Part III: Differential geometry (Michaelmas 2019)

Example Sheet 3

Throughout the course you are expected to use standard results from analysis without proof. The questions or parts marked with * are not necessarily harder, but go slightly beyond the lectured material and will not be examined.

1. Let E, E' be vector bundles over M, with $\Gamma(E), \Gamma(E')$ respectively the spaces of their smooth sections (over M), and suppose that $\alpha : \Gamma(E) \to \Gamma(E')$ is a map which is linear over $C^\infty(M)$. In this exercise you will verify that α is induced by a bundle morphism $F : E \to E'$.
   (i) Let $p \in M$ and $v \in E_p$. Show that there is an $s \in \Gamma(E)$ such that $s(p) = v$.
   [Work in a local trivialization and use a smooth cut-off function.]
   (ii) Now let $s \in \Gamma(E)$. Show that if s vanishes in a neighbourhood of p then so does $\alpha(s)$. Deduce that if $s(p) = 0$ then $\alpha(s)(p) = 0$.
   [For the first use cut-off functions, for the second work in a local trivialization.]
   (iii) Set $F(v) = \alpha(s)(p)$, where $s \in \Gamma(E)$ and $s(p) = v$. Show that F is a well-defined (independent of s) smooth bundle morphism, and finally show $\alpha(s) = F \circ s$.

2. Show that every (real) vector bundle can be given a positive definite inner product, varying smoothly with the fibres, i.e. given in each local trivialization (U_α, Φ_α) by a smooth map $g_\alpha : x \in U_\alpha \to g_\alpha(x) \in \mathrm{Sym}_+(n, \mathbb{R})$. Here $n = \operatorname{rank} E$ and $\mathrm{Sym}_+(n, \mathbb{R})$ denotes the set of all real positive-definite $n \times n$ symmetric matrices.
   [Hint: you might like to use a partition of unity.]
   Deduce that every real vector bundle admits an $O(n)$-structure. What about complex vector bundles?

3. (i) Consider the tangent bundle of S^2 endowed with an inner product on the fibres. Let Y denote the subset of the total space of TS^2 consisting of all the tangent vectors of unit length. Show that the tangent bundle TS^2 induces on Y the structure of a principal S^1-bundle over S^2.
   (ii)* Show further that the 3-dimensional manifold Y is diffeomorphic to $\mathbb{R}P^3$.
   [Hint for (ii): Find a correspondence between points of Y and *ordered pairs* of orthogonal unit vectors in $\mathbb{R}^3$. Recall also that $\mathbb{R}P^3$ is diffeomorphic to $SO(3)$.]

4. Let $E \to B$ be a rank 1 complex vector bundle with a unitary structure and A a unitary connection on E and $F(A)$ the curvature of A. Prove that the 2-form $\frac{i}{2\pi}F(A)$ on B is closed, and its de Rham cohomology class is well-defined independent of the choice of A. (This class therefore depends only on the bundle E; it is sometimes called the *first Chern class of* E, denoted $c_1(E)$.)

5*. Consider the situation as in the previous question but now suppose in addition that the base B of E is a compact oriented 2-dimensional manifold. Show that the *first Chern number of* E, $\int_B c_1(E) = \frac{i}{2\pi}\int_B F(A)$, is well-defined (independent of the choice of connection A). Find the first Chern number of the Hopf vector bundle over $\mathbb{C}P^1$.
   [You will need to consider a unitary structure on Hopf bundle, as given in the lectures, and construct a unitary connection. If the curvature form of this connection vanishes on some open neighbourhood in $\mathbb{C}P^1$ then the integral can be effectively computed over an open domain in $\mathbb{R}^2$, or $\mathbb{C}$, using Green's formula. Answer: -1.]

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http://www.dpmms.cam.ac.uk/∼agk22/dge3.pdf

6. Prove the following *integrability theorem* for flat connections. If E is a vector bundle over the open hypercube $H = \{x \in \mathbb{R}^n : \max_i |x_i| < 1\}$ and A is a flat connection on E then there is a bundle isomorphism taking E to the trivial bundle over H and A to the trivial (product) connection.
[Hint: it is a good idea to use induction in n. Killing the coefficient of A at any dx^k amounts to solving a linear ODE (with parameters).]

7. (i) Assuming the integrability theorem stated in question 6, deduce that if a vector bundle admits a flat connection then there is a choice of local trivializations of this bundle, so that the corresponding transition functions are *constant*, $\psi_{\beta\alpha}(x) \equiv h_{\beta\alpha} \in GL(V)$, for all $x \in U_\beta \cap U_\alpha$ (here V denotes the typical fibre).

   (ii) Show further that a flat connection on a vector bundle E over a simply-connected base manifold B, determines an isomorphism between E and a product bundle, i.e. a trivialization of E over all of B.
[Hint: covariant-constant sections.]

8. Let $\pi : P \to B$ be a principal G-bundle, $G = GL(m, \mathbb{R})$, with $R_g : q \in P \to qg \in P$ the smooth free right action ($g \in G$). The definitions of vertical and horizontal subspaces Tv_qP and S_q of T_qP are identical to those for vector bundles in the lectures. Show that for all g, $(dR_g)_q$ maps every horizontal S_q onto a horizontal subspace.

   We say that a field of horizontal subspaces $S = S_q$ is a **connection on** P if it is G-equivariant: $(dR_g)_q S_q = S_{qg}$, for all $q \in P$ and $g \in G$. Show that then in each local trivialization of P over a coordinate neighbourhood $U \subset B$ with coordinates x^k, $S_q = \cap_{i,j} \operatorname{Ker}(\theta^i_j)_q$, where $(\theta^i_j)_q = dc^i_j + \Gamma^i_{sk}(x) c^s_j dx^k$ for some $\Gamma^i_{sk} \in C^\infty(U)$ and c^i_j are the standard coordinates on G. Show further that these Γ^i_{sk} also define a connection on the vector bundle E associated to P via the usual action of G on $\mathbb{R}^m$.

   Show that for every connection S on P, the matrix-valued 1-forms $C^{-1}\theta = C^{-1}dC + C^{-1}AC$, where $C = (c^i_j)$, $\theta = (\theta^i_j)$ and $A = \Gamma^i_{jk} dx^k$, agree on the intersections $\pi^{-1}(U) \cap \pi^{-1}(U')$ and patch together to define a global 1-form on P.

9. Verify that if M is a submanifold of $\mathbb{R}^N$ then the Euclidean inner product restricts to define a Riemannian metric on M.
Let the symbol dS^2 denote the expression for the induced metric on the unit sphere $S^{n-1} \subset \mathbb{R}^n$. Show that the Euclidean metric on $\mathbb{R}^n \setminus \{0\}$ can be expressed in polar coordinates as $g = dr^2 + r^2 dS^2$, where $r = |x|$, $x \in \mathbb{R}^n$.

10. Suppose that $\mathbf{r} = \mathbf{r}(u, v)$ is a regular parameterization of a surface S in the affine space $\mathbb{R}^3$. There is a standard choice of a 'moving frame' (a basis of the tangent space $T_{\mathbf{r}}\mathbb{R}^3$) $\mathbf{r}_u$, $\mathbf{r}_v$, $\mathbf{n}$ at every point $\mathbf{r}$ of S, where $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v / |\mathbf{r}_u \times \mathbf{r}_v|$ is a unit normal vector to S. (Here the subscripts u and v at $\mathbf{r}$ are used to denote the respective partial derivatives.) Then there is a unique way to write the second derivatives of $\mathbf{r}$ as

$$\mathbf{r}_{uu} = \Gamma^1_{11}\mathbf{r}_u + \Gamma^2_{11}\mathbf{r}_v + L\mathbf{n}, \quad \mathbf{r}_{uv} = \Gamma^1_{12}\mathbf{r}_u + \Gamma^2_{12}\mathbf{r}_v + M\mathbf{n}, \quad \mathbf{r}_{vv} = \Gamma^1_{22}\mathbf{r}_u + \Gamma^2_{22}\mathbf{r}_v + N\mathbf{n},$$

   for some functions Γ^i_{jk}, L, M, N on S. By deducing the expressions for Γ^i_{jk} in terms of the first fundamental form of S, or otherwise, show that Γ^i_{jk} are the Christoffel symbols for the Levi–Civita connection of the metric induced on S by restriction from the ambient $\mathbb{R}^3$.

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