**Part III: Differential geometry (Michaelmas 2019)**

# Example Sheet 1

Throughout the course you are expected to use standard results from analysis without proof. The questions marked with * are not necessarily harder, but may go slightly beyond the lectured material and will not be examined.

1. (*i*) Let $M_1$ and $M_2$ be smooth manifolds of dimension $m_1$ and $m_2$ respectively. Show that $M_1 \times M_2$ is naturally a manifold of dimension $m_1 + m_2$ and the projections $p_i : M_1 \times M_2 \to M_i$ are smooth maps. Show also that if $N$ is another manifold then a map $f : N \to M_1 \times M_2$ is smooth if and only if $p_i \circ f$ is smooth for each $i = 1, 2$.

   (*ii*) Let $M$ be a smooth manifold, $U \subset M$ is a non-empty open subset and $f : U \to \mathbb{R}$ a smooth function. Must $f$ extend smoothly to $M$? Justify your answer.

2. Do the charts $\varphi_1(x) = x$ and $\varphi_2(x) = x^3$ $(x \in \mathbb{R})$ belong to the same $C^\infty$ structure on $\mathbb{R}$? Let $R_j$, $j = 1, 2$, denote the manifold obtained by using the chart $\varphi_j$ on the topological space $\mathbb{R}$. Are $R_1$ and $R_2$ diffeomorphic?

3. (*i*) Let $S^n \subset \mathbb{R}^{n+1}$ be the unit $n$-sphere about the origin and consider
$$\varphi_i^\pm : (x_0, \ldots, x_n) \in V_i^\pm \to (x_0, \ldots \widehat{x}_i \ldots, x_n) \in \mathbb{R}^n$$
   where $V_i^+ = \{(x_0, x_1, \ldots, x_n) : x_i > 0\}$, $V_i^- = \{(x_0, x_1, \ldots, x_n) : x_i < 0\}$. Verify that $\varphi_i^\pm$ define charts inducing a $C^\infty$ structure on $S^n$. Are these charts compatible (belong to the same $C^\infty$ structure) with stereographic projections as in the lectures?

   (*ii*) Show that the natural map $(x_0, \ldots, x_n) \in S^n \to x_0 : \ldots : x_n \in \mathbb{R}P^n$ is smooth.

4. By adapting the construction of charts via exponential mapping, as defined in the lectures, or otherwise, show that the following are Lie groups

   (*i*) special linear group $SL(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det A = 1\}$;
   (*ii*) unitary group $U(n) = \{A \in GL(n, \mathbb{C}) : AA^* = I\}$, where $A^*$ denotes the conjugate transpose of $A$ and $I$ is the $n \times n$ identity matrix;
   (*iii*) $Sp(m) = \{A \in U(2m) : AJA^t = J\}$, where $A^t$ denotes the transpose of $A$ (no complex conjugation!) and $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$.

   Identify the respective Lie algebras and find the dimensions of the above Lie groups.

5. Show that

   (*i*) the complex projective line $\mathbb{C}P^1$ is diffeomorphic to the sphere $S^2$;
   (*ii*) $SU(2)$ is diffeomorphic to $S^3$;
   (*iii*) $TS^1$ is diffeomorphic to $S^1 \times \mathbb{R}$;
   (*iv*) $TS^3$ is diffeomorphic to $S^3 \times \mathbb{R}^3$ (and, more generally, if $G$ is a Lie group then $TG$ is diffeomorphic to $G \times \mathbb{R}^d$, where $d = \dim G$).

   In (*iii*) and (*iv*) it is understood that the diffeomorphism should convert the tangent bundle projection into the first projection of the product.

---

http://www.dpmms.cam.ac.uk/∼agk22/dge1.pdf

6. Let $f : M \to N$ be a diffeomorphism of manifolds and $X$, $Y$ two smooth vector fields on $M$. Show that the differential map $df$ commutes with the Lie brackets, i.e. that $(df)([X,Y]) = [(df)X, (df)Y]$ as vector fields on $N$.

7. Construct an embedding of the $n$-dimensional torus $T^n$ in $\mathbb{R}^{n+1}$.

8. For which values of $c \in \mathbb{R}$ is the zero locus in $\mathbb{R}^3$ of the polynomial
$$z^2 - (x^2 + y^2)^2 + c$$
an *embedded* submanifold of $\mathbb{R}^3$ and for which $c$ is it an *immersed* submanifold?

9. Prove that a compact manifold $M$ of dimension $n$ can be embedded in $\mathbb{R}^N$, for some $N$. [Hint: use a finite open cover of $M$ by coordinate patches $U_1, \ldots, U_m$, with $\varphi_i$ the charts on $U_i$, and use cut-off functions $\beta_i \in C^\infty(M)$, $0 \le \beta_i \le 1$, so that $\beta_i|_{M \setminus U_i} \equiv 0$, $\beta_i|_{W_i} \equiv 1$ for some open $W_i \subset M$, with $\cup_{i=1}^m W_i = M$. Consider a map $p \in M \to (\ldots,\ \beta_i(p)\varphi_i(p),\ \beta_i(p),\ \ldots) \in \mathbb{R}^{m(n+1)}$.]

10. Prove that the map
$$\rho(x:y:z) = \frac{1}{x^2+y^2+z^2}(x^2, y^2, z^2, xy, yz, zx)$$
gives a well-defined embedding of $\mathbb{R}P^2$ into $\mathbb{R}^6$. Find on $\mathbb{R}^6$ a finite system of polynomials, of degree $\le 2$, whose common zero locus is precisely the image of $\rho$. Compare the codimension $\dim \mathbb{R}^6 - \dim \mathbb{R}P^2 = 4$ and the number of polynomials you required. (The map $\rho$ is a variant of 'Veronese embedding' important in Algebraic Geometry.) Construct an embedding of $\mathbb{R}P^2$ in $\mathbb{R}^4$.

11.* Show that $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$. [Hint: every rotation $A \in SO(3)$ may be written as a composition of two reflections in the planes orthogonal to unit vectors, say $\mathbf{a}$, then $\mathbf{b}$. After $\mathbf{a}$ is chosen, $\mathbf{b}$ is determined up to a $\pm 1$ factor. Express the desired diffeomorphism using the map $(\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{a} \times \mathbf{b}\,,\ \mathbf{a} \cdot \mathbf{b})$ onto $S^3/\pm 1$.]
Deduce that $T(\mathbb{R}P^3)$ is diffeomorphic to $\mathbb{R}P^3 \times \mathbb{R}^3$.

12.* (Calabi–Rosenlicht) Let $X = \{(x,y,z) \in \mathbb{R}^3 : xz = 0\}$ be the union of the $(x,y)$- and $(y,z)$-coordinate planes and define a family $U_a$, $a \in \mathbb{R}$, of subsets of $X$ by $U_a = \{(x,y,z) \in X : x \neq 0 \text{ or } y = a\}$ (you might find it helpful to make a sketch of $U_a$). What is $U_a \cap U_b$ for $a \neq b$?
Now let the map $h_a : U_a \to \mathbb{R}^2$ to the plane $\mathbb{R}^2$ with the coordinates $(u_a, v_a)$ be given by
$$u_a = x, \qquad v_a = \begin{cases} (y-a)/x, & \text{if } x \neq 0 \\ z & \text{if } x = 0 \end{cases}.$$
Show that $h_a$ is a bijection onto $\mathbb{R}^2$, find its inverse, and obtain the formula for $h_b \circ h_a^{-1} : h_a(U_a \cap U_b) \to h_b(U_a \cap U_b)$, for any $a, b \in \mathbb{R}$. Deduce that the family of charts $h_a$, $a \in \mathbb{R}$, defines a smooth structure on $X$ and that $X$ (with the induced topology from this smooth structure) is Hausdorff, connected, but *not* second countable.

Comments welcome at any time. `A.G.Kovalev@dpmms.cam.ac.uk`