

Lemma 3.2. *There exists a constant $n_0 = n_0(g)$ such that at most n_0 iterates of a contractible closed geodesic of length $\leq t$ can have index i or $i-1$ if $t/2 < i \leq t$.*

From the lemma it follows that $N_g^0(t) \geq \max_{t/2<i\leq t} b_i(t)/n_0$. Since $N_g^0(t) \geq N_g^0(t/2)$ we obtain $N_g^0(t) \geq \max_{1<i\leq t} b_i(t)/n_0$.

Proposition A is now a consequence of the following lemma:

Lemma 3.3. *There exists a constant $C = C(g) > 0$ such that $b_i(t) \geq R_i$ for $n+1 < i \leq C\,t$.*

Proof. First note that $H_i(\Lambda_0^t) \cong H_i(\Lambda_0^t, M)$ for $i > n+1$, so we only need to prove that $b_i(\Lambda_0^t) \geq R_i$ for $i \leq C\,t$.

Now take an element $e \in H_i(\Lambda(X))$. By Theorem 3.1 (we consider $X \subset M$ with the induced metric) there exists $C > 0$ (depending on g) such that whenever $C\,t \geq i$, there exists a cycle $\eta \in \Lambda^t(X)$ such that η represents e. Since ι preserves energy, $\iota \circ \eta$ is a cycle in Λ_0^t and the lemma follows.

□

4. Proof of Theorem B

Proof. Let $\widetilde{X_1}$ be a finite covering of X_1 of degree $k \geq 3$. Consider the covering N of $M = X_1 \# X_2$ given by the connected sum of $\widetilde{X_1}$ with k copies of X_2. Since the exponential growth rate of closed geodesics is invariant under finite coverings, we only need to work with N. We now want to find a submanifold of N to play the role of X in Proposition A. If D is an n-dimensional disc then $X := D \# 3X_2$ appears as a submanifold of N.

Lemma 4.1. *The inclusion map $\iota : X \to N$ induces a map*

$$\iota_* : H_*(\Lambda(X), k_p) \to H_*(\Lambda(N), k_p)$$

whose rank grows exponentially for some p.

Proof. There is a map $p : M \# 3X_2 \to 3X_2$ obtained by collapsing the exterior and the boundary of a disc $D \subset M$ (which contains the 3 points where the connected sums are taken) to a point. If $j = p \circ \iota$ it is enough to prove that the rank of the map induced by j in the free loop space homology grows exponentially.

Clearly, j is just the inclusion of $D \# 3X_2$ in $3X_2 = S^n \# 3X_2$. In particular, the computations done in [18] apply to this problem. If X_2 is not a homotopy sphere, there is a minimal degree in which its homology is non-trivial for some coefficient field k_p. In $3X_2$ we have then three cohomology classes in this minimal degree which are independent and whose cup product vanish. These cohomology classes imply the existence of elements in the corresponding DGA's which satisfy the hypothesis of [18, Proposition 9]. Of course, j sends these cohomology classes to corresponding ones in $3X_2 - D$. Then one looks at the map induced in Hochschild homology and the exponential growth of the rank follows.

□

Theorem B follows directly from the previous lemma and Proposition A.

□

5. Appendix: Proof of Theorem 3.1 when ∂X is non-empty

Proof. : We will only modify the proof in [21] to show that it works in the case of the free loop space. Assume that the boundary of X is non-empty. We have a collar of the boundary diffeomorphic to $\partial X \times [0,2)$. Let Y be the manifold obtained by deleting $\partial X \times [0,1)$ from X. Of course, Y is diffeomorphic to X. Now find a finite number of convex subsets of X which cover Y. Call them V_α, $1 \leq \alpha \leq k_0$. Let $\mathbf{T}$ be a triangulation of Y. For a point $p \in Y$, let $F(p)$ be the closed cell of lowest dimension containing p and let $O(p)$ be the union of all closed cells intersecting $F(p)$. Note that $O(p)$ is a compact subset of Y. Similarly, for any subset $K \subset Y$ one can define $F(K)$ as the union of $F(p)$ for $p \in K$ and $O(K)$ as the union of $O(p)$ for $p \in K$. There is a positive number δ (depending on the covering $\{V_\alpha\}$ and g) such that, after taking some barycentric subdivisions, we can take $\mathbf{T}$ so that for every subset K of diameter bounded by δ, $O(K)$ is contained in one of the V_α's.

Now we define the open subset Λ_k of $\Lambda(\cup V_\alpha)$ as the set of all paths ω in $\cup V_\alpha$ with $\omega(0) = \omega(1)$ so that for all j between 1 and 2^k,

$$O(\{\omega(j - 1/2^k), \omega(j/2^k)\}) \ \cup \ \omega[j - 1/2^k, j/2^k]$$

is contained in one of the V_α's. It is easy to see that $\Lambda(Y)$ is contained in the union of the Λ_k's.

Let B_k be the set of sequences $p_0, ..., p_{2^k}$ of points in Y such that $p_0 = p_{2^k}$ and for each j between 1 and 2^k $O(\{p_{j-1}, p_j\})$ is contained in one of the V_α's. Let $\Lambda_k^Y \subset \Lambda_k$ be the set of paths in Λ_k for which all 2^k points $\omega(j/2^k)$ are in Y. Then B_k is naturally identified with a subset of Λ_k^Y (an element of B_k uniquely determines a broken geodesic which sends $j/2^k$ to p_j) and it is actually a deformation retract of Λ_k^Y.

Given a cycle representing a homology class in $\Lambda(X)$ it can be represented by a cycle in $\Lambda(Y)$ which is therefore contained in Λ_k^Y for some k. Therefore we can retract it to B_k. But B_k is easily identified with a subset of Y^{2^k}. Moreover, under this identification if a point $(p_1, ..., p_{2^k}) \in B_k$, then the whole $F(p_1) \times ... \times F(p_{2^k})$ is contained in B_k. This implies that $\mathbf{T}$ induces a cell decomposition in B_k. Hence the i-homology class can be represented by a combination of cells of dimension i. A cell in B_k is a product of cells in each coordinate. The dimension of such a cell is the sum of the dimensions of the corresponding cells, of course. If the total dimension is i then there can be at most i cells of positive dimension. Since X is simply connected there exists a smooth map $f : X \to X$ which is smoothly homotopic to the identity and which sends the union of the images of all the geodesic segments joining vertices in the triangulation to a point. The norm of the differential of f is bounded since X is compact and f induces a map $\hat{f} : \Lambda(X) \to \Lambda(X)$ which is homotopic to the identity. Now paths belonging to an i-cell of B_k are formed by pieces joining vertices of the triangulation and at most $2i$ pieces in which one of the points is not a vertex. Under the map $\hat{f}$ the former are sent to a point and the latter to a path of length bounded

in terms of the norm of the differential of f and the diameter of the V_α's. Therefore, there exists a constant $C(g)$ such that the image of the i-skeleton of B_k is sent by $\hat{f}$ to the subset of paths with energy bounded by $C(g)\ i$. The theorem follows.

□

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