Part III: Differential geometry (Michaelmas 2004)

Alexei Kovalev (A.G.Kovalev@dpmms.cam.ac.uk)

2 Vector bundles.

Definition. Let B be a smooth manifold. A manifold E together with a smooth submersion[1] $\pi : E \to B$, onto B, is called a **vector bundle of rank k over B** if the following holds:

(i) there is a k-dimensional vector space V, called **typical fibre** of E, such that for any point $p \in B$ the fibre $E_p = \pi^{-1}(p)$ of π over p is a vector space isomorphic to V;

(ii) any point $p \in B$ has a neighbourhood U, such that there is a diffeomorphism

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ U & = & U \end{array}$$

and the diagram commutes, which means that every fibre E_p is mapped to $\{p\} \times V$.

Φ_U is called a **local trivialization** of E over U and U is a **trivializing neighbourhood** for E.

(iii) $\Phi_U|_{E_p} : E_p \to V$ is an isomorphism of vector spaces.

Some more terminology: B is called the **base** and E the **total space** of this vector bundle.

$\pi : E \to B$ is said to be a *real* or *complex* vector bundle corresponding to the typical fibre being a real or complex vector space. Of course, the linear isomorphisms etc. are understood to be over the respective field $\mathbb{R}$ or $\mathbb{C}$. In what follows vector bundles are taken to be real vector bundles unless stated otherwise.

Definition. Any smooth map $s : B \to E$ such that $\pi \circ s = \mathrm{id}_B$ is called a **section** of E. If s is only defined over a neighbourhood in B it is called a **local section**.

Examples. 0. A trivial, or product, bundle $E = B \times V$ with π the first projection. Sections of this bundle are just the smooth maps $C^\infty(B; V)$.

1. The tangent bundle TM of a smooth manifold M has already been discussed in Chapter 1. It is a real vector bundle of rank $n = \dim M$ which in general is not trivial.[2] The sections of TM are the vector fields. In a similar way, the cotangent bundle T^*M and,

[1] A smooth map is called a *submersion* if its differential is surjective at each point.

[2] Theorems 1.9 and 1.26 in Chapter 1 imply that TS^{2n} cannot be trivial.

more generally, the bundle of differential p-forms $\Lambda^p T^*M$ are real vector bundles of rank $\binom{n}{p}$ with sections being the differential 1-forms, respectively p-forms. Exercise: verify that the vector bundles $\Lambda^p T^*M$ ($1 \le p \le \dim M$) will be trivial if TM is so.

2. 'Tautological vector bundles' may be defined over projective spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$ (and, more generally, over the Grassmannians). Let $B = \mathbb{C}P^n$ say. Then let E be the disjoint union of complex lines through the origin in $\mathbb{C}^{n+1}$, with π assigning to a point in $p \in E$ the line ℓ containing that point, so $\pi(p) = \ell \in \mathbb{C}P^n$. We shall take a closer look at one example (Hopf bundle) below and show that the tautological construction indeed gives a well-defined (and non-trivial) complex vector bundle of rank 1 over $\mathbb{C}P^1$.

Structure group of a vector bundle.

It follows from the definition of a vector bundle E that one can define over the intersection of two trivializing neighbourhoods U_β, U_α a composite map

$$\Phi_\beta \circ \Phi_\alpha^{-1}(b, v) = (b, \psi_{\beta\alpha}(b)v),$$

$(b, v) \in (U_\beta \cap U_\alpha) \times \mathbb{R}^k$. For every fixed b the above composition is a linear isomorphism of $\mathbb{R}^k$ depending smoothly on b. The maps $\psi_{\beta\alpha} : U_\beta \cap U_\alpha \to GL(k, \mathbb{R})$. are called the **transition functions** of E.

It is not difficult to see that transition functions $\psi_{\alpha\beta}$ satisfy the following relations, called 'cocycle conditions'

$$\left.\begin{aligned} &\psi_{\alpha\alpha} = \mathrm{id}_{\mathbb{R}^k}, \\ &\psi_{\alpha\beta}\psi_{\beta\alpha} = \mathrm{id}_{\mathbb{R}^k}, \\ &\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = \mathrm{id}_{\mathbb{R}^k}\,. \end{aligned}\right\} \tag{2.1}$$

The left-hand side is defined on the intersection $U_\alpha \cap U_\beta$, for the second of the above equalities, and on $U_\alpha \cap U_\beta \cap U_\gamma$ for the third. (Sometimes the name 'cocycle condition' refers to just the last of the equalities (2.1); the first two may be viewed as notation.)

Now it may happen that a vector bundle $\pi : E \to B$ is endowed with a system of trivializing neighbourhoods U_α covering the base and such that all the corresponding transition functions $\psi_{\beta\alpha}$ take values in a *subgroup* $G \subseteq GL(k, \mathbb{R})$, $\psi_{\beta\alpha}(b) \in G$ for all $b \in U_\beta \cap U_\alpha$, for all α, β, where k is the rank of E. Then this latter system $\{(U_\alpha, \Phi_\alpha)\}$ of local trivializations over U_α's is said to define a G-**structure** on vector bundle E.

Examples. 0. If G consists of just one element (the identity) then E has to be a trivial bundle $E = B \times \mathbb{R}^k$.

1. Let $G = GL_+(k, \mathbb{R})$ be the subgroup of matrices with positive determinant. If the typical fibre $\mathbb{R}^k$ is considered as an oriented vector space then the transition functions $\psi_{\beta\alpha}$ preserve the orientation. The vector bundle E is then said to be **orientable**.

A basic example arises from a system of coordinate charts giving an orientation of a manifold M. The transition functions of TM are just the Jacobians and so M is orientable precisely when its tangent bundle is so.

2. A more interesting situation occurs when $G = O(k)$, the subgroup of all the non-singular linear maps in $GL(k, \mathbb{R})$ which preserve the Euclidean inner product on $\mathbb{R}^k$. It

follows that the existence of an $O(k)$ structure on a rank k vector bundle E is equivalent to a well-defined positive-definite inner product on the fibres E_p. This inner product is expressed in any trivialization over $U \subseteq B$ as a symmetric positive-definite matrix depending smoothly on a point in U.

Conversely, one can *define* a vector bundle with inner product by modifying the definition on page 20: replace every occurrence of 'vector space' by 'inner product space' and (linear) 'isomorphism' by (linear) 'isometry'. This will force all the transition functions to take value in $O(k)$ (why?).

A variation on the theme: an orientable vector bundle with an inner product is the same as vector bundle with an $SO(k)$-structure.

3. Another variant of the above: one can play the same game with rank k *complex* vector bundles and consider the $U(k)$-structures ($U(k) \subset GL(k, \mathbb{C})$). Equivalently, consider complex vector bundles with Hermitian inner product 'varying smoothly with the fibre'. Furthermore, complex vector bundles themselves may be regarded as rank $2k$ real vector bundles with a $GL(k, \mathbb{C})$-structure (the latter is usually called a complex structure on a vector bundle).

In the examples 2 and 3, if a trivialization Φ is 'compatible' with the given $O(k)$- or $SO(k)$-structure (respectively $U(k)$-structure) $\{(U_\alpha, \Phi_\alpha)\}$ in the sense that the transition functions $\Phi_\alpha \circ \Phi^{-1}$ take values in the orthogonal group (respectively, unitary group) then Φ is called an **orthogonal trivialization** (resp. **unitary trivialization**).

Principal bundles.

Let G be a Lie group. A **smooth free right action** of G on a manifold P is a smooth map $P \times G \to P$, $(p, h) \mapsto ph$, such that (1) for any $p \in P$, $ph = p$ if and only if h is the identity element of G; and (2) $(ph_1)h_2 = p(h_1 h_2)$ for any $p \in P$, any $h_1, h_2 \in G$. (It follows that for each $h \in G$, $P \times \{h\} \to P$ is a diffeomorphism.)

Definition . A (smooth) **principal G-bundle** P over B is a smooth submersion $\pi : P \to B$ onto a manifold B, together with a smooth right free action $P \times G \to P$, such that the set of orbits of G in P is identified with B (as a set), $P/G = B$, and also for any $b \in B$ there exists a neighbourhood $U \subseteq B$ of b and a diffeomorphism $\Phi_U : \pi^{-1}(U) \to U \times G$ such that $\mathrm{pr}_1 \circ \Phi_U = \pi|_{\pi^{-1}(U)}$, i.e. the following diagram is commutative

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\Phi_U} & U \times G \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mathrm{pr}_1} \\ U & = & U \end{array} \tag{2.2}$$

and Φ_U commutes with the action of G, i.e. for each $h \in G$, $\Phi_U(ph) = (b, gh)$, where $(b, g) = \Phi_U(p)$, $\pi(p) = b \in U$.

A **local section** of the principal bundle P is a smooth map $s : U \to P$ defined on a neighbourhood $U \subset B$ and such that $\pi \circ s = \mathrm{id}_U$.

For a pair of overlapping trivializing neighbourhoods U_α, U_β one has

$$\Phi_\beta \circ \Phi_\alpha^{-1}(b,g) = (b, \psi_{\beta\alpha}(b,g)),$$

where for each $b \in U_\alpha \cap U_\beta$, the $\psi_{\beta\alpha}(b,\cdot)$ is a map $G \to G$. Then, for each b in the domain of $\psi_{\beta\alpha}$, we must have $\psi_{\beta\alpha}(b,g)h = \psi_{\beta\alpha}(b,gh)$ for all $g,h \in G$, in view of (2.2). It follows (by taking g to be the unit element 1_G) that the map $\psi_{\beta\alpha}(b,\cdot)$ is just the *multiplication on the left* by $\psi_{\beta\alpha}(b,1_G) \in G$. It is sensible to slightly simplify the notation and write $g \mapsto \psi_{\beta\alpha}(b)g$ for this left multiplication. We find that, just like vector bundles, the principal G-bundles have transition functions $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ between local trivializations. In particular, $\psi_{\beta\alpha}$ for a principal bundle satisfy the same cocycle conditions (2.1).

A principal G-bundle over B may be obtained from a system of $\psi_{\beta\alpha}$, corresponding to an open cover of B and satisfying (2.1), — via the following 'Steenrod construction'. For each trivializing neighbourhood $U_\alpha \subset B$ for E consider $U_\alpha \times G$. Define an equivalence relation between elements $(b,h) \in U_\alpha \times G$, $(b',h') \in U_\beta \times G$, so that $(b,h) \sim (b',h')$ precisely if $b' = b$ and $h' = \psi_{\beta\alpha}(b)h$. Now let

$$P = \sqcup_\alpha \big(U_\alpha \times G\big)/\sim \tag{2.3}$$

the disjoint union of all $U_\alpha \times G$'s glued together according to the equivalence relation.

Theorem 2.4. *P defined by* (2.3) *is a principal G-bundle.*

Remark. The $\psi_{\beta\alpha}$'s can be taken from some vector bundle E over B, then P will be 'constructed from E'. The construction can be reversed, so as to start from a principal G-bundle P over a base manifold B and obtain the vector bundle E over B. Then E will be automatically given a G-structure.

In either case the data of transition functions is *the same* for the principal G-bundle P and the vector bundle P. The difference is in the action of the structure group G on the typical fibre. G acts on itself by left translations in the case of the principal bundle and G acts as a subgroup of $GL(k,\mathbb{R})$ on $\mathbb{R}^k$ in the case of vector bundle E. [3] The vector bundle E is then said to be **associated** to P via the action of G on $\mathbb{R}^k$.

Example: Hopf bundle.

Hopf bundle may be defined as the 'tautological' (see page 21) rank 1 complex vector bundle over $\mathbb{C}P^1$. The total space E of Hopf bundle, as a set, is the disjoint union of all (complex) lines passing through the origin in $\mathbb{C}^2$. Recall that every such line is the fibre over the corresponding point in $\mathbb{C}P^1$. We shall verify that the Hopf bundle is well-defined by working its transition functions, so that we can appeal to Theorem 2.4.

For a covering system of trivializing neighbourhoods in $\mathbb{C}P^1$, we can choose the coordinate patches of the smooth structure of $\mathbb{C}P^1$, defined in Chapter 1. Thus

$$\mathbb{C}P^1 = U_1 \cup U_2, \qquad U_i = \{z_1 : z_2 \in \mathbb{C}P^1, z_i \neq 0\}, \qquad i = 1,2,$$

[3] G need not be explicitly a subgroup of $GL(k,\mathbb{R})$, it suffices to have a *representation* of G on $\mathbb{R}^k$.

with the local complex coordinate $z = z_2/z_1$ on U_1, and $\zeta = z_1/z_2$ on U_2, and $\zeta = 1/z$ when $z \neq 0$. We shall denote points in the total space E as (wz_1, wz_2), with $|z_1|^2 + |z_2|^2 \neq 0$, $w \in \mathbb{C}$, so as to present each point as a vector with coordinate w relative to a basis (z_1, z_2) of a fibre. (This clarification is needed in the case when $(wz_1, wz_2) = (0, 0) \in \mathbb{C}^2$.) An 'obvious' local trivialization over U_i may be given, say over U_1, by $(w, wz) \in \pi^1(U_1) \to (1 : z, w) \in U_1 \times \mathbb{C}$, —but in fact this is not a very good choice. Instead we define

$$\begin{aligned}\Phi_1 : (w, wz) \in \pi^{-1}(U_1) &\to (1 : z,\ w\sqrt{1 + |z|^2}) \in U_1 \times \mathbb{C},\\ \Phi_2 : (w\zeta, w) \in \pi^{-1}(U_2) &\to (\zeta : 1,\ w\sqrt{1 + |\zeta|^2}) \in U_2 \times \mathbb{C}.\end{aligned}$$

Calculating the inverse, we find

$$\Phi_1^{-1}(1 : z, w) = \left(\frac{w}{\sqrt{1 + |z|^2}}, \frac{wz}{\sqrt{1 + |z|^2}}\right)$$

and so

$$\begin{aligned}\Phi_2 \circ \Phi_1^{-1}(1 : z, w) &= \Phi_2\left(\frac{w}{\sqrt{1 + |z|^2}}, \frac{wz}{\sqrt{1 + |z|^2}}\right)\\ &= \Phi_2\left(\frac{|\zeta| w}{\zeta\sqrt{|\zeta|^2 + 1}}\,\zeta, \frac{|\zeta| w}{\zeta\sqrt{|\zeta|^2 + 1}}\right) = \big(\zeta : 1, \frac{|\zeta|}{\zeta} w\big) = \big(1 : z, \frac{z}{|z|} w\big)\end{aligned}$$

giving the transition function $\tau_{2,1}(1 : z) = (z/|z|)$, for $1 : z \in U_1 \cap U_2$ (i.e. $z \neq 0$). The $\tau_{2,1}$ takes values in the *unitary group* $U(1) = S^1 = \{z \in \mathbb{C} : |z| = 1\}$, a subgroup of $GL(1, \mathbb{C}) = \mathbb{C} \setminus \{0\}$ (it is for this reason the square root factor was useful in the local trivialization). Theorem 2.4 now ensures that Hopf bundle E is a well-defined vector bundle, moreover a vector bundle with a $U(1)$-structure. Hence there is a *invariantly defined* notion of *length* of any vector in any fibre of E. The length of (wz_1, wz_2) may be calculated in the local trivializations Φ_1 or Φ_2 by taking the modulus of the second component of Φ_i (in $\mathbb{C}$). For each $i = 1, 2$, this coincides with the familiar Euclidean length $\sqrt{|wz_1|^2 + |wz_2|^2}$ of (wz_1, wz_2) in $\mathbb{C}^2$.

We can now use $\tau_{2,1}$ to construct the principal S^1-bundle (i.e. $U(1)$-bundle) $P \to \mathbb{C}P^1$ associated to Hopf vector bundle E, cf. Theorem 2.4. If $U(1)$ is identified with a unit circle $S^1 \subset \mathbb{C}$, any fibre of P may be considered as the unit circle in the respective fibre of E. Thus P is identified as the space of all vectors in E of length 1, so $P = \{(w_1, w_2) \in \mathbb{C}^2 \mid w_1\bar{w}_1 + w_2\bar{w}_2 = 1\}$ is the 3-dimensional sphere and the bundle projection is

$$\pi : (w_1, w_2) \in S^3 \to w_1 : w_2 \in S^2, \qquad \pi^{-1}(p) \cong S^1,$$

where we used the diffeomorphism $S^2 \cong \mathbb{C}P^1$ for the target space. (Examples 1, Q3(ii).) This principal S^1-bundle S^3 over S^2 is also called Hopf bundle. It is certainly not trivial, as S^3 is not diffeomorphic to $S^2 \times S^1$. (The latter claim is not difficult to verify, e.g. by showing that the de Rham cohomology $H^1(S^3)$ is trivial, whereas $H^1(S^2 \times S^1)$ is not. Cf. Examples 2 Q5.)

Pulling back vector bundles and principal bundles

Let P be a principal bundle over a base manifold B and E an associated vector bundle over B. Consider a smooth map $f : M \to B$.

The **pull-back of a vector (respectively, principal) bundle** is a bundle f^*E (f^*P) over M such that there is a commutative diagram (vertical arrows are the bundle projections)

$$\begin{array}{ccc} f^*E & \xrightarrow{F} & E \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & B, \end{array} \tag{2.5}$$

such that the restriction of F to each fibre $(f^*E)_p$ over $p \in M$ is an isomorphism onto a fibre $E_{f(p)}$.

A very basic special case of the above is when f maps M to a point in B; then the pull-back f^*E (and f^*P) is necessarily a trivial bundle (exercise: write out a trivialization map $f^*E \to M \times$ (typical fibre)). As a slight generalization of this example consider the case when $M = B \times X$, for some manifold X with $f : B \times X \to B$ the first projection. Then f^*E (resp. f^*P) may be thought of as bundles 'trivial in the X direction', e.g. $f^*E \cong E \times X$, with the projection $(e, x) \in E \times X \to (\pi(e), x) \in B \times X$.

The construction may be extended to a general vector bundle, by working in local trivialization. Then one has to ensure that the pull-back must be well-defined independent of the choice of local trivialization. To this end, let $\{\psi_{\beta\alpha}\}$ be a system of transition functions for E. Define

$$f^*\psi_{\beta\alpha} = \psi_{\beta\alpha} \circ f$$

and $f^*\psi_{\beta\alpha}$ is a system of functions on M satisfying the cocycle condition (2.1). Therefore, by Theorem 2.4 and a remark following this theorem, the $f^*\psi_{\beta\alpha}$ are transition functions for a well-defined vector bundle and principal bundle over M. Steenrod construction shows that these are indeed the pull-back bundles f^*E and f^*P as required by (2.5).

2.1 Bundle morphisms and automorphisms.

Let (E, B, π) and (E', B', π') be two vector bundles, and $f : B \to B'$ a smooth map.

Definition. A smooth map $F : E \to E'$ is a **vector bundle morphism covering** f if for any $p \in B$ F restricts to a linear map between the fibres $F : E_p \to E'_{f(p)}$ for any $p \in B$, so that

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi'}\big\downarrow \\ B & \xrightarrow{f} & B' \end{array}$$

is a commutative diagram, $\pi' \circ F = f \circ \pi$.

More explicitly, suppose that the local trivializations $\Phi : \pi^{-1}(U) \to U \times V$, $\Phi' : \pi'^{-1}(U') \to U' \times V'$ are such that $f(U) \subseteq U'$. Then the restriction $F_U = \Phi' \circ F|_{\pi^{-1}(U)} \circ \Phi^{-1}$ is expressed as

$$F_U : (b, v) \mapsto (f(b), h(b)v), \tag{2.6}$$

for some smooth $h : U \to L(V, V')$ family of linear maps between vector spaces V,V' depending on a point in the base manifold. In particular, it is easily checked that a composition of bundle morphisms $E \to E'$, $E' \to E''$ is a bundle morphism $E \to E''$.

Examples. 1. If $\varphi : M \to N$ is a smooth map between manifolds M,N then its differential $d\varphi : TM \to TN$ is a morphism of tangent bundles.

2. Recall the pull-back of a given vector bundle (E, B, π) via a smooth map $f : M \to B$. For every local trivialization $E|_U$ of E, the corresponding local trivialization of the pull-back bundle is given by $f^*E|_{f^{-1}(U)} \to (f^{-1}(U)) \times V$, where V is the typical fibre of E (hence also of f^*E). Thus the pull-back construction gives a well-defined map $F : f^*E \to E$ which restricts to a linear isomorphism between any pair of fibres $(f^*E)_p$ and $E_{f(p)}$, $p \in M$. (This isomorphism becomes just the identity map of the typical fibre V in the indicated local trivializations.) It follows that F is a bundle morphism covering the given map $f : M \to B$.

3. Important special case of bundle morphisms occurs when f is a diffeomorphism of B onto B'. A morphism $F : E \to E'$ between two vector bundles over B covering f is called an **isomorphism of vector bundles** if F restricts to a linear isomorphism $E_p \to E_{f(p)}$, for every fibre of E.

An isomorphism from a vector bundle E to itself covering the identity map id_B is called a **bundle automorphism** of E. The set $\operatorname{Aut} E$ of all the bundle automorphisms of E forms a group (by composition of maps). If $E = B \times V$ is a trivial bundle then any automorphism of E is defined by a smooth maps $B \to GL(V)$, so $\operatorname{Aut} E = C^\infty(B, GL(V))$.

If a vector bundle E has a G-structure ($G \subseteq GL(V)$) then it is natural to consider the group of G-bundle automorphisms of E, denoted $\operatorname{Aut}_G E$ and defined as follows. Recall that a G-structure means that there is a system of local trivializations over neighbourhoods covering the base B and with the transition functions of E taking values in G. Now a bundle automorphism $F \in \operatorname{Aut} E$ of $E \to B$ is determined in any local trivialization over open

$U \subseteq B$ by a smooth map $h : U \to GL(V)$, as in (2.6). Call F a G-**bundle automorphism** if for any of the local trivializations defining the G-structure this map h takes values in the subgroup G. It follows that $\operatorname{Aut}_G E$ is a subgroup of $\operatorname{Aut} E$. (In the case of trivial bundles the latter statement becomes $C^\infty(B, G) \subseteq C^\infty(B, GL(V))$.)

Remark. The group $\operatorname{Aut}_G E$ for the vector bundle E with G-structure has the same significance as the group of all self-diffeomorphisms of M for a smooth manifold M or the group of all linear isometries of an inner product vector space. I.e. $\operatorname{Aut}_G E$ is the 'group of natural symmetries of E' and properties any objects one considers on the vector bundle are geometrically meaningful if they are preserved by this symmetry group.

One more remark on bundle automorphisms. A map h_α giving the local expression over $U_\alpha \subseteq B$ for a bundle automorphism may be interchangeably viewed as a *transformation from one system of local trivializations to another*. Any given local trivialization, say Φ_α over U_α, is replaced by Φ'_α. We have $\Phi'_\alpha(e) = h_\alpha(\pi(e))\Phi_\alpha(e)$, $e \in E$. Respectively, the transition functions are replaced according to $\psi'_{\beta\alpha} = h_\beta \psi_{\beta\alpha} h_\alpha^{-1}$ (point-wise group multiplication in the right-hand side). This is quite analogous to the setting of linear algebra where one can either rotate some vector space with respect to a fixed basis or rotate the basis of a fixed vector space—both operations being expressed as a non-singular matrix.

In Mathematical Physics (and now also in some areas of Differential Geometry) the group of G-bundle automorphisms is also known as the group of **gauge transformations**[4], sometimes denoted $\mathcal{G}$.

[4]...and informally the 'group of gauge transformations' is often abbreviated as the 'gauge group' of E, although the 'gauge group' is really a different object! (It is the structure group G of vector bundle.) Alas, there is a danger of confusion.

2.2 Connections.

Sections of a vector bundle generalize vector-valued functions on open domains in $\mathbb{R}^n$. Is there a suitable version of *derivative* for sections, corresponding to the differential in multivariate calculus? In order to propose such a derivative, it is necessary at least to understand which sections are to have zero derivative, corresponding to the constant functions on $\mathbb{R}^n$. (Note that a section which is expressed as a constant in one local trivialization need not be constant in another.)

Vertical and horizontal subspaces.

Consider a vector bundle $\pi : E \to B$ with typical fibre $\mathbb{R}^m$ and $\dim B = n$. Let $U \subseteq B$ be a coordinate neighbourhood in B and also a trivializing neighbourhood for E. Write x^k, $k = 1, \dots, n$ for the coordinates on U and a^j, $j = 1, \dots, m$ for the standard coordinates on $\mathbb{R}^m$. Then with the help of local trivialization the tangent space T_pE for any point p, such that $\pi(p) \in U$, has a basis $\{\frac{\partial}{\partial x^k}, \frac{\partial}{\partial a^j}\}$. The kernel of the differential $(d\pi)_p : T_pE \to T_bB$, $b = \pi(p)$, is precisely the tangent space to the fibre $E_b \subset E$, spanned by $\{\frac{\partial}{\partial a^j}\}$.

Definition. The vector space $\mathrm{Ker}\,(d\pi)_p$ is called the **vertical subspace** of T_pE, denoted Tv_pE. A subspace S_p of T_pE is called a **horizontal subspace** if $S_p \cap Tv_pE = \{0\}$ and $S_p \oplus Tv_pE = T_pE$.

Thus any horizontal subspace at p is isomorphic to the quotient T_pE/Tv_pE and has the dimension $\dim(T_pE) - \dim(Tv_pE) = \dim B$. Notice that, unlike the vertical tangent space, a horizontal space can be chosen in many different ways (e.g. because there are many choices of local trivialization near a given point in B).

It is convenient to specify a choice of a horizontal subspace at every point of E as the kernel of a system of differential 1-forms on E, using the following

> *Fact from linear algebra*: if $\theta^1, \dots, \theta^m \in (\mathbb{R}^{n+m})^*$ are linear functionals then one will have $\dim(\cap_{i=1}^m \mathrm{Ker}\,\theta^i) = n$ if and only if $\theta^1, \dots, \theta^m$ are linearly independent in $(\mathbb{R}^{n+m})^*$.

Now let $\theta_p^1, \dots, \theta_p^m$ be linearly independent 'covectors' in T_p^*E, $p \in \pi^{-1}(U)$, and define

$$S_p := \{v \in T_pE |\ \theta_p^i(v) = 0,\ i = 1, \dots, m\}.$$

We can write, using local coordinates on U,

$$\theta_p^i = f_k^i dx^k + g_j^i da^j, \qquad i = 1, \dots, m, \quad f_k^i, g_j^i \in \mathbb{R}. \tag{2.7}$$

and any tangent vector in T_pE as $v = B^k(\frac{\partial}{\partial x^k})_p + C^i(\frac{\partial}{\partial a^i})_p$, $B^k, C^i \in \mathbb{R}$. The θ_p^j cannot all vanish on a *vertical* vector, i.e. on a vector having $B^k = 0$ for all k. That is,

$$\text{if} \qquad g_j^i C^j = 0 \ \text{ for all } i = 1, \dots, m \qquad \text{then} \qquad C^i = 0 \quad \text{for all } i = 1, \dots, m.$$

Therefore the $m \times m$ matrix $g = (g^i_j)$ must be *invertible*. Denote the inverse matrix by $c = g^{-1}$, $c = (c^i_l)$. Replace θ^i by $\tilde{\theta}^i = c^i_l \theta^l = da^i + e^i_k dx^k$, this does not change the space S_p.

The above arrangement can be made for every $p \in \pi^{-1}(U)$, with $f^i_k(p)$ and $g^i_j(p)$ in (2.7) becoming functions of p. Call a map $p \mapsto S_p$ a **field of horizontal subspaces** if the functions $f^i_k(p)$ and $g^i_j(p)$ are smooth. To summarize,

Proposition 2.8. *Let $S = S_p$, $p \in E$, be an arbitrary smooth field of horizontal subspaces in TE. Let x^k, a^j be local coordinates on $\pi^{-1}(U)$ arising, as above, from some local trivialization of E over a coordinate neighbourhood U. Then S_p is expressed as $S_p = \cap_{j=1}^m \mathrm{Ker}\, \theta^j_p$, where*

$$\theta^j = da^j + e^j_k(x, a) dx^k, \tag{2.9}$$

for some smooth functions $e^j_k(x, a)$. These $e^j_k(x, a)$ are uniquely determined *by a local trivialization.*

Definition. A field of horizontal subspaces $S_p \subset T_pE$ is called a **connection on** E if in every local trivialization it can be written as $S_p = \mathrm{Ker}\,(\theta^1_p, \ldots, \theta^m_p)$ as in (2.9), such that the functions $e^i_k(p) = e^i_k(x, a)$ are *linear* in the fibre variables,

$$e^i_k(x, a) = \Gamma^i_{jk}(x) a^j. \tag{2.10a}$$

and so

$$\theta^i_p = da^i + \Gamma^i_{jk}(x) a^j dx^k, \tag{2.10b}$$

where $\Gamma^i_{jk} : U \subset B \to \mathbb{R}$ are smooth functions called the **coefficients of connection** S_p in a given local trivialization.

As we shall see below, the linearity condition in a^i ensures that the horizontal sections (which are to become the analogues of constant vector-functions) form a linear *subspace* of the vector space of all sections of E. (A very reasonable thing to ask for.)

I will sometimes use an abbreviated notation

$$\theta^i_p = da^i + A^i_j a^j, \qquad \text{where } A^i_j = \Gamma^i_{jk} dx^k,$$

So Proposition 2.8 identifies a connection with a system of matrices $A = (A^i_j)$ of differential 1-forms, assigned to trivializing neighbourhoods $U \subset B$.

The transformation law for connections.

Now consider another coordinate patch $U' \subset X$, $U' = \{(x^{k'})\}$ and a local trivialization $\Phi' : \pi^{-1}(U') \to U' \times \mathbb{R}^m$ with $x^{k'}, a^{i'}$ the coordinates on $U' \times \mathbb{R}^m$.

Notation: throughout this subsection, the apostrophe $'$ will refer to the local trivialization of E over U', whereas the same notation without $'$ refers to similar objects in the local trivialization of E over U. In particular, the transition (matrix-valued) functions from U

to U' are written as $\psi_i^{i'}$ and from U' to U as $\psi_{i'}^i$, thus the matrix $(\psi_{i'}^i)$ is inverse to $(\psi_i^{i'})$. Likewise, $(\partial x^k/\partial x^{k'})$ denotes the inverse matrix to $(\partial x^{k'}/\partial x^k)$.

Recall that on $U' \cap U$ we have

$$x^{k'} = x^{k'}(x), \qquad a^{i'} = \psi_i^{i'}(x)a^i. \tag{2.11}$$

Then

$$dx^{k'} = \frac{\partial x^{k'}}{\partial x^k}dx^k, \qquad da^{i'} = d\psi_i^{i'} a^i + \psi_i^{i'} da^i = \frac{\partial \psi_i^{i'}}{\partial x^k} a^i dx^k + \psi_i^{i'} da^i,$$

so

$$\theta^{i'} = da^{i'} + \Gamma^{i'}_{j'k'} a^{j'} dx^{k'} = d\psi_i^{i'} a^i + \psi_i^{i'} da^i + \frac{\partial x^{k'}}{\partial x^k}\Gamma^{i'}_{j'k'} a^{j'} dx^k,$$

and

$$\psi_{i'}^i \theta^{i'} = da^i + (\psi_{i'}^i \cdot \frac{\partial \psi_j^{i'}}{\partial x^k} + \psi_{i'}^i \cdot \frac{\partial x^{k'}}{\partial x^k} \cdot \Gamma^{i'}_{j'k'} \cdot \psi_j^{j'})a^j dx^k.$$

But then $\psi_{i'}^i \theta^{i'} = \theta^i$ and we find, by comparing with (2.10b), that

$$\Gamma^i_{jk} = \Gamma^{i'}_{j'k'} \psi_{i'}^i \psi_j^{j'} \frac{\partial x^{k'}}{\partial x^k} + \psi_{i'}^i \frac{\partial \psi_j^{i'}}{\partial x^k} \tag{2.12a}$$

and, using $A_j^i = \Gamma^i_{jk} dx^k$,

$$A_{j'}^{i'} = \psi_i^{i'} A_j^i \psi_{j'}^j + \psi_i^{i'} d\psi_{j'}^i, \tag{2.12b}$$

Writing A^Φ and $A^{\Phi'}$ for the matrix-valued 1-forms expressing the connection A in the local trivializations respectively Φ and Φ' and abbreviating (2.11) to $\Phi' = \psi\Phi$ for the transition function ψ we obtain from the above that

$$A^{\psi\Phi} = \psi A^\Phi \psi^{-1} + \psi d\psi^{-1} = \psi A^\Phi \psi^{-1} - (d\psi)\psi^{-1}. \tag{2.12c}$$

The above calculations prove.

Theorem 2.13. *Any system of functions* Γ^i_{jk}, $i, j = 1, \dots, m$, $k = 1, \dots, n$ *attached to the local trivializations and satisfying the transformation law* (2.12) *defines on* E *a connection* A*, whose coefficients are* Γ^i_{jk}.

Remark. Suppose that we fix a local trivialization Φ and a connection A on E and regard ψ as a *bundle automorphism* of E, $\psi \in \operatorname{Aut} E$. With his shift of view, the formula (2.12c) expresses the action of the group $\operatorname{Aut} E$ on the space of connections on E. (Cf. the remark on bundle automorphisms and linear algebra, page 27.)

Before considering the third view on connections we need a rigorous and systematic way to consider 'vectors and matrices of differential forms'.

The endomorphism bundle End E. Differential forms with values in vector bundles.

Let (E, B, π) be a vector bundle with typical fibre V and transition functions $\psi_{\beta\alpha}$. If G is a linear map $V \to V$, or **endomorphism** of V, $G \in \operatorname{End} V$, in a trivialization labelled by α, then the same endomorphism in trivialization β will be given by $\psi_{\beta\alpha} G \psi_{\alpha\beta}$ (recall that $\psi_{\alpha\beta} = \psi_{\beta\alpha}^{-1}$).

This may be understood in the sense that the structure group of V acts linearly on $\operatorname{End} V$. Exploiting, once again, the idea of Theorem 2.4 and the accompanying remarks one can construct from E a new vector bundle $\operatorname{End} E$, with the same structure group as for E and with a typical fibre $\operatorname{End} V$. This is called the **endomorphism bundle** of a vector bundle E and denoted $\operatorname{End} E$.

One can further extend the above construction and define over B the vector bundle whose fibres are linear maps $T_bB \to E_b$, $b \in B$ or, more generally, the antisymmetric multilinear maps $T_bB \times \ldots \times T_bB \to E_b$ on r-tuples of tangent vectors. (So the typical fibre of the corresponding bundle is the tensor product $\Lambda^r(\mathbb{R}^n)^* \otimes V$ i.e. the space of antisymmetric multilinear maps $(\mathbb{R}^n)^r \to V$.) The sections of these bundles are called differential 1-forms (respectively r-forms) with values in E and denoted $\Omega^r_B(E)$. In any local trivialization, an element of $\Omega^r_B(E)$ may be written as a vector whose entries are differential r-forms. (The 'usual' differential forms correspond in this picture to the case when $V = \mathbb{R}$ and $E = B \times \mathbb{R}$.)

In a similar manner, one introduces the differential r-forms $\Omega^r_B(\operatorname{End} E)$ with values in the vector bundle $\operatorname{End} E$. These forms are given in a local trivialization as $m \times m$ matrices whose entries are the usual differential r-forms. The operations of products of two matrices, or of a matrix and a vector, extend to $\Omega^r_B(E)$ and $\Omega^r_B(\operatorname{End} E)$, in the obvious way, using wedge product between the entries.

Now from the examination of the transformation law (2.12) we find that although a connection is expressed by a differential form in a local trivialization, **a connection is not in general a well-defined differential form**. The difference between two connections however *is* a well-defined 'matrix of 1-forms', more precisely an element in $\Omega^1_B(\operatorname{End} E)$.

Thus the space of all connections on a given vector bundle E is naturally an *affine space*. Recall that an affine space of points and has a vector space assigned to it and the operation of 'adding a vector to a point to obtain another point' (with certain 'usual' properties). The latter vector space and affine space may be identified, but not canonically—one needs to choose where to map the zero vector. The vector space assigned to the affine space of connections on E is $\Omega^1_B(\operatorname{End} E)$.

A remark on principal bundles

In the case of a *principal G-bundle* $\pi : P \to B$, with $G \subseteq GL(m, \mathbb{R})$ a subgroup, the definitions of vertical and horizontal subspaces Tv_qP and S_q of T_qP, like for the vector bundles above, still make sense with E now replaced by P. A connection on this principal bundle may in fact be realized as a matrix-valued differential 1-form, θ say,—though the entries of θ will be 1-forms on the total space P, rather than on the base. More precisely,

θ will determine a connection on P if and only if θ is (1) horizontal and (2) G-equivariant with respect to the smooth free right action on P.

We may determine a choice of a field of horizontal subspaces by using the kernels of local matrices of 1-forms $\theta_p = (\theta^i_j)_p$ defined around each $p \in P$, where $(\theta^i_j)_p \in \Omega^1(\pi^{-1}(U))$ for $i, j = 1, \dots, m$ and $U \subset B$ a trivializing neighbourhood (and a coordinate neighbourhood) $U \subset B$. Denote by c^i_j the 'obvious' coordinates on the space $\mathrm{Matr}(m, \mathbb{R})$ and require that $(\theta^i_j)_p = dc^i_j + \Gamma^i_{sk}(x) c^s_j dx^k$ for some functions $\Gamma^i_{sk} \in C^\infty(U)$ satisfying the transformation law (2.12). In a more abbreviated notation, $\theta = dC + AC$, where $C = (c^i_j)$ and $A = (A^i_j) = (\Gamma^i_{jk} dx^k)$. One can verify, noting (2.12) and $C' = \psi C$ (cf. (2.11) and page 23), that the forms $C^{-1}\theta = C^{-1}dC + C^{-1}AC$ agree on the intersections $\pi^{-1}(U) \cap \pi^{-1}(U')$ and patch together to define a global 1-form θ on P. Cf. Sheet 3 Q8.

Note the construction of θ does *not* extend to the vector bundles.

Covariant derivatives

Definition . A **covariant derivative** on a vector bundle E is a $\mathbb{R}$-linear operator $\nabla^E : \Gamma(E) \to \Gamma(T^*B \otimes E)$ satisfying a 'Leibniz rule'

$$\nabla^E(fs) = df \otimes s + f\nabla^E s \tag{2.14}$$

for any $s \in \Gamma(E)$ and function $f \in C^\infty(B)$.

Here I used $\Gamma(\cdot)$ to denote the space of sections of a vector bundle. Thus $\Gamma(E) = \Omega^0_B(E)$ and $\Gamma(T^*B \otimes E) = \Omega^1_B(E)$.

Example. Consider a connection A and put $\nabla^E = d_A$ defined in a local trivialization by

$$d_A s = ds + As, \quad s \in \Gamma(E).$$

More explicitly, one can write $s = (s^1, \dots, s^m)$ with the help of a local trivialization, where s^j are smooth functions on the trivializing neighbourhood, and then

$$d_A(s^1, \dots, s^m) = \big((\frac{\partial s^1}{\partial x^k} + \Gamma^1_{jk} s^j) dx^k, \dots, (\frac{\partial s^m}{\partial x^k} + \Gamma^m_{jk} s^j) dx^k \big).$$

The operator d_A is well-defined as making a transition to another trivialization we have $s = \psi s'$ and $A = \psi A' \psi^{-1} - (d\psi)\psi^{-1}$, which yields the correct transformation law for $d_A s = ds + As = d(\psi s') + (\psi A' \psi^{-1} - (d\psi)\psi^{-1}) s' = \psi(ds' + A's') = \psi(d_A s)'$.

Theorem 2.15. *Any covariant derivative ∇^E arises as d_A from some connection A.*

Proof (gist). Firstly, any covariant derivative ∇^E is a *local* operation, which means that is s_1, s_2 are two sections which are equal over an open neighbourhood U of $b \in B$ then $(\nabla^E s_1)|_b = (\nabla^E s_2)|_b$. Indeed, let U_0 be a smaller neighbourhood of b with the closure $\overline{U_0} \subset U$ and consider a cut-off function $\alpha \in C^\infty(B)$, so that $0 \le \alpha \le 1$, $\alpha|_{U_0} = 1$, $\alpha|_{B \setminus U} = 0$. Then $0 = d(\alpha(s_1 - s_2)) = (s_1 - s_2) \otimes d\alpha + \alpha \nabla^E(s_1 - s_2)$, whence $(\nabla^E s_1)|_b = (\nabla^E s_2)|_b$ as required. So, it suffices to consider ∇^E in some local trivialization of E.

The proof now simply produces the coefficients Γ^i_{jk} of the desired A in an arbitrary local trivialization of E, over U say. Any local section of E defined over U may be written, with respect to the local trivialization, as a vector valued function $U \to \mathbb{R}^m$ (m being the rank of E). Let $\mathbf{e}_j$, $j = 1, \dots m$ denote the sections corresponding in this way to the constant vector-valued functions on U equal to the j-th standard basis vector of $\mathbb{R}^m$. Then the coefficients of the connection A are (uniquely) determined by the formula

$$\Gamma^i_{jk} = \big((\nabla^E \mathbf{e}_j)(\frac{\partial}{\partial x_k})\big)^i, \tag{2.16}$$

where $\nabla^E(\mathbf{e}_j)$ is a vector of differential 1-forms which takes the argument a vector field. We used a local coordinate vector field $\frac{\partial}{\partial x_k}$ (which is well-defined provided that a trivializing neighbourhood U for E is also a coordinate neighbourhood for B) and obtained a local section of E, expressed as a smooth map $U \to V$. Then $\Gamma^i_{jk} \in C^\infty(U)$ is the i-th component of this map in the basis $\mathbf{e}_j$ of V.

It follows, from the $\mathbb{R}$-linearity and Leibniz rule for ∇^E, that for an arbitrary local section we must have

$$\nabla^E s = \nabla^E(s^j \mathbf{e}_j) = (ds^i + s^j \Gamma^i_{jk} dx^k)\mathbf{e}_i = d_A s$$

where as usual $A = (A^i_j) = (\Gamma^i_{jk} dx^k)$, so we recover the d_A defined above. It remains to verify that Γ^i_{jk}'s actually transform according to (2.12) in any change of local trivialization, so we get a well-defined connection. The latter calculation is straightforward (and practically equivalent to verifying that d_A is well-defined independent of local trivialization.) □

The definition of covariant derivative further extends to differential forms with values in E by requiring the Leibniz rule, as follows

$$d_A(\sigma \wedge \omega) = (d_A\sigma) \wedge \omega + (-1)^q \sigma \wedge (d\omega), \qquad \sigma \in \Omega^q_B(E),\ \omega \in \Omega^r(B),$$

with $\wedge$ above extended in an obvious way to multiply vector-valued differential forms and usual differential forms. It is straightforward to verify, considering local bases of sections $\mathbf{e}_j$ and differential forms dx^k, that in any local trivialization one has $d_A\sigma = d\sigma + A \wedge \sigma$, where $\sigma \in \Omega^q_B(E)$.

Remark (Parallel sections). A section s of E is called *parallel*, or *covariant constant*, if $d_A s = 0$. In a local trivialization over coordinate neighbourhood U the section s is expressed as $s = (s^1(x), \dots, s^m(x))$, $x \in U$ and the graph of s is respectively $\Sigma = \{(x^k, s^j(x)) \in U \times \mathbb{R}^m : x \in B\}$, a submanifold of $U \times \mathbb{R}^m$. The tangent spaces to Σ are spanned by $\frac{\partial}{\partial x^k} + \frac{\partial s^j}{\partial x^k}\frac{\partial}{\partial a^j}$, for $k = 1, \dots, n$. We find that the 1-forms $\theta^i_{s(x)} = da^i + \Gamma^i_{jk} dx^k s^j$, $i = 1, \dots, m$, vanish precisely on the tangent vectors to Σ.

This is just the horizontality condition for a tangent vector to E and we see that *a section s is covariant constant if and only if any tangent vector to the graph of s is horizontal.* Another form of the same statement: $s : B \to E$ defines an embedding of B in E as the graph of s and the tangent space to the submanifold $s(B)$ at $p \in E$ is the horizontal subspace at p (relative to A) if and only if $d_A s = 0$.

To sum up, a connection on a vector bundle E can be given in **three equivalent ways**:

(1) as a (smooth) field of horizontal subspaces in TE depending linearly on the fibre coordinates, as in (2.10a);
or

(2) as a system of matrix-valued 1-forms A^i_j (a system of smooth functions Γ^i_{jk}) assigned to every local trivialization of E and satisfying the transformation rule (2.12) on the overlaps;
or

(3) as a covariant derivative ∇^E on the sections of E and, more generally, on the differential forms with values in E.

2.3 Curvature.

Let A be a connection on vector bundle E and consider the repeated covariant differentiation of an arbitrary section (or r-form) $s \in \Omega^r_B(E)$ (assume $r = 0$ though). Calculation in a local trivialization gives

$$d_A d_A s = d(ds+As)+A\wedge(ds+As) = (dA)s-A\wedge(ds)+A\wedge(ds)+A\wedge(As) = (dA+A\wedge A)s.$$

Thus $d_A d_A$ is a linear *algebraic* operator, i.e. unlike the differential operator but d_A, the $d_A d_A$ commutes with the multiplication by smooth functions.

$$d_A d_A(fs) = f d_A d_A s, \qquad \text{for any } f \in C^\infty(B). \tag{2.17}$$

Notice that the formula (2.17) does *not* make explicit reference to any local trivialization. We find that $(d_A d_A s)$ at any point $b \in B$ is determined by the value $s(b)$ at that point. It follows that the operator $d_A d_A$ is a multiplication by an endomorphism-valued differential 2-form. (This 2-form can be recovered explicitly in coordinates similarly to (2.16), using a basis $\mathbf{e}_i$ say of local sections and a basis of differential 1-forms dx^k in local coordinates on B.)

Definition. The form

$$F(A) = dA + A \wedge A \in \Omega^2(B; \operatorname{End} E).$$

is called the **curvature form** of a connection A.

Definition. A connection A is said to be **flat** is its curvature form vanishes $F(A) = 0$.

Example. Consider a trivial bundle $B \times \mathbb{R}^m$, so the space of sections is just the vector-functions $C^\infty(B; \mathbb{R}^m)$. Then exterior derivative applied to each component of a vector-function is a well-defined linear operator satisfying Leibniz rule (2.14). The corresponding connection is called **trivial**, or **product connection**. It is clearly a flat connection.

The converse is only true with an additional topological condition that the base B is *simply-connected*; then any flat connection on E induces a (global) trivialization $E \cong B \times \mathbb{R}^m$ (Examples 3, Q7(ii)) and will be a product connection with respect to this trivialisation.

Bianchi identity.

Covariant derivative on a vector bundle E with respect to a connection A can be extended, in a natural way, to any section of B of $\operatorname{End} E$ by requiring the following formula to hold,

$$(d_A B)s = d_A(Bs) - B(d_A s),$$

for every section s of E. Notice that this is just a suitable form of Leibniz rule.

The definition further extends to differential forms with values $\operatorname{End} E$, by setting for every $\mu \in \Omega^p(B; \operatorname{End} E)$ and $\sigma \in \Omega^q(B; E)$,

$$(d_A\mu) \wedge \sigma = d_A(\mu \wedge \sigma) - (-1)^p \mu \wedge (d_A\sigma).$$

(Write $\mu = \sum_k B_k\omega_k$ and $\sigma = \sum_j s_j\eta_j$.) It follows that a suitable Leibniz rule also holds when $\mu_1 \in \Omega^p(B; \operatorname{End} E)$, $\mu_2 \in \Omega^q(B; \operatorname{End} E)$ to give $d_A(\mu_1 \wedge \mu_2) = (d_A\mu_1) \wedge \mu_2 + (-1)^p\mu_1 \wedge (d_A\mu_2)$. In the special case of trivial vector bundle, $E = B \times \mathbb{R}$ and with $d_A = d$ the exterior differentiation, the above formulae recover the familiar results for the usual differential forms.

In particular, any $\mu \in \Omega^2(B; \operatorname{End} E)$ in a local trivialization becomes a matrix of 2-forms and its covariant derivative is a matrix of 3-forms given by

$$d_A\mu = d\mu + A \wedge \mu - \mu \wedge A.$$

Now, for any section s of E, we can write $d_A(d_Ad_A)s = (d_Ad_A)d_As$, i.e. $d_A(F(A)s) = F(A)d_As$ and comparing with the Leibniz rule above we obtain.

Proposition 2.18 (Bianchi identity). *Every connection A satisfies $d_AF(A) = 0$.*

2.4 Orthogonal and unitary connections

Recall that an *orthogonal structure* on a (real) vector bundle E defined as a family Φ_α of local trivializations (covering the base) of this bundle so that all the transition functions $\psi_{\beta\alpha}$ between these take values in the orthogonal group $O(m)$, $m = \operatorname{rk} E$. These trivializations Φ_α are then referred to as *orthogonal trivializations.* There is a similar concept of a *unitary structure* and *unitary trivializations* of a complex vector bundle.

Given a choice of orthogonal (unitary) structure on E, the standard Euclidean (Hermitian) inner product on the typical fibre $\mathbb{R}^m$ ($\mathbb{C}^m$) induces, with the help of the orthogonal (unitary) local trivializations, a well-defined inner product on the fibres of E. Cf. Examples 3 Q2.

Definition. We say that A is an *orthogonal connection* relative to an orthogonal structure, respectively a *unitary connection* relative to a unitary structure, on a vector bundle E if

$$d\langle s_1, s_2\rangle = \langle d_As_1, s_2\rangle + \langle s_1, d_As_2\rangle$$

for any two sections s_1, s_2 of E, where $\langle\cdot,\cdot\rangle$ denotes the inner product on the fibres of E.

Proposition 2.19. *An orthogonal connection has skew-symmetric matrix of coefficients in any orthogonal local trivialization. A unitary connection has skew-Hermitian matrix of coefficients in any unitary local trivialization.*

Proof.

$$0 = \langle d_A(s_1^i \mathbf{e}_i), \overline{s_2^j}\mathbf{e}_j\rangle + \langle s_1^i \mathbf{e}_i, \overline{d_A(s_2^j \mathbf{e}_j)}\rangle - d\langle s_1^i \mathbf{e}_i, \overline{s_2^j}\mathbf{e}_j\rangle = (A_j^i + \overline{A_i^j}) s_1^i \overline{s_2^j} \quad \text{for any } s_1^i, s_2^j,$$

where $\mathbf{e}_i$ is the standard basis of $\mathbb{R}^m$ or $\mathbb{C}^m$. □

Corollary 2.20. *The curvature form $F(A)$ of an orthogonal (resp. unitary) connection A is skew-symmetric (resp. skew-Hermitian) in any orthogonal (unitary) trivialization.*

2.5 Existence of connections.

Theorem 2.21. *Every vector bundle $E \to B$ admits a connection.*

Proof. It suffices to show that there exists a well-defined *covariant derivative* ∇^E on sections of E. We shall construct a example of ∇^E using a *partition of unity.*

Let W_α be an open covering of B by trivializing neighbourhoods for E and Φ_α the corresponding local trivializations. Then on each restriction $E|_{W_\alpha}$ we may consider a trivial product connection $d_{(\alpha)}$ defined using Φ_α. Of course, the expression $d_{(\alpha)}s$ will only make sense over all of B if a section $s \in \Gamma(E)$ is equal to zero away from W_α. Now consider a partition of unity ρ_i subordinate to W_α. The expressions $\rho_i s$, $\rho_i d_{(i)} s$ make sense over all of B as we may extend by zero away from W_i. Now define

$$\nabla^E s := \sum_{i=1}^{\infty} d_{(i)}(\rho_i s) = \sum_{i=1}^{\infty} \rho_i d_{(i)} s, \tag{2.22}$$

where for the second equality we used Leibniz rule for $d_{(i)}$ and the property $\sum_{i=1}^{\infty} \rho_i = 1$ (so $\sum_{i=1}^{\infty} d\rho_i = 0$). The ∇^E defined by (2.22) is manifestly linear in s and Leibniz rule for ∇^E holds because it does for each $d_{(i)}$. □