BASIC PROPERTIES OF HOLOMORPHIC FUNCTIONS

We shall need some basic results from the theory of functions of a single complex variable. The theorems reviewed below are all covered in IB Complex Analysis. One excellent text to consult if you did not take the course, or for further details, is 'Complex Analysis' by Lars V. Ahlfors. You are expected to know the statements of the theorems below but will not, in Topics in Analysis, be directly examined on their proofs.

1 Complex integration

If $f : [a, b] \to \mathbb{C}$ is a continuous function then its integral is defined as

$$\int_a^b f(t)dt = \int_a^b \mathrm{Re}(f(t))dt + i\int_a^b (\mathrm{Im}\, f(t))dt.$$

Proposition 1.1.

$$\left|\int_a^b f(t)dt\right| \le (b-a) \sup_{a\le t\le b} |f(t)|$$

with equality if and only if f is constant.

A path in $\mathbb{C}$ was defined earlier in the lectures as a continuous function from $[a, b]$ to $\mathbb{C}$. In Complex Analysis one restricts attention to piecewise continuously differentiable paths.

Definition 1.2. A *piecewise C^1 path* in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$ for which there is a subdivision $a = a_0 < a_1 < \ldots < a_n = b$, such that for each $k = 1, \ldots, n$, the restriction of γ to $[a_{k1}, a_k]$ is continuously differentiable, where the derivatives at a_{k-1} and a_k are one-sided (right and left, respectively).

Remark that for a given piecewise C^1 path γ the choice of subdivision $\{a_k\}$ in the above definition is not unique.

> In what follows, *all paths will be assumed piecewise C^1*, with $a_0, a_1, \ldots, a_n$ chosen as in Definition 1.2.

The *length* of a path γ is defined as the integral

$$\mathrm{length}(\gamma) = \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} |\gamma'(t)|dt.$$

and is independent of the choice of subdivision $\{a_k\}$.

Definition 1.3. Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be continuous and suppose that the image of a path γ is contained in U. Then the integral of f along γ is

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} f(\gamma(t))\, \gamma'(t)\, dt \tag{1.4}$$

The integral (1.4) will change sign if γ is 'traversed in the opposite direction'. That is, if $\gamma^- : [-b,-a] \to \mathbb{C}$ is given by $\gamma^-(t) = \gamma(-t)$, then $\int_{\gamma^-} f(z)dx = -\int_\gamma f(z)dx$.

Another important property of the path integral (1.4) is that it is invariant under a *change of parameterization.* More precisely, this means if $\varphi : [a', b'] \to [a, b]$ is a piece-wise C^1, increasing function mapping $[a', b']$ onto $[a, b]$ and we set $\delta(t) = \gamma\varphi(t))$, then $\int_\delta f(z)dz = \int_\gamma f(z)dz$. (This is not difficult to check using the change of variables formula for real integrals.) The integral (1.4) is also independent of the choice of $\{a_k\}$.

We shall sometimes abuse notation and identity a curve with its image, e.g. by saying 'a point z in not on γ' or 'γ is a path in $S \subset \mathbb{C}$'.

Proposition 1.5. *if $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be continuous and γ is a path in U, then*

$$\left| \int_\gamma f(z)dz \right| \leq \text{length}(\gamma) \sup_{a \leq t \leq b} |f(\gamma(t))|.$$

We say that a path $\gamma : [a, b] \to \mathbb{C}$ is *closed* if $\gamma(a) = \gamma(b)$. For a closed piecewise C^1 path γ and a point z not on γ, the *winding number* of γ about z (defined earlier in the lectures) can be computed by the integral

$$w(\gamma; z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}.$$

The following concepts will be convenient later.

Definition 1.6. A *domain D* in $\mathbb{C}$ is a path-connected open set of complex numbers.

Definition 1.7.

(i) A closed path γ in a domain D is *homologous to zero in D* if $w(\gamma; z) = 0$ for each $z \in \mathbb{C} \setminus D$.

(ii) A domain D is *simply-connected* if every closed path in D is homologous to zero in D.

For example, a disc, a half-plane, a region between two parallel lines are all simply-connected. An annulus is not.

2 Holomorphic functions and the Cauchy–Riemann equations

Definition 2.8. Let $U \subseteq \mathbb{C}$ be an open set. A function $f : U \to \mathbb{C}$ is *holomorphic on U* if it is complex-differentiable at each point in U, i.e. if

$$f'(z) = \lim_{w \to z} \frac{f(w) - f(z)}{w - z}$$

exists for all $z \in U$.

For $S \subseteq \mathbb{C}$ an arbitrary set of complex numbers, a function f is said to be *holomorphic on S* if f is defined and holomorphic on some open set containing S.

Thus e.g. 'holomorphic at a point $z_0 \in \mathbb{C}$' means 'holomorphic on some open disc about z_0'.

The formal rules for derivatives of a sum, product, quotient, inverse function and the chain rule for functions of one real variable carry over to functions of one complex variable.

Complex polynomials are holomorphic on all of $\mathbb{C}$. A rational function $p(z)/q(z)$ (where p, q are complex polynomials and q is not identically zero) is holomorphic on the complement of the (finite) set of roots of q.

Given a complex-valued function $f : U \subset \mathbb{C} \to \mathbb{C}$, we may write $f(x+iy) = u(x,y) + iv(x,y)$, where $u, v : U \to \mathbb{R}$ are the real and imaginary parts of f and we identified $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way.

Theorem 2.9. *Let U be an open subset of $\mathbb{C}$.*

(i) *A function $f = u + iv : U \to \mathbb{C}$ is complex-differentiable at $z_0 = a + ib \in U$ if and only if u and v are differentiable at (a,b) and satisfy the* Cauchy–Riemann equations

$$u_x(a,b) = v_y(a,b), \qquad u_y(a,b) = -v_x(a,b) \tag{2.10}$$

where the subscripts x and y denote the respective partial derivatives.

(ii) *If the functions $u, v : U \to \mathbb{R}$ have continuous partial derivatives on U and satisfy the Cauchy–Riemann equations* (2.10) *on U, then the function $f = u + iv$ is holomorphic on U.*

3 Some foundational theorems

Theorem 3.11 (Cauchy's theorem)**.** *Let D be a domain in $\mathbb{C}$ and suppose that f is a holomorphic function on D. Then*

$$\int_\gamma f(z)dz = 0$$

for every closed path γ homologous to zero in D.

Corollary 3.12. *If D is a simply-connected domain and f is holomorphic on D, then*

$$\int_\gamma f(z)dz = 0$$

for every closed path in D.

Theorem 3.13 (Cauchy Integral Formula)**.** *Let D be a domain in $\mathbb{C}$, $f : D \to \mathbb{C}$ a holomorphic function and suppose that a closed path $\gamma : [\alpha, \beta] \to D$ is homologous to zero in D. Then*

$$w(\gamma; z)\, f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z}$$

holds for each $z \in U \setminus \mathrm{Image}(\gamma)$.

Theorem 3.14 (Morera's theorem)**.** *If $B \subset \mathbb{C}$ is an open disc, $f : B \to \mathbb{C}$ is continuous and $\int_\gamma f(x)dz = 0$ for every closed path γ in B, then f is holomorphic on B.*

Corollary 3.15 (A uniform limit of holomorphic functions is holomorphic)**.** *If U is an open subset in $\mathbb{C}$, $f_n : U \to \mathbb{C}$ is holomorphic for each $n = 1, 2, \dots$ and $f_n \to f$ uniformly on U, then f is holomorphic on U.*

Theorem 3.16 (Maximum modulus principle)**.** *Let f be a holomorphic function on a domain $D \subset \mathbb{C}$. If there is a $z_0 \in D$ such that $|f(z)| \le |f(z_0)|$ for each $z \in D$, then f is constant on D.*

Corollary 3.17. *Let D be a bounded domain in $\mathbb{C}$. Denote by $\overline{D}$ the closure of D and by $\partial D = \overline{D} \setminus D$ the boundary of D. If $f : \overline{D} \to \mathbb{C}$ is continuous on $\overline{D}$ and holomorphic on D, then*

$$\sup_{\overline{D}} |f(z)| = \sup_{\partial D} |f(z)|$$

Theorem 3.18 (Taylor's theorem)**.** *Let U be an open subset of $\mathbb{C}$, f a holomorphic function on U and $z_0 \in U$. There there is a unique set of complex numbers $c_0, c_1, \ldots,$ such that whenever an open disc $B_\rho(z_0) = \{z : |z - z_0| < \rho\}$ is contained in U, we have*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

for every $z \in B_\rho(z_0)$.

Furthermore, the convergence of the series is uniform on every compact subset of $B_\rho(z_0)$.

Notice that it follows from Taylor's theorem that if f is holomorphic on U, then f can be uniformly approximated *locally*, i.e. on some open disc about each point in U, by complex polynomials. However, without further hypotheses, uniform approximation of f by polynomials on all of U need not hold.

Since we may differentiate power series term-wise, we also have the following.

Corollary 3.19. *If U is an open subset of $\mathbb{C}$ and f is a holomorphic function on U, then f has complex derivatives of all orders everywhere on U (and these are holomorphic on U).*