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The 'Identity Theorem'

Definition. *Let $E \subset \mathbb{C}$. We say that a point $w \in E$ is an* **isolated** *if there is an open disc $D(w, \varepsilon)$, $\varepsilon > 0$, such that $D(w, \varepsilon) \cap E = \{w\}$.*

The opposite notion is that of **accumulation point**: $w \in \mathbb{C}$ is an accumulation point of $E \subset \mathbb{C}$ if every open disc with centre $w \in D$ meets E in infinitely many points. (In this case, there will be a sequence $w_n \neq w$ converging to w as $n \to \infty$.)

Theorem 2.20 (Principle of isolated zeros)**.** *Assume that $f : D(w, R) \to \mathbb{C}$ is a holomorphic function which is not identically zero. Then there is an $r > 0$ such that $f(z) \neq 0$ whenever $0 < |z - w| < r \leq R$.*

Proof. If $f(w) \neq 0$ then a required $r > 0$ exists since f is continuous.

Let $f(w) = 0$. By Theorem 2.17, f is represented on $D(w, R)$ by convergent power series, $f(z) = \sum_{m=0}^{\infty} c_m (z - w)^m$ valid for all $z \in D(w, R)$. and $c_0 = 0$. Let $n > 0$ be the smallest such that $c_n \neq 0$. Then $f(z) = (z - w)^n \sum_{m=n}^{\infty} c_m (z - w)^{m-n} = (z - w)^n g(z)$, where g is holomorphic on $D(w, R)$ and $g(w) \neq 0$. Then g does not vanish on $D(w, r)$ for some $r > 0$, thus f does not vanish on $D(w, r) \setminus \{0\}$. $\square$

We say that f has at w a **zero of order** n if $f(z) = (z-w)^n g(z)$ holds on some disc around w with g is holomorphic and $g(w) \neq 0$. Thus if f is a non-constant holomorphic function on some open set and $f(w) = 0$, then there is an integer $n > 0$, the order of this zero.

Here is another important consequence of Theorem 2.20.

Corollary 2.21 ('Identity Theorem')**.** *Let $D \subset \mathbb{C}$ be a domain and f, g holomorphic functions on D. If the set $E = \{z \in D : f(z) = g(z)\}$ contains a non-isolated (i.e. accumulation) point, then $f(z) = g(z)$ for all $z \in D$.*

Proof. The function $h(z) = f(z) - g(z)$ is holomorphic on D. If $w \in E$ is not isolated then h must vanish on some disc $D(w, \varepsilon)$, $\varepsilon > 0$ (in fact on any disc centred at w and contained in the domain D), otherwise there is a contradiction to Theorem 2.20.

Suppose $a \in D$ is a point not in $D(w, \varepsilon)$. As D is path-connected, we may consider a path $\gamma : [0, 1] \to D$ with $\gamma(0) = w$, $\gamma(1) = a$. Let $t_0 = \sup\{t \in [0, 1] : h(\gamma(s)) = 0 \text{ for all } s \in [0, t]\}$, this is well-defined as the set in question is non-empty (contains zero) and bounded. Then $h(\gamma(t_0)) = 0$ as $h \circ \gamma$ is continuous. So $\gamma(t_0)$ is a non-isolated zero of h and (noting the previous argument) $h \circ \gamma$ must vanish on $[t_0, t_0 + \delta)$ for some $\delta > 0$. This contradicts the definition of t_0 unless $t_0 = 1$. Thus $h(a) = h(\gamma(1)) = 0$ and the result follows. $\square$