Part III: Differential geometry (Michaelmas 2019)

Example Sheet 2

Throughout the course you are expected to use standard results from analysis without proof. The questions or parts marked with * are not necessarily harder, but go slightly beyond the lectured material and will not be examined.

1. (*i*) Show that an $\omega \in \Lambda^2((\mathbb{R}^n)^*)$ can be written as a wedge product of co-vectors $\omega = \eta_1 \wedge \eta_2$, $\eta_i \in (\mathbb{R}^n)^*$, if and only if $\omega \wedge \omega = 0$ (n is arbitrary). If instead $\omega \in \Lambda^3((\mathbb{R}^n)^*)$ satisfies $\omega \wedge \omega = 0$, must ω be a wedge product of a co-vector and a 2-form?
   (*ii*) Let α be a nowhere-zero 1-form. Prove that for a $(p+1)$-form β $(p \geq 0)$, one has $\alpha \wedge \beta = 0$ if and only if $\beta = \alpha \wedge \gamma$ for some p-form γ. [You might like to do it on $\mathbb{R}^n$ first. Partition of unity is useful in the general case.]

2. Prove that $\mathbb{R}P^n$ is orientable if and only if n is odd.
   [Hint: consider the $2:1$ map $S^n \to \mathbb{R}P^n$ and a suitable choice of orientation n-form on S^n.]

3. Show that if a manifold is simply-connected, then it is orientable.

4. Prove the identity $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$, for a 1-form ω and vector fields X, Y. *Can you generalize this relation to the case when ω is a p-form?

5. Show that
   $$d\omega = 0, \qquad \text{where} \quad \omega = \frac{-y dx + x dy}{x^2+y^2},$$
   and that on the half-plane $U = \{(x,y) \in \mathbb{R}^2 : x > 0\}$, $\omega = df$ for some smooth function $f : U \to \mathbb{R}$. Show that ω cannot be written as df for any smooth function f on $\mathbb{R}^2 \setminus \{0\}$.
   Hence or otherwise deduce that the de Rham cohomology of the circle is $H^1(S^1) = \mathbb{R}$.

6. (*i*) Show that every closed 1-form on S^2 is exact.
   (*ii*) Construct a linear isomorphism $H^k(S^n) \cong H^{k-1}(S^{n-1})$, for all $k, n > 1$. Calculate the de Rham cohomology $H^k(S^n)$ for every k, n.
   [You may assume a generalized version of the Poincaré Lemma: for M any smooth manifold, $H^k(M \times \mathbb{R}) \cong H^k(M)$ for all k.]

7. Construct a nowhere-vanishing (smooth) vector field on S^{2n+1} for each n.

8. Prove that a principal G-bundle $P \to B$ has a smooth global section if and only if P is (isomorphic to) a trivial bundle.

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http://www.dpmms.cam.ac.uk/∼agk22/dge2.pdf

9. Let $\pi : P \to B$ be a principal G-bundle. Show that $\pi^* P \to P$ is a trivial principal G-bundle.

10. Show that map $(x_0 : x_1 : x_2 : x_3) \in \mathbb{R}P^3 \to (x_0 + ix_1) : (x_2 + ix_3) \in \mathbb{C}P^1$ defines a principal $U(1)$-bundle, the two standard coordinate patches on $\mathbb{C}P^1$ may be taken as trivializing neighbourhoods, and the transition function then is given by $\psi(z : 1) = (z/|z|)^2$.

11. Let G be a Lie group and X_i, $i = 1, \dots, d = \dim G$, a system of linearly independent left-invariant vector fields on G induced by a basis of $T_I G$. Show that the condition

$$\omega^i(X_j) = \delta^i_j \quad \text{identically on } G$$

defines a system of smooth 1-forms ω^i on G which are linearly independent at each point. Show further that the 1-forms ω^i are *left-invariant* in the sense that

$$L_g^*(\omega^i) = \omega^i, \qquad \text{for every } g \in G.$$

(More precicely, $L_g^*(\omega^i(h)) = \omega^i(g^{-1}h)$. Here $L_g(h) = gh$ for each $h \in G$.) Let C^k_{ij} be a set of real constants determined by $[X_i, X_j] = C^k_{ij} X_k$ (the summation convention is assumed). Show that

$$d\omega^k = -\sum_{i<j} C^k_{ij} \omega^i \wedge \omega^j.$$

12.* (*i*) A *flow* on a manifold M may be defined as a smooth one-parameter family of diffeomorphisms A_t ($t \in \mathbb{R}$) of M onto itself, satisfying $A_{t+s} = A_t \circ A_s$ and $A_{-t} = A_t^{-1}$ and $A_0 = \mathrm{id}_M$. Show that $X(p) = \frac{d}{dt}(A_t(p))|_{t=0}$ ($p \in M$) defines a (smooth) vector field on M. (In this sense, vector fields are the 'infinitesimal diffeomorphisms' of M.)

(*ii*) Recall that any diffeomorphism A of M converts a function f on M into a new function $f \circ A$ and a vector field Y into a new vector field $(dA)Y$. If A_t and X are as defined in (*i*) then the operation $\frac{d}{dt}(f \circ A_t)|_{t=0}$, resp. $\frac{d}{dt}\big((dA_t)Y\big)|_{t=0}$, is denoted $\mathcal{L}_X$ and called the *Lie derivative* in the direction of X. Verify that

$$\mathcal{L}_X f = Xf, \qquad \text{for } f \in C^\infty(M),$$
$$\mathcal{L}_X Y = [Y, X], \qquad \text{for a vector field } Y \text{ on } M.$$

(*iii*) (The 'infinitesimal Stokes' formula'.) The Lie derivative of any differential form ω of degree $r > 0$ is defined in a similar manner to (*ii*), i.e. $\mathcal{L}_X \omega = \frac{d}{dt}(A_t^* \omega)|_{t=0}$. Let $X \lrcorner\, \omega$ denote the interior product, i.e. a $(r-1)$-form $(X \lrcorner\, \omega)(X_1, \dots, X_{r-1}) = \omega(X, X_1, \dots, X_{r-1})$. Show that the Lie derivative of ω may be computed as

$$\mathcal{L}_X \omega = X \lrcorner\, d\omega + d(X \lrcorner\, \omega).$$

13. Modify the construction of Hopf bundle the lectures replacing $\mathbb{C}$ everywhere by $\mathbb{R}$ to obtain a rank one real vector bundle over S^1. The total space of this $\mathbb{R}$-analogue of Hopf (vector) bundle is thus a surface (2-dimensional manifold). Can you identify this surface?

Comments welcome at any time. `A.G.Kovalev@dpmms.cam.ac.uk`