A brief summary of the theory

Gabriel P. Paternain

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UNIVERSITY OF CAMBRIDGE

We started assuming knowledge about power series (radius of convergence) from Analysis I and uniform convergence from Analysis II.

Def of holomorphic function $\iff$ Cauchy-Riemann eqns

$$f = u + iv, \qquad u_x = v_y, \;\; u_y = -v_x.$$

Power series can be differentiated ∞ many times.

New concept: integration along curves.

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Cauchy Thm for Δs: $f : U \to \mathbb{C}$ holomorphic, $T \subset U$

$$\int_{\partial T} f(z)\, dz = 0.$$

(Subdivision argument.)

Plus

Antiderivative Thm: $f : U \to \mathbb{C}$ continuous with

$$\int_{\partial T} f(z)\, dz = 0$$

for all $T \subset U$ star-domain, then there is F s.t. $F' = f$.

Gives via the the FTC

Convex Cauchy: $f : U \to \mathbb{C}$ hol in star-domain U, then

$$\int_{\gamma} f(z)\, dz = 0 \quad \text{for all closed } \gamma.$$

Convex Cauchy applied to $\frac{f(z)-f(w)}{z-w}$ in $D(a,r)$ gave CIF (addendum):

$$f(w) = \frac{1}{2\pi i}\int_{|z-a|=\rho} \frac{f(z)}{z-w}\,dz, \quad \rho < r, \quad w \in D(a,\rho).$$

$$\Longrightarrow$$

Liouville, Maximum modulus principle, Mean value property, FTA.

$\Longrightarrow$ Using $\frac{1}{z-w} = \sum \frac{(w-a)^n}{(z-a)^{n+1}}$ we got

Taylor expansion

f is ∞ differentiable

Morera (converse Cauchy)

uniform limit of holomorphic functions is holomorphic

Riemann zeta function : $\zeta(s)$

Principle of isolated zeros $\Longrightarrow$ Analytic continuation

Topological ideas: Index and homotopy

$$I(\gamma, w) := \frac{\theta(b) - \theta(a)}{2\pi} \in \mathbb{Z},$$

where we write $\gamma(t) = w + r(t)e^{i\theta(t)}$, with θ continuous (γ closed and $w \notin$ image(γ).

Alternative way of writing the index:

$$I(\gamma, w) = \frac{1}{2\pi} \int_\gamma \frac{dz}{z - w}.$$

Constant on connected components of $\mathbb{C} \setminus$ image(γ).

Notion of homotopy: continuous deformations of curves.

A theme: "from local to global". This gave Homotopy form of Cauchy thm: $f : U \to \mathbb{C}$ holomorphic with ϕ and ψ homotopic, then

$$\int_\phi f(z)\,dz = \int_\psi f(z)\,dz.$$

Applying this to $\frac{f(z)-f(w)}{z-w}$ and two concentric circles bounding an annulus we got

The Laurent Series: $f(z) = \sum_{-\infty}^{\infty} c_n(z-a)^n$

$$\Longrightarrow$$

Isolated singularities: removable, poles and essential

$$c_{-1} = \mathrm{Res}_{z=a} f.$$

$f : U \setminus \{z_1, \ldots, z_k\} \to \mathbb{C}$ holomorphic, U simply connected.

Residue theorem:

$$\int_\gamma f(z)\, dz = 2\pi i \sum_j I(\gamma, z_j) \mathrm{Res}_{z=z_j} f.$$

$$\Longrightarrow$$

Computing definite integrals via residues (Jordan's lemma etc)

$$\Longrightarrow$$

Looking at $\frac{f'}{f}$ and $I(f \circ \gamma, 0) = N - P$ we got

Argument principle, local degree, open mapping thm, Rouché.